CONGRUENCES RELATING REGULAR PARTITION FUNCTIONS, A GENERALIZED TAU FUNCTION, AND PARTITION FUNCTION WEIGHTED COMPOSITION SUMS

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Abstract

Let n and t be positive integers with $t \geq 2$. Let $R_t(n)$ be the number of t-regular partitions of n. A class of functions, denoted $\tau_k(n)$, is defined as follows:

$$q\prod_{m=1}^{\infty}(1-q^m)^k = \sum_{n=1}^{\infty}\tau_k(n)q^n,$$

where k is an integer. We express $\tau_k(n)$ as a binomial coefficient weighted partition sum. Consequently, we obtain congruence identities that relate $\tau_k(n)$, $R_t(n)$ and partition function weighted composition sums.

1. Introduction

Euler [3] considered the following product-to-sum representation:

$$\prod_{m=1}^{\infty}(1-q^m) = \sum_{n=0}^{\infty}\omega(n)q^n,$$

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and found that

$$\omega(n) = \begin{cases} (-1)^l & \text{if } n = \frac{3l^2 \pm l}{2}; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

This is the celebrated Euler's pentagonal number theorem.

Ramanujan [15] considered the following product-to-sum representation:

$$q \prod_{m=1}^{\infty} (1 - q^m)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n,$$

and made the following conjectures:

1. $\tau(nm) = \tau(n)\tau(m)$ if $\gcd(m, n) = 1$,
2. for prime p and integer $r \geq 1$: $\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$,
3. for prime p: $|\tau(p)| \leq 2p^{\frac{11}{2}}$.

The first two were established by Mordell [12]. Delinge [2] established the third. The function $\tau(n)$ defined above is known as Ramanujan's tau function.

The following common generalization of the aforementioned functions of Ramanujan and Euler is the object of study in this article.

Definition 1. Let $k \neq 0$ be an integer. We define an arithmetical function, denoted $\tau_k(n)$, in the following way:

$$q \prod_{m=1}^{\infty} (1 - q^m)^k = \sum_{n=1}^{\infty} \tau_k(n) q^n. \tag{2}$$

Newman [14] and Kostant [10] were concerned with the polynomial representation of $\tau_k(n)$. Serre [17] and Heim et al. [7] examined the natural density of the set $\{k \in \mathbb{N} : \tau_k(n) \neq 0\}$ at several instances of n.

The main objective of this paper is to explore various arithmetic properties of $\tau_k(n)$. We will first use the logarithmic derivative method to determine some arithmetic properties of $\tau_k(n)$ (at specific instances of k). This forms the core part of Section 2. In Section 3, $\tau_k(n)$ is expressed as a binomial coefficient weighted partition sum. As a result, congruence relations involving $\tau_k(n)$ and t-regular partition functions are obtained (at certain instances of k and t). In Section 4, an expression for

$$\sum_{\substack{n=a_1+a_2+\cdots a_k \\ a_i \in \mathbb{N}\cup\{0\}}} p(a_1)p(a_2)\cdots p(a_k) \text{ modulo } l$$

is obtained, where l is an odd prime number, and $p(n)$ is the number of partitions of n.

2. Divisibility Properties of $\tau_k(n)$ Using Logarithmic Differentiation

In this section, we will discuss several congruence properties of $\tau_k(n)$ when the modulus belongs to the set $\{k-1\} \cup \{d \in \mathbb{N} : d|k\}$.

2.1. $\tau_k(n)$ Modulo $k-1$ When $k-1$ Is a Prime Number

Proposition 1. *Let n be a positive integer, and let $k-1$ be a prime number. If $0 \le n-1-\frac{3r^2 \pm r}{2} \not\equiv 0 \pmod{(k-1)}$ for every non-negative integer r, then*

$$\tau_k(n) \equiv 0 \pmod{(k-1)}.$$

Proof. The pentagonal number theorem of Euler [3] allows us to write

$$\begin{aligned} \sum_{n=1}^{\infty} \tau_k(n) q^n &= q \prod_{m=1}^{\infty} (1-q^m)^k \\ &= \left(\sum_{r=1}^{\infty} \tau_{k-1}(r) q^r \right) \left(\sum_{s=0}^{\infty} \omega(s) q^s \right), \end{aligned} \tag{3}$$

where $\omega(s)$ is as in (1). When equating the coefficients of q^n at the extremes of the chain of equalities (3), we obtain the following identity:

$$\sum_{i=0}^{n-1} \tau_{k-1}(n-i)\omega(i) = \tau_k(n).$$

If $\tau_{k-1}(n-i) \equiv 0 \pmod{(k-1)}$ whenever $\omega(i) = \pm 1$, then it follows that $\tau_k(n) \equiv 0 \pmod{(k-1)}$. So a criterion for

$$\tau_{k-1}(n) \equiv 0 \pmod{(k-1)}$$

is a requisite to proceed further. To that end, we define

$$T_{k-1}(q) = \prod_{m=1}^{\infty} (1-q^m)^{k-1} = \sum_{n=1}^{\infty} \tau_{k-1}(n) q^{n-1}.$$

Now performing the operation $q\frac{d}{dq}(\log T_{k-1}(q))$ and considering the Lambert's series expansion for the sum of positive divisors of n (denoted $\sigma(n)$), we obtain

$$n\tau_{k-1}(n+1) = -(k-1) \left(\sum_{i=1}^{n} \tau_{k-1}(i) \sigma(n+1-i) \right). \tag{4}$$

Now we observe from the Identity (4) that, if $\gcd(n, k-1) = 1$, then $\tau_{k-1}(n+1) \equiv 0 \pmod{(k-1)}$. Since $k-1$ is a prime number, the condition $\gcd(n, k-1) = 1$ is equivalent to the condition $n \not\equiv 0 \pmod{(k-1)}$. Thus, if $n \not\equiv 0 \pmod{(k-1)}$ then $\tau_{k-1}(n+1) \equiv 0 \pmod{(k-1)}$. This completes the proof. $\square$

Remark 1. Since the τ function was introduced, an in-depth study over the value $\tau(n)$ modulo 23 is an important consideration. Mordell [13] gave the following criterion for the divisibility of $\tau(n)$ by 23:

$$\tau(23n+m) \equiv 0 \pmod{23}$$

for $m \in \{5,7,10,11,14,15,17,19,20,21,22\}$. Here an application of Proposition 1 gives the following criterion: if $0 \leq n-1-\frac{3r^2\pm r}{2} \not\equiv 0 \pmod{23}$ for every non-negative integer r, then

$$\tau(n) \equiv 0 \pmod{23}.$$

2.2. $\tau_k(n)$ Modulo Divisors of k

Interestingly, by substituting any formal power series with integer coefficients, say $f(q) \in \mathbb{Z}[[q]]$, for $\prod_{m=1}^{\infty}(1-q^m)$, one can further generalize Definition 1. Denote

$$f(q)^k = a_0 + a_1 q + a_2 q^2 + \cdots .$$

On differentiating with respect to q, we have

$$kf(q)^{k-1}f'(q) = a_1 + 2a_2 q + \cdots .$$

This gives the relation

$$na_n \equiv 0 \pmod{|k|}.$$

Now fixing $f(q) = \prod_{m=1}^{\infty}(1-q^m)$ we have $a_n = \tau_k(n+1)$. This observation yields the following result.

Proposition 2. *Let n be a positive integer, and let k be an integer with $|k| \geq 2$. Then we have*

$$n\tau_k(n+1) \equiv 0 \pmod{|k|}.$$

The following result is a straightforward application of Proposition 2.

Proposition 3. *Let $m \geq 0$ and k be integers such that $|k| \geq 2$. Then we have*

$$\tau_k(|k|m + dr + 1) \equiv 0 \left(mod\ \frac{|k|}{d} \right)$$

for every $d \mid |k|$ such that $d < |k|$, and for every integer r such that $\gcd(r, \frac{|k|}{d}) = 1$.

Proof. We have the congruence below based on Proposition 2:

$$n\tau_k(n+1) \equiv 0 \pmod{|k|}.$$

Given the aforementioned congruence and $\gcd(n, |k|) = d$, it can be deduced that

$$\frac{n}{d}\tau_k(n+1) \equiv 0 \left(\bmod \frac{|k|}{d}\right).$$

Since $\gcd(n, |k|) = d$, for a positive integer r satisfying $\gcd\left(r, \frac{|k|}{d}\right) = 1$, the integer $\frac{n}{d}$ must have the form $\frac{n}{d} = \frac{|k|}{d}m + r$ for some integer $m \geq 0$. As a result, n assumes the form $n = |k|m + dr$. $\square$

The following list of congruences for $\tau(n)$ modulo the divisors of 24 is obtained by substituting 24 for k.

Proposition 4. *For every integer $m \geq 0$, we have*

1. $\tau(24m + r + 1) \equiv 0 \pmod{24}$ *for each* $r \in \{1, 5, 7, 11, 13, 17, 19, 23\}$*;*
2. $\tau(24m + r + 1) \equiv 0 \pmod{12}$ *for each* $r \in \{4, 20\}$*;*
3. $\tau(24m + r + 1) \equiv 0 \pmod{8}$ *for each* $r \in \{3, 9, 6, 15\}$*;*
4. $\tau(24m + r + 1) \equiv 0 \pmod{6}$ *for each* $r \in \{8, 16\}$*;*
5. $\tau(24m + 13) \equiv 0 \pmod{4}$*.*

3. Representation of $\tau_k(n)$ as a Partition Sum Involving Binomial Coefficients

In this section, congruence properties of $\tau_k(n)$ are derived using a partition sum representation (involving binomial coefficients) of $\tau_k(n)$. Presenting the main results of this section requires the following definitions of partition theory.

Definition 2. Let n be a positive integer. By a *partition of* n, we mean a non-increasing sequence of positive integers whose sum equals n. Each element of the sequence is called a *part*. If each part, say a_i, appears f_i times in a partition of n then we denote that partition by $n = a_1^{f_1} \cdots a_r^{f_r}$. In this case $f_1, \cdots, f_r$ are said to be the *frequencies* of the partition $a_1^{f_1} \cdots a_r^{f_r}$.

Definition 3. Let n and $t \geq 2$ be positive integers. If all of the parts of a partition of n are not divisible by t, then the partition is called a *t-regular partition*. We denote the number of t-regular partitions of n by $R_t(n)$.

We note that the number of partitions of n with parts from the set $\mathbb{N} \setminus t\mathbb{N}$ equals the number of t-regular partitions of n, from which the following equalities arise:

$$\begin{aligned}\sum_{n=0}^{\infty} R_t(n)q^n &= \prod_{r\in\mathbb{N}\setminus t\mathbb{N}} \frac{1}{1-q^r} \\ &= \prod_{s\in\mathbb{N}} \frac{1-q^{ts}}{1-q^{ts}} \prod_{r\in\mathbb{N}\setminus t\mathbb{N}} \frac{1}{1-q^r} \\ &= \prod_{m=1}^{\infty} \frac{1-q^{tm}}{1-q^m}.\end{aligned}$$

This insight is one we utilize frequently in this section. We express $\tau_k(n)$ as a binomial-coefficient-weighted partition sum in the following result.

Theorem 1. *Let k be a positive integer. We have*

(a)

$$\tau_k(n+1) = \sum_{\substack{n=a_1^{f_1}\cdots a_r^{f_r};\\ f_i\le k.}} (-1)^{f_1+\cdots f_r} \binom{k}{f_1}\cdots\binom{k}{f_r}, \tag{5}$$

(b)

$$\tau_{-k}(n+1) = \sum_{n=a_1^{f_1}\cdots a_r^{f_r}} \binom{f_1+k-1}{k-1}\cdots\binom{f_r+k-1}{k-1}. \tag{6}$$

Proof. In light of the binomial theorem, we may write

$$\begin{aligned}\sum_{n=1}^{\infty}\tau_k(n)q^n &= q\prod_{m=1}^{\infty}(1-q^m)^k \\ &= q\prod_{m=1}^{\infty}\left(1-\binom{k}{1}q^{1\cdot m}+\binom{k}{2}q^{2\cdot m}-\cdots+(-1)^k\binom{k}{k}q^{k\cdot m}\right).\end{aligned}$$

The above equality suggests that the value $(-1)^{f_1+f_2+\cdots+f_r}\binom{k}{f_1}\binom{k}{f_2}\cdots\binom{k}{f_r}$ contributes to the coefficient of q^{n+1} for each partition of n of the form $n = a_1^{f_1}\cdots a_r^{f_r}$ with the restriction $1 \le f_i \le k$, and vice versa. Therefore, (a) is implied.

By using binomial expansion, we can write

$$\begin{aligned}\sum_{n=1}^{\infty}\tau_{-k}(n)q^n &= q\prod_{m=1}^{\infty}(1-q^m)^{-k} \\ &= q\prod_{m=1}^{\infty}\left(\binom{k-1}{k-1}+\binom{(k-1)+1}{k-1}q^m+\binom{(k-1)+2}{k-1}q^{2m}+\cdots\right).\end{aligned}$$

The aforementioned equality suggests that the value $\binom{f_1+k-1}{k-1}\cdots\binom{f_r+k-1}{k-1}$ contributes to the coefficient of q^{n+1} for each partition of n of the type $n = a_1^{f_1}\cdots a_r^{f_r}$, and vice versa. Thus, (b) is implied. □

3.1. Parity Results Connecting $\tau_k(n)$ and $R_t(n)$ at Specific Instances of k and t

We can get a parity result for $\tau_k(n)$ (which involves a partition function) using the partition sum representation mentioned in Theorem 1.

Definition 4. Let n be a positive integer and let A be a set of positive integers. We define $F_A(n)$ to be the number of partitions of n having each frequency from the set A.

Theorem 2. *Let k be a positive integer. Let $A = \{a \in \mathbb{N} : a \le k, \binom{k}{a} \equiv 1 \pmod 2\}$. We have*

$$\tau_k(n+1) \equiv F_A(n) \pmod 2. \tag{7}$$

Proof. From the representation given in (5) of Theorem 1, the proof follows immediately. □

Using Theorem 2, we obtain a parity result for the 4-regular partition function.

Theorem 3. *Let n be a positive integer. We have*

$$R_4(n) \equiv \begin{cases} 1 \pmod 2 & \text{if } n = \frac{m(m+1)}{2}; \\ 0 \pmod 2 & \text{otherwise.} \end{cases} \tag{8}$$

Proof. We observe that $\binom{24}{k} \equiv 1 \pmod 2$ if, and only if, $k \in \{8, 16, 24\}$. We may now write

$$\tau(n+1) \equiv F_{\{8,16,24\}}(n) \pmod 2,$$

in accordance with Theorem 2. As a result, $\tau(n+1) \equiv 0 \pmod 2$ for each $n \not\equiv 0 \pmod 8$. Therefore, the n such that $n \equiv 0 \pmod 8$ is our primary concern.

The partitions of n that the function $F_{\{8,16,24\}}(n)$ counts when $n \equiv 0 \pmod 8$ can be expressed as follows:

$$n = a_1^8 a_2^{16} a_3^{24}.$$

Alternatively expressed,

$$\frac{n}{8} = a_1^1 a_2^2 a_3^3.$$

Consequently, $F_{\{8,16,24\}}(8m)$ counts the number of partitions of m whose frequencies do not exceed 3. We denote the number of such partitions by $d_3(m)$.

As can be seen,

$$\begin{aligned}\sum_{n=0}^{\infty} d_3(n)q^n &= (1+q+q^2+q^3)(1+q^2+q^4+q^6)(1+q^3+q^6+q^9)\cdots \\ &= \prod_{m=1}^{\infty} \frac{1-q^{4m}}{1-q^m} \qquad (9)\\ &= \prod_{m=1}^{\infty} (1+q^m)(1+q^{2m}).\end{aligned}$$

Considering that

$$\sum_{n=0}^{\infty} q(n)q^n = \prod_{m=1}^{\infty} (1+q^m)$$

is the generating function for the number of partitions of n with distinct parts (denoted $q(n)$), the equation

$$d_3(n) = \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} q(n-2s)q(s) \tag{10}$$

is obtained from the above chain of equalities.

We obtain

$$q(s) \equiv \omega(s) \pmod 2,$$

in light of Euler's Pentagonal Number theorem. Upon substituting this in Equation (10), we obtain

$$d_3(n) \equiv \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} q(n-2s)\omega(s) \pmod 2. \tag{11}$$

In view of Theorem 3 (i) in [1], we have

$$\sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} q(n-2s)\omega(s) = \begin{cases} 1 & \text{if } \delta_t(n) = 1; \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

where

$$\delta_t(n) = \begin{cases} 1 & \text{if } n = \frac{m(m+1)}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

At this point, we can see from Equation (9) that $d_3(n) = R_4(n)$. Now the result follows from (11) and (12). □

Remark 2. The congruence $d_3(n) \equiv 1 \pmod 2$ is true only when $n = \frac{m(m+1)}{2}$. This suggests that $\tau(n+1)$ is odd only if $\frac{n}{8} = \frac{m(m+1)}{2}$. Since $\frac{n}{8} = \frac{m(m+1)}{2}$ simplifies to $n+1 = (2m+1)^2$, it follows that $\tau(n+1)$ is odd only if $n+1$ is an odd square. This is an established result. Ewell [4] previously provided a proof.

Remark 3. The number of 4-regular partitions of n is equal to the number of partitions of n with frequencies from the set $\{1, 2, 3\}$, as we have concluded in the previous theorem's proof. This equinumerous statement can be generalized in the way that follows: the number of $(t+1)$-regular partitions of n is equal to the number of partitions of n with frequencies from the set $\{1, 2, \cdots, t\}$. This generalization may be validated by the subsequent equalities. If one denotes the number of partitions of n with frequencies not greater than t by $d_t(n)$, then:

$$\begin{aligned}1 + \sum_{n=1}^{\infty} d_t(n)q^n &= \prod_{m=1}^{\infty} (1 + q^m + q^{2m} \cdots + q^{tm}) \\ &= \prod_{m=1}^{\infty} \frac{1 - q^{(t+1)m}}{1 - q^m} \\ &= 1 + \sum_{n=1}^{\infty} R_{t+1}(n)q^n.\end{aligned}$$

Remark 4. We find that

$$\tau_{2k}(2n) \equiv 0 \pmod 2$$

without the use of Theorem 2. This is deduced from a general property of $f(q) \in \mathbb{Z}[[q]]$. Denote $f(q) = a_0 + a_1 q + a_2 q^2 + \cdots$. Then

$$\begin{aligned}f(q)^2 &= a_0a_0 + (a_0a_1 + a_0a_1)q + (a_0a_2 + a_1a_1 + a_2a_0)q^2 + \cdots \\ &\equiv a_0^2 + a_1^2 q^2 + \cdots \pmod 2 \\ &\equiv a_0 + a_1 q^2 + \cdots \pmod 2.\end{aligned}$$

This gives

$$f(q)^2 \equiv f(q^2) \pmod 2.$$

Consequently, we have

$$qf(q)^{2k} \equiv qf(q^2)^k \pmod 2.$$

Now plugging $\prod_{m=1}^{\infty} (1 - q^m)$ in place of $f(q)$, we have $\tau_{2k}(2n) \equiv 0 \pmod 2$.

It is not so simple to determine the parity of $\tau_{2k}(2n+1)$. We obtain parity expressions for $\tau_{14}(2n+1)$ and $\tau_6(2n+1)$ in the following result.

Theorem 4. *Let n be a positive integer. We have*

(a) $\tau_{14}(2n+1) \equiv R_8(n) \pmod 2$;

(b) $\tau_6(2n+1) \equiv \begin{cases} 1 \pmod 2 & \text{if } n = \frac{m(m+1)}{2}; \\ 0 \pmod 2 & \text{otherwise.} \end{cases}$

Proof. Define $A = \{1 \leq a_i \leq 14 : \binom{14}{a_i} \equiv 1 \pmod 2\} = \{2, 4, 6, 8, 10, 12, 14\}$. Considering Theorem 2, we can now write

$$\tau_{14}(n+1) \equiv F_{\{2,4,6,8,10,12,14\}}(n) \pmod 2.$$

All partitions of n that $F_{\{2,4,6,8,10,12,14\}}(n)$ counts while n is even are of the type

$$n = a_1^2 a_2^4 a_3^6 a_4^8 a_5^{10} a_6^{12} a_7^{14}.$$

Alternatively expressed,

$$\frac{n}{2} = a_1^1 a_2^2 a_3^3 a_4^4 a_5^5 a_6^6 a_7^7.$$

As a result, $F_{\{2,4,6,8,10,12,14\}}(n)$ counts the number of $\frac{n}{2}$ partitions with frequencies that do not exceed 7. Consequently, $F_{\{2,4,6,8,10,12,14\}}(n) = d_7(\frac{n}{2}) = R_8(\frac{n}{2})$. Thus, considering the even n, we have

$$\begin{aligned}\tau_{14}(n+1) &\equiv F_{\{2,4,6,8,10,12,14\}}(n) \pmod 2\\ &\equiv R_8\left(\frac{n}{2}\right) \pmod 2.\end{aligned}$$

Now (a) follows. A similar search together with the parity result of $R_4(n)$ (mentioned in Theorem 3) gives (b). $\square$

The following parity result relates $R_{2^s}(n)$ and $\tau_{2^s-1}(n)$. This follows from a parity result concerning binomial coefficients.

Theorem 5. *For every positive integer s, we have*

$$R_{2^s}(n) \equiv \tau_{2^s-1}(n+1) \pmod 2. \tag{13}$$

Proof. We note that $\frac{2^s-2}{2} = 2^{s-1} - 1$ is the largest integer that does not exceed $\frac{2^s-1}{2}$. Stated in another way,

$$\lfloor \frac{2^s - 1}{2} \rfloor = 2^{s-1} - 1.$$

Using the result of James Glaisher [9], we obtain

$$\binom{n}{k} \equiv \begin{cases} 0 \pmod 2 & \text{if } n \text{ is even and } k \text{ is odd;}\\ \binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor} \pmod 2 & \text{otherwise.}\end{cases} \tag{14}$$

We obtain the following congruence relation by substituting $2^s - 1$ for n in (14):

$$\binom{2^s-1}{k} \equiv \binom{2^{s-1}-1}{\lfloor \frac{k}{2} \rfloor} \pmod 2.$$

After using the previously specified modulo 2 reduction $s-1$ times, taking the right-side term for subsequent reduction, we obtain

$$\binom{2^s-1}{k} \equiv 1 \pmod 2.$$

Now that the aforementioned observation has been made, we may write

$$\begin{aligned}\sum_{n=0}^{\infty} \tau_{2^s-1}(n+1)q^n &= \prod_{m=1}^{\infty}(1-q^m)^{2^s-1} \\ &\equiv \prod_{m=1}^{\infty}(1+q^m+\cdots+q^{2^{(s-1)m}}) \pmod 2 \\ &\equiv \prod_{m=1}^{\infty}\frac{1-q^{2^s m}}{1-q^m} \pmod 2 \\ &\equiv \sum_{n=0}^{\infty} R_{2^s}(n)q^n \pmod 2.\end{aligned}$$

The proof is now completed. □

3.2. Ramanujan's Tau Function Modulo 3, 5, 7, 11, 13, 17, 23, and 25

This section is concerned with deriving a simple expression for $\tau(n)$ modulo m when $m \in \{3,5,7,11,13,17,23,25\}$. The derivations of this section just rely on some arithmetic properties of $\binom{24}{s}$.

Theorem 6. *Let n be a positive integer. We have*

$$\tau(n+1) \equiv \begin{cases} R_9\left(\frac{n}{3}\right) \pmod 3 & \text{if } 3 \mid n; \\ 0 \pmod 3 & \text{otherwise.}\end{cases} \tag{15}$$

Proof. Given the following observations:

1. $\binom{24}{k} \equiv 0 \pmod 3$ when $k \in \{1,2,4,5,7,8,10,11,13,14,16,17,19,20,22,23\}$,
2. $\binom{24}{k} \equiv -1 \pmod 3$ when $k \in \{3,9,15,21\}$,
3. $\binom{24}{k} \equiv 1 \pmod 3$ when $k \in \{6,12,18,24\}$,

we may write

$$\begin{aligned}\sum_{n=0}^{\infty} \tau(n+1)q^n &\equiv \prod_{m=1}^{\infty}(1+q^{3m}+q^{6m}+\cdots+q^{24m}) \pmod 3 \\ &\equiv \prod_{m=1}^{\infty}\frac{1-q^{9\times 3m}}{1-q^{3m}} \pmod 3.\end{aligned}$$

Since

$$\prod_{m=1}^{\infty} \frac{1-q^{9m}}{1-q^m} = \sum_{n=0}^{\infty} R_9(n) q^n,$$

in view of the above observation, we obtain the following congruence:

$$\tau(n+1) \equiv \begin{cases} R_9\left(\frac{n}{3}\right) \pmod 3 & \text{if } 3 \mid n, \\ 0 \pmod 3 & \text{otherwise.} \end{cases}$$

□

As an immediate consequence of the theorem above, we obtain the following result of Ramanujan.

Corollary 1 (Ramanujan [16])**.** *Let n be a positive integer. We have*

$$\tau(3n) \equiv 0 \pmod 3.$$

Proof. Theorem 6 allows us to write

$$\begin{aligned} \tau(3n) &= \tau(3n-1+1) \\ &\equiv 0 \pmod 3. \end{aligned}$$

□

As another consequence of Theorem 6, we obtain the following expression for $R_9(n)$ modulo 3.

Corollary 2. *Let n be a positive integer. We have*

$$R_9(n) \equiv \sigma(3n+1) \pmod 3. \tag{16}$$

Proof. We have

$$\tau(n) \equiv n\sigma(n) \pmod 3$$

from the works of Ramanujan [6, p. 112]. It follows therefrom that

$$\tau(n) \equiv \begin{cases} 0 \pmod 3 & \text{if } 3 \mid n; \\ \sigma(n) \pmod 3 & \text{if } \gcd(n,3) = 1. \end{cases}$$

Given the above observation, Theorem 6 allows us to write

$$\begin{aligned} R_9(n) &\equiv \tau(3n+1) \pmod 3 \\ &\equiv \sigma(3n+1) \pmod 3. \end{aligned}$$

□

Theorem 7. *Let n be a positive integer. We have*

$$\tau(n+1) \equiv R_{25}(n) \pmod{5}. \tag{17}$$

Proof. We observe that

1. $\binom{24}{k} \equiv 1 \pmod{5}$ when $k \in \{0,2,4,6,8,10,12,14,16,18,20,22,24\}$,
2. $\binom{24}{k} \equiv -1 \pmod{5}$ when $k \in \{1,3,5,7,9,11,13,15,17,19,21,23\}$.

Based on these observations, we can write

$$\begin{aligned}\sum_{n=0}^{\infty} \tau(n+1) q^n &\equiv \prod_{m=1}^{\infty} (1+q^m+q^{2m}+\cdots+q^{24m}) \pmod{5}\\ &\equiv \prod_{m=1}^{\infty} \frac{1-q^{25m}}{1-q^m} \pmod{5}.\end{aligned}$$

Since

$$\prod_{m=1}^{\infty} \frac{1-q^{25m}}{1-q^m} = \sum_{n=0}^{\infty} R_{25}(n) q^n,$$

we obtain from the above observation that

$$\tau(n+1) \equiv R_{25}(n) \pmod{5}.$$

□

An expression for R_{25} modulo 5 can be obtained by applying the aforementioned theorem.

Corollary 3. *Let n be a positive integer. We have*

$$R_{25}(n) \equiv (n+1)\sigma(n+1) \pmod{5}. \tag{18}$$

Proof. Wilton [18] established that

$$\tau(n) \equiv n\sigma(n) \pmod{5}.$$

We may now write in light of Theorem 7:

$$\begin{aligned}R_{25}(n) &\equiv \tau(n+1) \pmod{5}\\ &\equiv (n+1)\sigma(n+1) \pmod{5}.\end{aligned}$$

□

Theorem 8. *Let n be a positive integer. We have*

$$\tau(n+1) \equiv \sum_{n=\frac{m(m+1)}{2}+7\frac{r(r+1)}{2}} (-1)^{m+r}(2m+1)(2r+1) \pmod{7}. \tag{19}$$

Proof. We observe that

$$\binom{24}{k} \equiv \begin{cases} 1 \pmod 7 & \text{if } k\text{=0, 3, 21, 24;} \\ 0 \pmod 7 & \text{if } k\text{=4, 5, 6, 11, 12, 13, 18, 19, 20;} \\ 3 \pmod 7 & \text{if } k\text{=1, 2, 22, 23;} \\ -4 \pmod 7 & \text{if } k\text{=7, 10, 14, 17;} \\ -12 \pmod 7 & \text{if } k\text{=8, 9, 25, 16.} \end{cases}$$

Based on this observation, we can write

$$\begin{aligned}
\sum_{n=0}^{\infty} \tau(n+1)q^n \equiv \prod_{m=1}^{\infty} \Big[& \left(1-3q^m+3q^{2m}-q^{3m}\right) + 4\left(q^{7m}-3q^{8m}+3q^{9m}-q^{10m}\right) \\
& - 4\left(q^{14m}-3q^{15m}+3q^{16m}-q^{17m}\right) \\
& - \left(q^{21m}-3q^{22m}+3q^{23m}-q^{24m}\right)\Big] \pmod 7
\end{aligned}$$

$$\begin{aligned}
\equiv \prod_{m=1}^{\infty} \Big[& \left(1-3q^m+3q^{2m}-q^{3m}\right) - 3\left(q^{7m}-3q^{8m}+3q^{9m}-q^{10m}\right) \\
& + 3\left(q^{14m}-3q^{15m}+3q^{16m}-q^{17m}\right) \\
& - \left(q^{21m}-3q^{22m}+3q^{23m}-q^{24m}\right)\Big] \pmod 7 \\
\equiv \prod_{m=1}^{\infty} & \Big[\left(1-3q^m+3q^{2m}-q^{3m}\right)\left(1-3q^{7m}+3q^{14m}-q^{21m}\right)\Big] \pmod 7 \\
\equiv \prod_{m=1}^{\infty} & (1-q^m)^3 \prod_{r=1}^{\infty}\left(1-q^{7r}\right)^3 \pmod 7.
\end{aligned}$$

Jacobi's triple product identity states that

$$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{s=0}^{\infty} a_s q^s, \tag{20}$$

where

$$a_s = \begin{cases} (-1)^t(2t+1) & \text{if } s = \frac{t(t+1)}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

The intended congruence will result from applying this identity to the tail end product of the preceding chain of expressions. □

Theorem 9. *Let n be a positive integer. We have*

$$\tau(n+1) \equiv \sum_{n=\frac{3l^2\pm l}{2}+\frac{3m^2\pm m}{2}+11\frac{3s^2\pm s}{2}+11\frac{3r^2\pm r}{2}} (-1)^{l+m+s+r} \pmod{11}. \tag{21}$$

Proof. Since

$$\binom{24}{k} \equiv \begin{cases} 1 \pmod{11} & \text{for } k{=}0,\ 2,\ 22,\ 24; \\ 2 \pmod{11} & \text{for } k{=}1,\ 11,\ 13,\ 23; \\ 4 \pmod{11} & \text{for } k{=}12; \\ 0 \pmod{11} & \text{for } k{=}3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 14,\ 15,\ 16,\ 17,\ 18,\ 19,\ 20,\ 21, \end{cases}$$

we can write

$$\begin{aligned} \sum_{n=0}^{\infty} \tau(n+1)q^n &\equiv \prod_{m=1}^{\infty} \left[\left(1-2q^m+q^{2m}\right) - 2\left(q^{11m}-2q^{12m}+q^{13m}\right)\right. \\ &\qquad \left. + \left(q^{22m}-2q^{23m}+q^{24m}\right)\right] \pmod{11} \\ &\equiv \prod_{m=1}^{\infty} \left[(1-q^m)^2(1-q^{11m})^2\right] \pmod{11}. \end{aligned}$$

Applying Euler's pentagonal number theorem now results in the following equality:

$$\prod_{m=1}^{\infty} \left[(1-q^m)^2(1-q^{11m})^2\right] = 1 + \sum_{n=\frac{3l^2\pm l}{2}+\frac{3m^2\pm m}{2}+11\frac{3s^2\pm s}{2}+11\frac{3r^2\pm r}{2}} (-1)^{l+m+s+r}q^n.$$

This insight will yield the expected congruence when applied to the tail end product of the aforementioned chain of expressions. □

Theorem 10. *Let n be a positive integer. We have*

$$\tau(n+1) \equiv \sum_{n=13\times\frac{3r^2\pm r}{2}+s} (-1)^r \tau_{11}(s) \pmod{13}. \tag{22}$$

Proof. We observe that

$$\binom{24}{k} \equiv \begin{cases} \binom{11}{k} \pmod{13} & \text{when } 0 \le k \le 11; \\ 0 \pmod{13} & \text{when } k = 12; \\ \binom{11}{24-k} \pmod{13} & \text{when } 13 \le k \le 24. \end{cases}$$

This observation enables us to write

$$\begin{aligned} \sum_{n=0}^{\infty} \tau(n+1)q^n &\equiv \prod_{m=1}^{\infty} \left[\left(\binom{11}{0} - \binom{11}{1}q^m + \cdots - \binom{11}{11}q^{11m}\right)\right. \\ &\qquad \left. - q^{13m}\left(\binom{11}{0} - \binom{11}{1}q^m + \cdots - \binom{11}{11}q^{11m}\right)\right] \pmod{13} \\ &\equiv \prod_{m=1}^{\infty} \left[(1-q^m)^{11}(1-q^{13m})\right] \pmod{13}. \end{aligned}$$

Now given the definition of $\tau_{11}(n)$ and Euler's pentagonal number theorem, we may write

$$\prod_{m=1}^{\infty}(1-q^m)^{11}(1-q^{13m}) = 1 + \left(\sum_{n=13\times\frac{3r^2\pm r}{2}+s}(-1)^r\tau_{11}(s)\right)q^n.$$

This observation will yield the expected congruence when applied to the tail end product of the aforementioned chain of expressions. □

Theorem 11. *Let n be a positive integer. We have*

$$\tau(n+1) \equiv \sum_{n=17\times\frac{3r^2\pm r}{2}+s}(-1)^r\tau_7(s) \pmod{17}. \tag{23}$$

Proof. Since

$$\binom{24}{k} \equiv \begin{cases} \binom{7}{k} \pmod{17} & \text{when } 0 \le k \le 7; \\ \binom{7}{24-s} \pmod{17} & \text{when } 0 \le 24-s \le 7; \\ 0 \pmod{17} & \text{otherwise,} \end{cases}$$

one can write

$$\begin{aligned} \sum_{n=0}^{\infty}\tau(n+1)q^n &\equiv \prod_{m=1}^{\infty}\Bigg[\left(\binom{7}{0}-\binom{7}{1}q^m+\cdots-\binom{7}{7}q^{7m}\right) \\ &\qquad - q^{17m}\left(\binom{7}{0}-\binom{7}{1}q^m+\cdots-\binom{7}{7}q^{7m}\right)\Bigg] \pmod{17} \\ &\equiv \prod_{m=1}^{\infty}(1-q^m)^7(1-q^{17m}) \pmod{17}. \end{aligned}$$

One can have

$$\prod_{m=1}^{\infty}(1-q^m)^7(1-q^{17m}) = 1 + \left(\sum_{n=17\times\frac{3r^2\pm r}{2}+s}(-1)^r\tau_7(s)\right)q^n$$

based on the definition of $\tau_7(n)$ and Euler's pentagonal number theorem. This observation will yield the expected congruence when applied to the tail end product of the aforementioned chain of expressions. □

Theorem 12. *Let n be a positive integer. We have*

$$\tau(n+1) \equiv \sum_{n=19\times\frac{3r^2\pm r}{2}+s}(-1)^r\tau_5(s) \pmod{19}. \tag{24}$$

Proof. Since

$$\binom{24}{k} \equiv \begin{cases} 1 \pmod{19} & \text{when } k = 0, 5, 19, 24; \\ 5 \pmod{19} & \text{when } k = 1, 4, 20, 23; \\ 10 \pmod{19} & \text{when } k = 2, 3, 21, 22; \\ 0 \pmod{19} & \text{otherwise,} \end{cases}$$

one can write

$$\begin{aligned} \sum_{n=0}^{\infty} \tau(n+1) q^n &\equiv \prod_{m=1}^{\infty} \left[\left(1 - 5q^m + 10q^{2m} - 10q^{3m} + 5q^{4m} - q^{5m}\right) \right. \\ &\qquad \left. - q^{19m} \left(1 - 5q^m + 10q^{2m} - 10q^{3m} + 5q^{4m} - q^{5m}\right) \right] \pmod{19} \\ &\equiv \prod_{m=1}^{\infty} \left[(1-q^m)^5 (1-q^{19m}) \right] \pmod{19}. \end{aligned}$$

Now from the definition of $\tau_5(n)$ and Euler's pentagonal number theorem, we have

$$\prod_{m=1}^{\infty} (1-q^m)^5 (1-q^{19m}) = 1 + \left(\sum_{n = 19 \times \frac{3r^2 \pm r}{2} + s} (-1)^r \tau_5(s) \right) q^n.$$

While applying this observation in the tail end product of the above chain of expressions, we get the expected congruence. □

Utilizing the following Ramanujan's formula [8, pp. 163-164] for $\tau(p^r)$:

$$\tau(p^r) = \frac{p^{\frac{11}{2} r}}{\sin \psi_p} \sin{(r+1)\psi_p},$$

where p is a prime number and $\cos \psi_p = \frac{\tau(p)}{2p^{\frac{11}{2}}}$, Lehmer [11] gave an expression for $\tau(n)$ modulo 23:

$$\tau(n) \equiv \sigma_{11}(n_1) 2^t 3^{\frac{-t}{2}} \prod_{i=1}^{t} \sin \frac{2\pi}{3} (1 + \alpha_i) \pmod{23}, \tag{25}$$

where $n = n_1 \prod_{i=1}^{t} p_i^{\alpha_i}$, p_is are the only prime factors of n which are not of the form $u^2 + 23v^2$ but are quadratic residues of 23, and α_i is the exponent of the highest power of p_i dividing n.

We provide an expression for $\tau(n)$ modulo 23 in the following result, which is quite simple in comparison to (25).

Theorem 13. *Let n be a positive integer. We have*

$$\tau(n+1) \equiv \sum_{n = \frac{3r^2 \pm r}{2} + 23 \times \frac{3s^2 \pm s}{2}} (-1)^{r+s} \pmod{23}. \tag{26}$$

Proof. Since

$$\binom{24}{k} \equiv \begin{cases} 1 \pmod{23} & \text{if } k = 0, 1, 23, 24; \\ 0 \pmod{23} & \text{otherwise,} \end{cases}$$

we can write

$$\begin{aligned} \sum_{n=0}^{\infty} \tau(n+1) q^n &\equiv \prod_{m=1}^{\infty} \left(1 - q^m - q^{23m} + q^{24m}\right) \pmod{23} \\ &\equiv \prod_{m=1}^{\infty} \left[(1 - q^m)(1 - q^{23m})\right] \pmod{23}. \end{aligned}$$

Write

$$\prod_{m=1}^{\infty} \left[(1 - q^m)(1 - q^{23m})\right] = \sum_{n=0}^{\infty} a_n q^n.$$

Now in accordance with Euler's pentagonal number theorem, we have

$$a_n = \begin{cases} (-1)^{r+s} & \text{if } n = \frac{3r^2 \pm r}{2} + 23 \times \frac{3s^2 \pm s}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

While applying this observation in the tail end product of the above chain of expressions, we get the intended congruence. □

Theorem 14. *Let n be a positive integer. We have*

$$\tau(n+1) \equiv \sum_{n = r + 5\frac{s(s+1)}{2} + 5\frac{3t^2 \pm t}{2}} (-1)^{s+t} (2s+1) R_5(r) \pmod{25}. \tag{27}$$

Proof. Since

$$\binom{24}{k} \equiv \begin{cases} 1 & \text{when } k\text{=0, 2, 4, 20, 22, 24;} \\ -1 & \text{when } k\text{=1, 3, 21, 23;} \\ 4 & \text{when } k\text{=5, 7, 9, 15, 17, 19;} \\ -4 & \text{when } k\text{=6, 8, 16, 18;} \\ 6 & \text{when } k\text{=10, 12, 14;} \\ -6 & \text{when } k\text{=11, 13,} \end{cases}$$

we can write

$$\begin{aligned} \sum_{n=0}^{\infty} \tau(n+1) q^n \equiv \prod_{m=1}^{\infty} \Big[&\left(1 + q^m + q^{2m} + q^{3m} + q^{4m}\right) \\ &- 4\left(q^{5m} + q^{6m} + q^{7m} + q^{8m} + q^{9m}\right) \\ &+ 6\left(q^{10m} + q^{11m} + q^{12m} + q^{13m} + q^{14m}\right) \\ &- 4\left(q^{15m} + q^{16m} + q^{17m} + q^{18m} + q^{19m}\right) \\ &+ \left(q^{20m} + q^{21m} + q^{22m} + q^{23m} + q^{24m}\right)\Big] \pmod{25} \end{aligned}$$

$$\begin{aligned}
&\equiv \prod_{m=1}^{\infty}\left[\frac{1-q^{5m}}{1-q^m} - 4q^{5m}\frac{1-q^{5m}}{1-q^m} + 6q^{10m}\frac{1-q^{5m}}{1-q^m} - 4q^{15m}\frac{1-q^{5m}}{1-q^m}\right.\\
&\qquad \left. +q^{20m}\frac{1-q^{5m}}{1-q^m}\right] \pmod{25}\\
&\equiv \prod_{m=1}^{\infty}\frac{1-q^{5m}}{1-q^m}\prod_{r=1}^{\infty}(1-q^{5r})^4 \pmod{25}.
\end{aligned}$$

Given the generating function of $R_5(n)$, Euler's pentagonal number theorem and Jacobi's triple product identity, we may write

$$\prod_{m=1}^{\infty}\frac{1-q^{5m}}{1-q^m}\prod_{r=1}^{\infty}(1-q^{5r})^4 = 1 + \sum_{\substack{n\in\mathbb{N}\\ n=r+5\frac{s(s+1)}{2}+5\frac{3t^2\pm t}{2}}}(-1)^{s+t}(2s+1)R_5(r)q^n.$$

While applying this observation in the tail end product of the above chain of expressions, we get the intended congruence. □

3.3. Prime Moduli

Let p be a prime number. This section provides an expression for $\tau_k(n)$ modulo p when $k \in \{p^s : s \in \mathbb{N}\} \cup \{2p, 2p+1, p^2+1\}$.

Theorem 15. *Let p be a prime number, and let s be a positive integer. We have*

$$\tau_{p^s}(n+1) \equiv \begin{cases}(-1)^t \pmod{p} & \text{if } n = \frac{p^s(3t^2\pm t)}{2};\\ 0 \pmod{p} & \text{otherwise.}\end{cases} \tag{28}$$

Proof. Since

$$\binom{p^s}{t} \equiv 0 \pmod{p}$$

for every $t \in \{1, 2, \ldots, p^s - 1\}$, we can write

$$\begin{aligned}
\sum_{n=1}^{\infty}\tau_{p^s}(n)q^{n-1} &= \prod_{m=1}^{\infty}(1-q^m)^{p^s}\\
&\equiv \prod_{m=1}^{\infty}(1-q^{p^s m}) \pmod{p}.
\end{aligned}$$

Using Euler's pentagonal number theorem, we now obtain

$$\prod_{m=1}^{\infty}(1-q^{p^s m}) = \sum_{r=0}^{\infty}\omega_{p^s}(r)q^r,$$

where

$$\omega_{p^s}(r) = \begin{cases} (-1)^t & \text{if } \frac{r}{p^s} = \frac{(3t^2 \pm t)}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

While applying this observation in the tail end product of the above chain of expressions, we get the intended congruence. □

Definition 5. Let m be a positive integer and let p be a prime number. Let $\vartheta_p(m)$ be defined as a non-negative integer k such that $p^k \mid m$ but $p^{k+1} \nmid m$.

Theorem 16. *Let p be an odd prime number. We have*

$$\tau_{2p}(n+1) \equiv \sum_{n+1=p\left(\frac{3r^2 \pm r}{2} + \frac{3s^2 \pm s}{2}\right)} (-1)^{r+s} \pmod{p}. \tag{29}$$

Proof. Clearly $\binom{2p}{0} \equiv 1 \pmod{p}$ and $\binom{2p}{2p} \equiv 1 \pmod{p}$. We observe that $\vartheta_p(2p \times (2p-1) \times \cdots \times (2p-(k-1)) = 1$ and $\vartheta_p(1 \times 2 \times \cdots \times k) = 0$ when $1 \leq k \leq p-1$. Consequently, $\binom{2p}{k} \equiv 0 \pmod{p}$ when $1 \leq k \leq p-1$. Also, we have

$$\binom{2p}{p} = \frac{2p \times (2p-1) \times \cdots \times (p+1)}{1 \times 2 \times \cdots \times p} = \frac{2 \times (2p-1) \times \cdots \times (p+1)}{(p-1)!}.$$

Since $2p-1 \equiv -1 \pmod{p}$, $2p-2 \equiv -2 \pmod{p}$, $\cdots$, $p+1 \equiv -(p-1) \pmod{p}$, we obtain (in light of Wilson's theorem) that

$$\begin{aligned} 2 \times (2p-1) \times \cdots \times (p+1) &\equiv 2(p-1)! \pmod{p} \\ &\equiv -2 \pmod{p}. \end{aligned}$$

Consequently, $\binom{2p}{p}$ is of the form $\frac{rp-2}{kp-1}$. Given this form, we obtain $\binom{2p}{p} - 2 = \frac{rp-2}{kp-1} - 2 = \frac{(r-2k)p}{kp-1}$. Therefrom, it follows that $\binom{2p}{p} \equiv 2 \pmod{p}$. Moreover, since $\binom{2p}{k} = \binom{2p}{2p-k}$, we can write

$$\begin{aligned} \sum_{n=0}^{\infty} \tau_{2p}(n+1)q^n &\equiv \prod_{m=1}^{\infty} \left(1 + (-1)^p 2q^{pm} + (-1)^{2p} q^{2pm}\right) \pmod{p} \\ &\equiv \prod_{m=1}^{\infty} (1-q^{pm})^2 \pmod{p}. \end{aligned}$$

Since

$$\prod_{m=1}^{\infty} (1-q^{pm})^2 = 1 + \left(\sum_{n+1=p\left(\frac{3r^2 \pm r}{2} + \frac{3s^2 \pm s}{2}\right)} (-1)^{r+s} \right) q^n,$$

the result follows as a consequence of the above observation. □

Corollary 4. *Let p be an odd prime number. If $p \nmid n+1$ then*

$$\tau_{2p}(n+1) \equiv 0 \pmod{p}.$$

Theorem 17. *Let p be an odd prime number. Then we have*

$$\tau_{2p+1}(n+1) \equiv \sum_{n+1=\frac{3r^2\pm r}{2}+p\left(\frac{3s^2\pm s}{2}+\frac{3t^2\pm t}{2}\right)} (-1)^{r+s+t} \pmod{p}. \tag{30}$$

Proof. In general, the congruences $\binom{2p+1}{0} \equiv \binom{2p+1}{1} \equiv 1 \pmod{p}$ are true. Consider the product:

$$\prod_{k=1}^{2p+1} \frac{2p+1-(k-1)}{k}. \tag{31}$$

Here, the product of the first r terms gives the value $\binom{2p+1}{r}$. Based on this observation, we deduce that $\binom{2p+1}{r} \equiv 0 \pmod{p}$ for r limited to the bound $2 \le r \le p-1$.

We will now show that $\binom{2p+1}{p} \equiv 2 \pmod{p}$. To that end, consider the product of the first p terms of (31):

$$\frac{2p+1}{1} \cdot \frac{2p}{2} \cdot \frac{2p-1}{3} \cdots \frac{(2p+1)-(p-1)}{p}.$$

After cancelling p in the above product, we obtain the following term:

$$\frac{(2p+1) \times 2 \times (2p-1) \times \cdots \times ((2p+1)-(p-1))}{(p-1)!}.$$

Wilson's theorem allows us to write

$$\begin{aligned}(2p+1) \times 2 \times (2p-1) \times \cdots \times ((2p+1)-(p-1)) &\equiv -2 \times (p-2)! \pmod{p} \\ &\equiv -2 \pmod{p}.\end{aligned}$$

The form of $\binom{2p+1}{p}$ is thus $\frac{sp-2}{rp-1}$. From this, we have $\binom{2p+1}{p} - 2 = \frac{sp-2}{rp-1} - 2 = \frac{(s-2r)p}{rp-1} \equiv 0 \pmod{p}$.

Considering that $\binom{2p+1}{k} = \binom{2p+1}{2p+1-k}$, we can write

$$\begin{aligned}\sum_{n=0}^{\infty} \tau_{2p+1}(n+1)q^n &\equiv \prod_{m=1}^{\infty} \Big(1 - q^m - 2q^{pm} + 2q^{(p+1)m} \\ &\qquad + q^{2pm} - q^{(2p+1)m}\Big) \pmod{p} \\ &\equiv \prod_{m=1}^{\infty} \left[(1-q^m)(1-q^{pm})^2\right] \pmod{p}.\end{aligned}$$

In light of Euler's pentagonal number theorem, we obtain

$$\prod_{m=1}^{\infty}(1-q^m)(1-q^{pm})^2 = 1+\left(\sum_{n=\frac{3r^2\pm r}{2}+p\left(\frac{3s^2\pm s}{2}+\frac{3t^2\pm t}{2}\right)}(-1)^{r+s+t}\right)q^n.$$

By using this in the congruence mentioned above, we obtain the intended congruence. $\square$

Theorem 18. *Let p be an odd prime number. We have*

$$\tau_{p^2+1}(n+1) \equiv \sum_{n+1=\frac{3r^2\pm r}{2}+p^2\,\frac{3s^2\pm s}{2}}(-1)^{r+s} \pmod{p}. \tag{32}$$

Proof. The following congruences are all evident: $\binom{p^2+1}{0}\equiv 1 \pmod{p}$, $\binom{p^2+1}{1}\equiv 1 \pmod{p}$, $\binom{p^2+1}{p^2}\equiv 1 \pmod{p}$ and $\binom{p^2+1}{p^2+1}\equiv 1 \pmod{p}$.

Consider the term

$$\binom{p^2+1}{k} = \frac{(p^2+1)\times p^2\times\cdots\times(p^2+1-(k-1))}{1\times 2\times\cdots\times k}.$$

Assume $1\le a\le p-1$. It can then be observed that, for $(a-1)p+1\le k-1\le ap$,

$$\vartheta_p((p^2+1)\times(p^2+1-1)\times\cdots\times(p^2+1-(k-1))) = a+1$$

and

$$\vartheta_p(1\times 2\times\cdots\times k) = a-1 \text{ or } a.$$

Assume $k-1\in\{(p-1)p+1,\cdots,p^2-2\}$. It is easy to see that

$$\vartheta_p((p^2+1)\times p^2\times\cdots\times(p^2-(k-1))) = p+1$$

and

$$\vartheta_p(1\times 2\times\cdots\times k) = p-1.$$

Consequently, $\binom{p^2+1}{k}\equiv 0 \pmod{p}$ when $2\le k\le p^2-1$. Based on this observation, we may write

$$\begin{aligned}\sum_{n=0}^{\infty}\tau_{p^2+1}(n+1)q^n &\equiv \prod_{m=1}^{\infty}\left(1-q^m+(-1)^{p^2}q^{p^2m}+(-1)^{p^2+1}q^{(p^2+1)m}\right) \pmod{p}\\ &\equiv \prod_{m=1}^{\infty}(1-q^m)(1-q^{p^2m}) \pmod{p}.\end{aligned}$$

Since

$$\prod_{m=1}^{\infty}(1-q^m)(1-q^{p^2m}) = 1+\left(\sum_{n=\frac{3r^2\pm r}{2}+p^2\,\frac{3s^2\pm s}{2}}(-1)^{r+s}\right)q^n,$$

the result follows. $\square$

3.4. Congruence Properties of $R_9(n)$ Modulo 3 and $R_p(n)$ Modulo p, Where p is a Prime Number

In this section, we apply Ewell's congruence for $\tau(n)$ to obtain a recursive congruence relation for $R_9(n)$ modulo 3. Additionally, for any prime number p, we derive an expression for $R_p(n)$ modulo p.

Theorem 19. *Let n be a positive integer. We have*

$$R_9(4n+1) \equiv R_9(n) \pmod{3}. \tag{33}$$

Proof. Theorem 6 allows us to write

$$R_9(n) \equiv \tau(3n+1) \pmod{3}.$$

Ewell [5] proved that

$$\tau(4n) \equiv \tau(n) \pmod{3}.$$

Given these insights, we may write

$$\begin{aligned} R_9(4n+1) &\equiv \tau(4(3n+1)) \pmod{3} \\ &\equiv \tau(3n+1) \pmod{3} \\ &\equiv R_9(n) \pmod{3}. \end{aligned}$$

Now the result follows. □

Corollary 5. *Let r and s be positive integers. We have*

$$R_9\left((r-1)4^{s-1} + \frac{4^s-1}{3}\right) \equiv R_9(r) \pmod{3}. \tag{34}$$

In particular,

$$R_9\left(\frac{4^s-1}{3}\right) \equiv 1 \pmod{3},$$

$$R_9\left(4^{s-1} + \frac{4^s-1}{3}\right) \equiv 2 \pmod{3}$$

and

$$R_9\left(2 \times 4^{s-1} + \frac{4^s-1}{3}\right) \equiv 0 \pmod{3}.$$

Proof. The recurrence relation

$$a_s = 4a_{s-1} + 1,$$

with initial condition $a_1 = r$ must be solved in order to apply the recursive congruence relation (33) of Theorem 19.

Let $F(x) = a_1x+a_2x^2+\cdots$ be the generating function for the sequence $a_1, a_2, \ldots$. Then the above recurrence relation yields

$$F(x) = \frac{(r-1)x}{1-4x} + \frac{x(1-x)}{1-4x}.$$

We obtain $a_s = (r-1)4^{s-1} + \frac{4^s-1}{3}$ when we expand the aforementioned expression using the geometric series expansion.

We now obtain

$$R_9(a_s) \equiv R_9(a_{s-1}) \pmod 3$$

according to Theorem 19. The preceding recursive congruence is then applied $s-1$ times, yielding the following congruence:

$$R_9(a_s) \equiv R_9(r) \pmod 3.$$

Now (34) follows. After fixing $r = 1, r = 2$, and $r = 3$, respectively, and noticing that $R_9(1) = 1$, $R_9(2) = 2$, and $R_9(3) = 3$, we obtain the particular cases. □

Theorem 20. *For every prime number p, we have*

$$R_p(n) \equiv \tau_{p-1}(n+1) \pmod p. \tag{35}$$

Proof. Since p is a prime number, for $1 \le k \le \lfloor \frac{p}{2} \rfloor$, we obtain

$$\begin{aligned}\binom{p-1}{k} &= \frac{tp + k!(-1)^k}{k!} \\ &\equiv (-1)^k \pmod p.\end{aligned}$$

Given this observation, we may write

$$\begin{aligned}\sum_{n=0}^{\infty} \tau_{p-1}(n+1)q^n &\equiv \prod_{m=1}^{\infty}\left(1 + q^m + \cdots + q^{(p-1)m}\right) \pmod p \\ &\equiv \prod_{m=1}^{\infty} \frac{1-q^{pm}}{1-q^m} \pmod p \\ &\equiv \sum_{n=1}^{\infty} R_p(n)q^n \pmod p.\end{aligned}$$

Now the result follows. □

4. Divisibility of Partition Function Weighted Composition Sums

The Ramanujan's congruences for the partition function are as follows:

(a) $p(5n+4) \equiv 0 \pmod 5$,

(b) $p(7n+5) \equiv 0 \pmod 7$,

(c) $p(11n+6) \equiv 0 \pmod{11}$.

It is rare to find such a simple congruence for other larger primes. The following congruence (modulo l, an odd prime), involving partition function values, is found in this section:

$$\sum_{\substack{n=a_1+a_2+\cdots a_k\\ a_i\in\mathbb{N}\cup\{0\}}} p(a_1)p(a_2)\cdots p(a_k) \equiv \sum_{n=t+ls} \tau_{l-k}(t)p(s) \pmod l.$$

The following lemma is crucial in obtaining the above congruence.

Lemma 1. *Let l be an odd prime number. Let k be an integer such that $1 \leq k < l$. Let $n \geq 0$ be an integer. Let r be the remainder obtained by division of n into l. We have*

$$\binom{n+k}{k} \equiv \begin{cases} (-1)^r\binom{l-k-1}{r} \pmod l & \text{when } r \in \{0,1,\cdots,l-k-1\};\\ 0 \pmod l & \text{when } r \in \{l-k,\cdots,l-1\}. \end{cases} \tag{36}$$

Proof. Consider the following expression:

$$\binom{n+k}{k} = \frac{(n+k)(n+k-1)\cdots(n+1)}{k!}.$$

Using this representation, we find an expression for $\binom{n+k}{k}$ modulo l.

Assume that $n = sl + r$ for some $r \in \{l-1, l-2, \cdots, l-k\}$. Then $(n+k)(n+k-1)\cdots(n+1) \equiv 0 \pmod l$. Since l is a prime number and $k < l$, we obtain $\frac{(n+k)(n+k-1)\cdots(n+1)}{k!} \equiv 0 \pmod l$. That is, $\binom{n+k}{k} \equiv 0 \pmod l$ when $r \in \{l-k, \cdots, l-1\}$. The second case follows.

Assume that $n = sl + r$ for some $r \in \{0, 1, \cdots, l-k-1\}$. Since l is a prime number and $k < l$, we obtain

$$\begin{aligned}\binom{n+k}{k} &= \frac{(sl+r+k)(sl+r+k-1)\cdots(sl+r+1)}{k!}\\ &= \frac{s^*l}{k!} + \frac{(r+k)(r+k-1)\cdots(r+1)}{k!}\\ &\equiv \binom{r+k}{k} \pmod l\\ &\equiv \binom{r+k}{r} \pmod l.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(-1)^r\binom{l-k-1}{r} &= (-1)^r\frac{(l-k-1)(l-k-2)\cdots(l-k-1-(r-1))}{r!}\\ &= (-1)^r\frac{(l-(k+1))(l-(k+2))\cdots(l-(k+r))}{r!}\\ &= (-1)^r\left(\frac{s^{'}l}{k!}+(-1)^r\frac{(k+1)(k+2)\cdots(k+r)}{r!}\right)\\ &\equiv (-1)^r(-1)^r\binom{r+k}{r} \pmod{l}.\end{aligned}$$

From the above two congruences, we have the relation

$$\binom{n+k}{k} \equiv (-1)^r\binom{l-k-1}{r} \pmod{l}$$

when $k < l$. The proof is now completed. □

Now we are equipped to prove the following main result of this section.

Theorem 21. *Let $k \geq 2$ be a positive integer. Let $l > k$ be an odd prime number. Let $p(n)$ be the number of partitions of n. We have*

$$\tau_{-k}(n+1) \equiv \sum_{n=t+ls} \tau_{l-k}(t)p(s) \pmod{l}. \tag{37}$$

In another notation,

$$\sum_{\substack{n=a_1+a_2+\cdots a_k\\ a_i\in\mathbb{N}\cup\{0\}}} p(a_1)p(a_2)\cdots p(a_k) \equiv \sum_{n=t+ls} \tau_{l-k}(t)p(s) \pmod{l}. \tag{38}$$

Proof. We have

$$\begin{aligned}\prod_{m=1}^{\infty}(1-q^m)^{-k} &= \sum_{n=0}^{\infty}\tau_{-k}(n+1)q^n\\ &= 1+\sum_{n=1}^{\infty}\left(\sum_{\substack{n=a_1+a_2+\cdots a_k\\ a_i\in\mathbb{N}\cup\{0\}}} p(a_1)p(a_2)\cdots p(a_k)\right)q^n.\end{aligned}$$

This equality permits us to present the congruence of this result in two different forms. Now in accordance with Lemma 1, we may write

$$\begin{aligned}
\prod_{m=1}^{\infty}(1-q^m)^{-k} &= \prod_{m=1}^{\infty}\left(\sum_{n=0}^{\infty}\binom{n+k-1}{k-1}q^{mn}\right)\\
&\equiv \prod_{m=1}^{\infty}\left[\left(\sum_{r=0}^{l-k}(-1)^r\binom{l-k}{r}q^{rm}\right)\left(1-q^{lm}\right)^{-1}\right] \pmod{l}\\
&\equiv \prod_{m=1}^{\infty}\left[(1-q^m)^{l-k}\left(1-q^{lm}\right)^{-1}\right] \pmod{l}.
\end{aligned}$$

Now the result follows. □

Corollary 6. *Let $p(n)$ be the number of partitions of n. We have*

$$\sum_{\substack{a+b=n\\ a,b\in\mathbb{N}\cup\{0\}}} p(a)p(b) \equiv \sum_{n=t+3s}\omega(t)p(s) \pmod{3}, \tag{39}$$

where $\omega(t)$ is given in (1).

Proof. Fix $l=3$ and $k=2$ in Theorem 21, then the result follows as a consequence of the observation $\tau_1(n)=\omega(n)$. □

Let $l \geq 5$ be a prime number. In view of Identity (20), we find a sufficient condition for the following congruence:

$$\sum_{\substack{n=a_1+a_2+\cdots+a_{l-3}\\ a_i\in\mathbb{N}\cup\{0\}}} p(a_1)p(a_2)\cdots p(a_{l-3}) \equiv 0 \pmod{l}.$$

Corollary 7. *Let $l \geq 5$ be a prime number. For non-zero integer n, let $R(n,l)$ denote the remainder obtained by division of n into l. If m is a non-negative integer such that $m \not\equiv s \pmod{l}$ for every*

$$s \in \{0\} \cup \left\{R\left((-1)^r\binom{l-3}{r}, l\right) : 0 \leq r \leq \frac{l-3}{2}\right\},$$

then for $n \equiv m \pmod{l}$ we have

$$\sum_{\substack{n=a_1+a_2+\cdots a_{l-3}\\ a_i\in\mathbb{N}\cup\{0\}}} p(a_1)p(a_2)\cdots p(a_{l-3}) \equiv 0 \pmod{l}. \tag{40}$$

Proof. Assume the following:

1. $m \not\equiv s \pmod{l}$ for every $s \in \{0\} \cup \left\{R\left((-1)^r\binom{l-3}{r}, l\right) : 0 \leq r \leq \frac{l-3}{2}\right\}$,

2. $n \equiv m \pmod{l}$,

3. $k = l - 3$.

Based on assumption 3 and Theorem 21, we may write

$$\sum_{\substack{n=a_1+a_2+\cdots a_{l-3}\\ a_i\in\mathbb{N}\cup\{0\}}} p(a_1)p(a_2)\cdots p(a_{l-3}) \equiv \sum_{n=t+ls} \tau_3(t)p(s) \pmod{l}. \tag{41}$$

In view of Identity (20), we can write

$$\tau_3(t) = \begin{cases} (-1)^j(2j+1) & \text{if } t = \frac{j(j+1)}{2};\\ 0 & \text{otherwise.} \end{cases}$$

So, based on assumption 2, (41) can be expressed as follows:

$$\begin{aligned} \sum_{\substack{n=a_1+a_2+\cdots a_{l-3}\\ a_i\in\mathbb{N}\cup\{0\}}} p(a_1)p(a_2)\cdots p(a_{l-3}) &\equiv \sum_{n=\binom{j+1}{2}+ls} (-1)^j(2j+1)p(s) \pmod{l}\\ &\equiv \sum_{lb+m=\binom{j+1}{2}+ls} (-1)^j(2j+1)p(s) \pmod{l} \end{aligned}$$

for some non-negative integer b. The index of the right extreme summation is non-empty only when $\binom{j+1}{2} \equiv m \pmod{l}$ for some j. In view of Lemma 1, $R\left(\binom{j+1}{2}, l\right)$ for $j = 1, 2\cdots$, constitute a subset of the set $\{0\} \cup \left\{R\left((-1)^r\binom{l-3}{r}, l\right) : 0 \le r \le \frac{l-3}{2}\right\}$. Based on assumption 1, we have $m \not\equiv s \pmod{l}$ for every $s \in \{0\} \cup \left\{R\left((-1)^r\binom{l-3}{r}, l\right) : 0 \le r \le \frac{l-3}{2}\right\}$. Given this observation, it follows that the index of the right extreme sum is empty. This leads to the conclusion that

$$\sum_{\substack{n=a_1+a_2+\cdots a_{l-3}\\ a_i\in\mathbb{N}\cup\{0\}}} p(a_1)p(a_2)\cdots p(a_{l-3}) \equiv 0 \pmod{l}.$$

Now the proof is completed. □

An interplay of Lemma 1 with Theorem 8 yields the following result of Ramanujan.

Corollary 8 (Ramanujan [16]). *Let n be a positive integer. We have*

$$\tau(7n) \equiv 0 \pmod{7}.$$

Proof. In accordance with Theorem 8, we can write

$$\begin{aligned}\tau(7n) &= \tau(7n-1+1)\\ &\equiv \sum_{7n-1=\frac{m(m+1)}{2}+7\frac{r(r+1)}{2}} (-1)^{m+r}(2m+1)(2r+1) \pmod 7. \end{aligned} \tag{42}$$

The index of the summation above suggests that

$$\binom{m+1}{2} = \binom{m-1+2}{2} \equiv 6 \pmod 7.$$

Since $6 = \binom{4}{2}$, in view of Lemma 1, we obtain that $m-1 \equiv 2 \pmod 7$. This implies that $2m+1 \equiv 0 \pmod 7$. Substituting this congruence in the sum in (42), we obtain that $\tau(7n) \equiv 0 \pmod 7$. $\square$

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