SERIES OF HEIGHT ONE MULTIPLE ZETA FUNCTIONS

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Abstract

We demonstrate that the multiple Hurwitz zeta function is the ordinary generating function for the sequence of height 1 multiple zeta functions. This principle is then used to evaluate various series involving such zeta functions and other important sequences of number theoretic and combinatorial nature.

1. Introduction

For positive integers $s_2, \ldots, s_j$, the multiple zeta function $\zeta(s, s_2, \ldots, s_j)$ may be considered as a single-variable function defined for $\Re(s) > 1$ by

$$\zeta(s, s_2, \ldots, s_j) = \sum_{n_1 > n_2 > \cdots > n_j > 0} \frac{1}{n_1^s n_2^{s_2} \cdots n_j^{s_j}}.$$

When $s = s_1 > 1$ is an integer, the value $\zeta(s_1, s_2, \ldots, s_j)$ is known as a multiple zeta value [21, 14] of weight $s_1 + \cdots + s_j$, of depth j, and of height $\#\{i : s_i > 1\}$; such values have been extensively studied. Following [20], we consider the function $\zeta(s, \{1\}^{j-1}) := \zeta(s, \underbrace{1, \ldots, 1}_{j-1})$ as the *height 1 multiple zeta function of depth* j, and we exhibit here various series involving the functions $\zeta(s, \{1\}^j)$ and Stirling, poly-Bernoulli, harmonic, hyperharmonic, and Roman harmonic numbers. These series are all derived by considering the *multiple Hurwitz zeta function* $\zeta_r(s)$ (see Section 2) as the ordinary generating function of the sequence $\{\zeta(s, \{1\}^j)\}_{j=0}^{\infty}$ of height 1 multiple zeta functions.

Among the most important algebraic properties of multiple zeta values is the duality relation, which in the height 1 case takes a simple symmetric form

$$\zeta(k+1, \{1\}^{j-1}) = \zeta(j+1, \{1\}^{k-1}) \tag{1.1}$$

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[21, 12, 14] for positive integers j, k. Consequently the height one MZVs have a well known symmetric generating function [12, 14]. Although here we consider the functions $\zeta(s, \{1\}^j)$ in a non-symmetric way by considering s as a complex variable, symmetry remains evident in many of our results. For example, in Corollary 1 below, we show that the values $C_j^{(-n)} = \mathbb{B}_j^{(-n)}(1)$ of the poly-Bernoulli polynomials, a sequence of positive integers with well-known symmetry and well-known combinatorial interpretations, serve as the coefficients of the expansion of the Riemann zeta function $\zeta(s)$ as a series in $\zeta(s+n, \{1\}^j)$.

After deriving the generating function and giving some multiple zeta function series in the next two sections, we further illustrate the method by generalizing the classical zeta series (2.24) of Goldbach to multiple zeta functions. Then in Section 3 we express the hyperharmonic zeta function $\sigma(r, s)$ in terms of multiple Hurwitz and height 1 multiple zeta functions. As applications of the main theorem, we conclude with some alternate expressions for series and constants related to height 1 multiple zeta functions, which were studied recently in [6, 20, 8].

2. Multiple Zeta Functions Generate Multiple Zeta Functions

For a positive integer r, the *multiple Hurwitz zeta function* [15, 18] of order r, denoted by $\zeta_r(s, a)$, is defined by the r-fold series

$$\zeta_r(s, a) := \sum_{t_1=0}^{\infty} \cdots \sum_{t_r=0}^{\infty} (a + t_1 + \cdots + t_r)^{-s} \tag{2.1}$$

for $\Re(s) > r$ and $\Re(a) > 0$, and continued meromorphically to $s \in \mathbb{C}$ with simple poles at $s = 1, 2, ..., r$. When $r = 1$ or $a = 1$ that part of the notation is often suppressed, so that $\zeta_1(s, 1) = \zeta(s)$ denotes the Riemann zeta function. These functions may equivalently be defined [18, eq. (3.3)] by the single Dirichlet series

$$\zeta_r(s, a) = \sum_{m=0}^{\infty} \binom{m + r - 1}{m} (m + a)^{-s}, \tag{2.2}$$

which has the advantage of allowing the order r to be any complex number, and defines $\zeta_r(s, a)$ as an analytic function of r for $\Re(a) > 0$ and $\Re(r) < \Re(s)$, which may be meromorphically continued to all $r \in \mathbb{C}$ [18, Section 3].

Theorem 1. *If* $\Re(s) > 1$ *and* $|r - 1| < |s - 1|$*, then*

$$\zeta_r(s) = \sum_{j=0}^{\infty} \zeta(s, \{1\}^j)(r - 1)^j.$$

Proof. The binomial coefficient in (2.2) may be expanded as

$$\binom{m+r-1}{m} = \frac{1}{m!}\sum_{k=0}^{m}\left[{m+1 \atop k+1}\right](r-1)^k \tag{2.3}$$

as a polynomial in $r-1$, whose coefficients are the (unsigned) Stirling numbers of the first kind, which satisfy

$$\left[{m+1 \atop k+1}\right] = m!\sum_{m\geqslant n_1>\cdots>n_k>0}\frac{1}{n_1\cdots n_k}. \tag{2.4}$$

Thus for $\Re(s)>1$ and any nonnegative integer j we evaluate

$$\begin{aligned}
D_r^j\zeta_r(s)\Big|_{r=1} &= D_r^j\sum_{m=0}^{\infty}\binom{m+r-1}{m}(m+1)^{-s}\Bigg|_{r=1}\\
&= D_r^j\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{m}\left[{m+1 \atop k+1}\right](r-1)^k(m+1)^{-s}\Bigg|_{r=1}\\
&= \sum_{m=0}^{\infty}\frac{j!}{m!}\left[{m+1 \atop j+1}\right](m+1)^{-s}\\
&= j!\sum_{m+1>n_1>\cdots>n_j>0}\frac{1}{(m+1)^s}\frac{1}{n_1\cdots n_j} = j!\zeta(s,\{1\}^j),
\end{aligned} \tag{2.5}$$

where $D_r = d/dr$ is the derivative operator. For fixed s with $\Re(s)>1$, the function $\zeta_r(s)$ is a meromorphic function of r [18, eq.(3.4)] with simple poles at $r = s, s+1, \ldots$, so in particular it is analytic on the disk $|r-1|<|s-1|$. The convergence of the series, and therefore the theorem, then follows from Taylor's theorem applied to the analytic function $\zeta_r(s)$ at $r=1$. $\square$

2.1. Integer Orders r

When the order r is taken to be a positive integer, the multiple zeta function $\zeta_r(s)$ reduces to a finite linear combination of Riemann zeta functions; this gives an expression of the Riemann zeta function as a series involving poly-Bernoulli numbers. The *poly-Bernoulli polynomials* $\mathbb{B}_n^{(k)}(x)$ are defined [5] by the generating function

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{-xt} = \sum_{n=0}^{\infty}\mathbb{B}_n^{(k)}(x)\frac{t^n}{n!} \tag{2.6}$$

(where $\mathrm{Li}_k(z) = \sum_{m=1}^{\infty}\frac{z^m}{m^k}$ is the polylogarithm function), and the values $C_n^{(k)} = \mathbb{B}_n^{(k)}(1)$ give the values $C_n^{(k)} = (-1)^n\xi_k(-n)$ at negative integers of the Arakawa-

Kaneko zeta function $\xi_k(s)$ [1, Theorem 6]. These values have the explicit formula

$$C_n^{(k)} = (-1)^n \sum_{i=0}^{n} \frac{(-1)^i i! \left\{ {n+1 \atop i+1} \right\}}{(i+1)^k} \tag{2.7}$$

in terms of the Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$. For negative integer orders, the values $C_n^{(-k)}$ are positive integers which have many combinatorial interpretations, including enumeration of certain subsets of 01 matrices [2, Theorems 2,4,7], of certain permutations [2, Theorems 10,12,16], of certain permutation tableaux [2, Theorem 6], and of acyclic graph orientations [2, Theorems 19,20]. Here they appear as the expansion coefficients which arise when $\zeta(s)$ is expressed as a series in $\zeta(s+n, \{1\}^j)$.

Corollary 1. *If n is a nonnegative integer and $\Re(s) > n+1$, we have*

$$\sum_{j=0}^{\infty} \zeta(s, \{1\}^j) n^j = \frac{1}{n!} \sum_{m=0}^{n} \left[{n \atop m} \right] \zeta(s-m).$$

Consequently, for $\Re(s) > 1$ and any nonnegative integer n,

$$\zeta(s) = \sum_{j=0}^{\infty} C_j^{(-n)} \zeta(s+n, \{1\}^j),$$

where $C_n^{(k)} = \mathbb{B}_n^{(k)}(1)$ are the values of poly-Bernoulli polynomials at $x = 1$.

Proof. For the first statement, expand the binomial coefficient in (2.2) as a polynomial in $m+1$ to obtain

$$\zeta_r(s) = \frac{1}{(r-1)!} \sum_{m=0}^{r-1} \left[{r-1 \atop m} \right] \zeta(s-m) \tag{2.8}$$

for positive integer orders r; then take $r = n+1$ in Theorem 1. For the second statement, we invert (2.8) according to the duality of Stirling numbers

$$f_n = \sum_{m=0}^{n} \left[{n \atop m} \right] (-1)^m g_m \quad \iff \quad g_n = \sum_{m=0}^{n} \left\{ {n \atop m} \right\} (-1)^m f_m \tag{2.9}$$

with $f_m = m! \zeta_{m+1}(s)$ and $g_m = (-1)^m \zeta(s-m)$, yielding

$$\begin{aligned} (-1)^n \zeta(s-n) &= \sum_{m=0}^{n} \left\{ {n \atop m} \right\} (-1)^m m! \zeta_{m+1}(s) \\ &= \sum_{j=0}^{\infty} \zeta(s, \{1\}^j) \sum_{m=0}^{n} (-1)^m m! \left\{ {n \atop m} \right\} m^j, \end{aligned} \tag{2.10}$$

for $\Re(s) > n+1$, via Theorem 1. Comparison with (2.7) shows that this inner sum over m in (2.10) is precisely $(-1)^n C_{n-1}^{(-j-1)}$. The symmetry $C_n^{(-k-1)} = C_k^{(-n-1)}$ [11, p. 76], and substituting $s \to s+n$, then finishes the proof. □

Example 1. Taking $n = 0$ yields the tautology $\zeta(s) = \zeta(s)$, while taking $n = 1$ yields the simple series

$$\zeta(s) = \sum_{j=0}^{\infty} \zeta(s+1, \{1\}^j) \qquad (\Re(s) > 1). \tag{2.11}$$

Taking $n = 3$ yields, for $\Re(s) > 4$,

$$\sum_{j=0}^{\infty} 3^j \zeta(s, \{1\}^j) = \frac{\zeta(s-1)}{3} + \frac{\zeta(s-2)}{2} + \frac{\zeta(s-3)}{6}, \tag{2.12}$$

with inverted series

$$\zeta(s-3) = \zeta(s) + 7\zeta(s,1) + 31\zeta(s,1,1) + 115\zeta(s,1,1,1) + \cdots, \tag{2.13}$$

with coefficients $C_j^{(-3)} = \{1, 7, 31, 115, 391, 1267, 3991, 12355, 37831, \ldots\}$.

Remark 1. The series of this corollary may be manipulated in countless ways. For example, since they are uniformly convergent series of analytic functions, one may apply operators such as d/ds to conclude

$$\sum_{j=0}^{\infty} \zeta'(s, \{1\}^j) = \zeta'(s-1) \quad (\Re(s) > 2). \tag{2.14}$$

Or, observing that $\lim_{s\to\infty} \zeta_r(s) = 1$, one may easily obtain the double series

$$\sum_{k=0}^{\infty}\sum_{j=1}^{\infty} \zeta(s+k, \{1\}^j) r^j = \frac{1}{r!} \sum_{m=1}^{r} \sum_{n=m}^{r} \begin{bmatrix} r \\ n \end{bmatrix} \zeta(s-m) - H_r, \tag{2.15}$$

where $H_r = 1 + \cdots + 1/r$ is the r-th harmonic number, by subtracting the $j = 0$ term from both sides of Corollary 1, summing over $k \geqslant 0$, and using the identities

$$\sum_{m=0}^{r} \begin{bmatrix} r \\ m \end{bmatrix} = r! \quad \text{and} \quad \sum_{m=0}^{r} m \begin{bmatrix} r \\ m \end{bmatrix} = r! H_r \tag{2.16}$$

to telescope the series.

The companion to Corollary 1 for negative integer orders r, which we include for sake of completeness, is considerably simpler. In this case the evaluation involves the *Roman harmonic numbers* $H_{n,k}$, which are defined [7, 8] by

$$H_{n,k} = \sum_{n \geqslant n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k \geqslant 1} \frac{1}{n_1 n_2 \cdots n_k}, \tag{2.17}$$

and may also be given by the single sum

$$H_{n,k} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} j^{-k}. \tag{2.18}$$

Observe that $H_{n,1} = H_n$ is the usual harmonic number.

Corollary 2. *If r is a positive integer and $\Re(s) > r+1$, then*

$$\sum_{j=0}^{\infty} \zeta(s, \{1\}^j)(-r)^j = \sum_{m=0}^{r-1} (-1)^m \binom{r-1}{m} (m+1)^{-s}.$$

In particular, for integers k, n with $k > n+1$, we have the rational sum

$$\sum_{j=0}^{\infty} \zeta(k, \{1\}^j)(-n)^j = \frac{H_{n,k-1}}{n}.$$

Proof. For negative integer orders, the series (2.2) defining the multiple Hurwitz zeta function reduces to the finite sum

$$\zeta_{-r}(s) = \sum_{j=0}^{r} (-1)^j \binom{r}{j} (j+1)^{-s}, \tag{2.19}$$

so the first statement then follows from Theorem 1. For an integer value $s = k$, the second statement then follows by comparison with (2.18). □

Example 2. Taking $r = 0, 1$ in this corollary produces

$$\sum_{j=0}^{\infty} (-1)^j \zeta(s, \{1\}^j) = 1 \quad (\Re(s) > 2), \tag{2.20}$$

$$\sum_{j=0}^{\infty} (-2)^j \zeta(s, \{1\}^j) = 1 - 2^{-s} \qquad (\Re(s) > 3). \tag{2.21}$$

As with Corollary 1, we may apply the operator d/ds (for example) to deduce various identities such as

$$\sum_{j=0}^{\infty} (-1)^j \zeta'(s, \{1\}^j) = 0 \quad (\Re(s) > 2), \tag{2.22}$$

$$\sum_{j=0}^{\infty} (-2)^j \zeta'(s, \{1\}^j) = 2^{-s} \log 2 \qquad (\Re(s) > 3). \tag{2.23}$$

These are just a few of the many results so obtainable.

2.2. Generalization of Goldbach's Series

In a 1729 letter to Daniel Bernoulli, Christian Goldbach gave the series

$$\sum_{n=2}^{\infty}(\zeta(n)-1)=1 \tag{2.24}$$

[16], which has the alternating counterpart

$$\sum_{n=2}^{\infty}(-1)^n(\zeta(n)-1)=\frac{1}{2}. \tag{2.25}$$

Generally, given a convergent sequence $\{a_n\}$ of real numbers with limit L, one may consider the series $\sum_n |a_n-L|$ and $\sum_n(-1)^n|a_n-L|$; here we give such evaluations where the a_j involve height 1 multiple zeta values, generalizing Goldbach's classical series.

Theorem 2. *For any positive integer k, we have*

$$\sum_{j=0}^{\infty}\left(k^j\zeta(k+1,\{1\}^j)-\frac{1}{k!k}\right)=\frac{1}{k!}\left(\sum_{m=0}^{k-1}\begin{bmatrix}k\\m\end{bmatrix}\zeta(k+1-m)+H_k\right),$$

where $H_k=1+\cdots+1/k$ is the harmonic number, with corresponding alternating series having rational sum

$$\sum_{j=0}^{\infty}(-1)^j\left(k^j\zeta(k+1,\{1\}^j)-\frac{1}{k!k}\right)=\frac{H_{k,k}}{k}-\frac{1}{2k!k}.$$

Proof. Fixing an integer $s>1$ and considering $\zeta_r(s)$ as a meromorphic function of r, we remove its leftmost pole by expanding $1/((s-1)!(s-r))$ as a geometric series and subtracting it from both sides of Theorem 1. This produces the series

$$\zeta_r(s)-\frac{1}{(s-1)!(s-r)}=\sum_{j=0}^{\infty}\left(\zeta(s,\{1\}^j)-\frac{1}{(s-1)!(s-1)^{j+1}}\right)(r-1)^j, \tag{2.26}$$

which now converges for $|r-1|<s$ since the leftmost pole of the left-hand side is now at $r=s+1$ [18, eq. (3.4)]. For integers $s>1$, the limit as $r\to s$ of the left side of (2.26) was evaluated in [19, Theorem 5.1] as

$$\lim_{r\to s}\left(\zeta_r(s)-\frac{1}{(s-1)!(s-r)}\right)=(s-1)!\gamma_s(1)-\gamma+H_{s-1}, \tag{2.27}$$

where γ denotes Euler's constant and $\gamma_r(1)$ denotes the order r generalized Euler constant defined by

$$\gamma_r(1):=\lim_{s\to r}\left(\zeta_r(s)-\frac{1}{(s-1)!(s-r)}\right), \tag{2.28}$$

with $\gamma_1(1) = \gamma$. But evaluating $\zeta_r(s)$ as $s \to r$ using (2.8) yields

$$\gamma_r(1) = \frac{\gamma}{(r-1)!} + \sum_{m=0}^{r-2} \begin{bmatrix} r-1 \\ m \end{bmatrix} \zeta(r-m), \tag{2.29}$$

so the first statement follows from (2.26), (2.27), and (2.29), upon replacing k with $s-1$. The second statement is obtained by evaluating (2.26) at $1-r = s-1$ using (2.19) and (2.18). $\square$

Example 3. By means of the duality (1.1), we have $\zeta(2, \{1\}^j) = \zeta(j+2)$, so the $k = 1$ case of this theorem reduces to the classical series (2.24) of Goldbach and its alternating counterpart (2.25). The cases $k = 2, 3, \ldots$ generalize these as follows.

$$\sum_{j=0}^{\infty} \left(2^j \zeta(3, \{1\}^j) - \frac{1}{4} \right) = \frac{\zeta(2)}{2} + \frac{3}{4}, \tag{2.30}$$

$$\sum_{j=0}^{\infty} (-1)^j \left(2^j \zeta(3, \{1\}^j) - \frac{1}{4} \right) = \frac{3}{4}, \tag{2.31}$$

$$\sum_{j=0}^{\infty} \left(3^j \zeta(4, \{1\}^j) - \frac{1}{18} \right) = \frac{\zeta(3)}{3} + \frac{\zeta(2)}{2} + \frac{11}{36}, \tag{2.32}$$

$$\sum_{j=0}^{\infty} (-1)^j \left(3^j \zeta(4, \{1\}^j) - \frac{1}{18} \right) = \frac{557}{648}. \tag{2.33}$$

3. Hyperharmonic Zeta Functions Generate Multiple Zeta Functions

The *hyperharmonic numbers* $H_m^{[r]}$ of order r are defined by $H_m^{[0]} = \frac{1}{m}$ for $m > 0$, $H_0^{[r]} = 0$, and the recursion

$$H_m^{[r]} = \sum_{i=1}^{m} H_i^{[r-1]} \tag{3.1}$$

for positive integers r (cf. [13, 9]). Thus $H_n = H_n^{[1]}$ denotes the usual harmonic number. They are also given by the derivative

$$H_m^{[r]} = D_r \binom{m+r-1}{m} \tag{3.2}$$

[17, eq. (2.5)], which has the advantage of allowing the order r to be any complex number. In [13, 9] it was shown that the hyperharmonic zeta function

$$\sigma(r, s) := \sum_{m=1}^{\infty} H_m^{[r]} m^{-s} \tag{3.3}$$

converges for $\Re(s) > r$, and for integers $0 \leqslant r < s$ its values may be expressed as rational polynomials in Riemann zeta values. Here we express the hyperharmonic zeta functions $\sigma(r,s)$ in terms of the multiple Hurwitz zeta functions and height 1 zeta functions.

Theorem 3. *We have*

$$\sigma(r,s) = D_r \zeta_r(s,a)\Big|_{a\to 0} = \sum_{j=0}^{\infty}(j+1)\zeta(s+1,\{1\}^j)r^j,$$

where the first equality gives the meromorphic continuation of $\sigma(r,s)$ to the complex plane, and the second equality is valid for $\Re(s) > 0$ and $|r| < |s|$.

Proof. For $\Re(r) < \Re(s)$, the first equality follows from (2.2) and (3.2); thus $\sigma(r,s)$ extends to a meromorphic function of r with poles at $r = s, s+1, \ldots$. A calculation identical to (2.5) at $(r,a) = (0,0)$ rather than $(1,1)$ then shows that

$$D_r^j \sigma(r,s)\Big|_{r=0} = D_r^{j+1}\zeta_r(s,a)\Big|_{r=0,a\to 0} = (j+1)!\zeta(s+1,\{1\}^j), \tag{3.4}$$

so the second equality expresses the Taylor series of the function $\sigma(r,s)$ at $r = 0$, which is analytic for $|r| < |s|$. □

Example 4. For $r = 0$ the Theorem reduces to the tautology $\zeta(s+1) = \zeta(s+1)$. For $r = 1$ we get

$$\sigma(1,s) = \sum_{m=1}^{\infty} H_m m^{-s} = \zeta(s,1) + \zeta(s+1) = \sum_{j=0}^{\infty}(j+1)\zeta(s+1,\{1\}^j). \tag{3.5}$$

A useful alternate version of Theorem 3 is given by the series

$$D_r \zeta_r(s) = \sum_{m=0}^{\infty} H_m^{[r]}(m+1)^{-s} = \sum_{j=1}^{\infty} j\zeta(s,\{1\}^j)(r-1)^{j-1}, \tag{3.6}$$

obtained by differentiating the series of Theorem 1 using (3.2), valid for $\Re(s) > 1$ and $|r-1| < |s-1|$. The two versions are related by the identity

$$\sum_{m=0}^{\infty} H_m^{[r]}(m+1)^{-s} = \sigma(r,s) - \sigma(r-1,s). \tag{3.7}$$

For positive integers $r < s$ the values $\sigma(r,s)$ were evaluated recursively in [13, Theorem 6], yielding

$$\sigma(r,s) - \sigma(r-1,s) = \frac{1}{r-1}\left(\sigma(r-1,s-1) - \zeta_r(s)\right) \tag{3.8}$$

for $s > r > 1$. Taking $r = 2, 3$ in (3.6), and using (3.5), (3.8), we obtain

$$\sum_{j=1}^{\infty} j\zeta(s,\{1\}^j) = \zeta(s-1,1) + \zeta(s) - \zeta(s-1) \quad (s \geqslant 3), \tag{3.9}$$

$$\begin{aligned}\sum_{j=1}^{\infty} j2^j\zeta(s,\{1\}^j) &= \zeta(s-1,1) + \zeta(s-2,1) + \zeta(s) \\ &\quad + \frac{\zeta(s-1)}{2} - \frac{3\zeta(s-2)}{2} \quad (s \geqslant 4).\end{aligned} \tag{3.10}$$

For negative integer orders, we observe from (3.2) that

$$H_m^{[-r]} = \begin{cases} (-1)^{m+1} H_m^{[r+1-m]}, & 0 < m \leqslant r+1; \\ \dfrac{(-1)^r}{m\dbinom{m-1}{r}}, & m \geqslant r+1 \end{cases} \tag{3.11}$$

for $r \geqslant 0$. Taking $r = 0$ in (3.6) and invoking the geometric series $m^{-1} = \sum_{n=1}^{\infty}(m+1)^{-n}$ as in [18, eq. (5.10)] produces

$$\begin{aligned}\sum_{j=1}^{\infty} (-1)^{j-1} j\zeta(s,\{1\}^j) &= \sum_{m=1}^{\infty} \frac{1}{m(m+1)^s} \\ &= \sum_{n=1}^{\infty} (\zeta(s+n) - 1) \\ &= s - \sum_{k=2}^{s} \zeta(k),\end{aligned} \tag{3.12}$$

the first two expressions valid for $\Re(s) > 2$ and the last for integers $s > 2$. Taking $r = -1$ in (3.6) and using a similar technique as in [18, eq. (5.11)] produces

$$\begin{aligned}\sum_{j=1}^{\infty} (-2)^{j-1} j\zeta(s,\{1\}^j) &= \frac{1}{2^s} + \sum_{n=1}^{\infty} (1 - 2^{n-1}) \left(\zeta(s+n) - 1 - \frac{1}{2^{s+n}}\right) \\ &= s - 1 - \frac{s-2}{2^{s+1}} - \sum_{k=2}^{s} \left(1 - \frac{1}{2^{s+1-k}}\right) \zeta(k),\end{aligned} \tag{3.13}$$

the first expression valid for $\Re(s) > 3$ and the second for integers $s > 3$.

Since $D_r\zeta_r(s)$ tends exponentially to zero as $s \to \infty$ for any fixed r, the series obtained from (3.6) may be summed over s to produce double series. For example, by summing (3.9), (3.10), and (3.12) over integers $s \geqslant 3$ we obtain

$$\sum_{k=3}^{\infty}\sum_{j=1}^{\infty} j\zeta(k,\{1\}^j) = 1, \tag{3.14}$$

$$\sum_{k=4}^{\infty}\sum_{j=1}^{\infty} j2^j\zeta(k,\{1\}^j) = \frac{5}{2} + \frac{\zeta(2)}{2} - 2\zeta(3), \tag{3.15}$$

$$\sum_{k=3}^{\infty}\sum_{j=1}^{\infty} (-1)^{j-1} j\zeta(k,\{1\}^j) = 2\zeta(2) - 3. \tag{3.16}$$

The evaluation of (3.14) applies the duality (1.1) and the identity (2.11) to the terms $\zeta(s-1,1)$ in (3.9), and telescopes the sum of the other two terms, yielding the value

$$\begin{aligned}\sum_{k=3}^{\infty}\sum_{j=1}^{\infty} j\zeta(k,\{1\}^j) &= \sum_{j=0}^{\infty}\zeta(3,\{1\}^j) + \sum_{k=3}^{\infty}[\zeta(k)-\zeta(k-1)] \\ &= \zeta(2) + [1-\zeta(2)] = 1. \end{aligned} \tag{3.17}$$

Identities (3.15), (3.16) (and many others) are similarly obtained.

4. Expressions of Multiple Zeta Constants

We conclude by considering the class of shifted alternating multiple zeta series

$$\nu_{k,p} := \sum_{n=2}^{\infty} \frac{(-1)^n}{n+k}\zeta(n,\{1\}^p)$$

for integers $p \geqslant 0$ and $k \geqslant -1$, which were studied in [6]. These constants are intimately connected to the singular behavior of height 1 multiple zeta functions, going back to Euler's 1731 series expression $\nu_{0,0} = \gamma$ for the constant term γ in the Laurent expansion of $\zeta(s)$ at $s = 1$. In [6] Coppo showed that

$$\nu_{0,1} - \nu_{-1,0} = \gamma_1 + \frac{\gamma^2}{2} - \frac{\pi^2}{12} \tag{4.1}$$

where γ_1 is the first Stieltjes constant (i.e., the linear coefficient in the Laurent expansion of $\zeta(s)$ near $s = 1$). This was recently generalized [20, Corollary 5.2] to height one zeta functions of arbitrary depth j; specifically we have

$$\nu_{0,j} - \nu_{-1,j-1} = \gamma_1^{[j]} + [s^{j+1}](\Gamma(s+1)^{-1}) \tag{4.2}$$

where $\gamma_1^{[j]}$ is the first height 1 Stieltjes constant of depth j, that is, the linear coefficient in the Laurent expansion of $\zeta(s,\{1\}^{j-1})$ at $s=1$, and $[s^{j+1}](\Gamma(s+1)^{-1})$ denotes the coefficient of s^{j+1} in the Taylor series for $1/\Gamma(s+1)$. The constants $\nu_{0,p}$ and $\nu_{-1,p}$ were also interpreted in terms of the Ramanujan summations of multiple harmonic star sums [20, Theorem 6.1]. We now observe the following integral expression for the constants $\nu_{k,p}$ which follows from Theorem 1 and the duality (1.1).

Theorem 4. *For $k \geqslant -1$ and $p \geqslant 0$ we have*

$$\nu_{k,p} = \int_0^1 (1-r)^{k+1} \zeta_r(p+2)\, dr.$$

Proof. For integers $k \geqslant -1$ and $p \geqslant 0$ we use Theorem 1 to calculate

$$\begin{aligned}\int_0^1 (1-r)^{k+1}\zeta_r(p+2)\,dr &= \sum_{j=0}^{\infty} (-1)^{k+1}\zeta(p+2,\{1\}^j)\int_0^1 (r-1)^{j+k+1} dr \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j \zeta(p+2,\{1\}^j)}{j+k+2} \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j \zeta(j+2,\{1\}^p)}{j+k+2} = \nu_{k,p}. \end{aligned} \tag{4.3}$$

□

As a corollary, from the expression (4.2) we have the following integral representation of the first height 1 Stieltjes constants.

Corollary 3. *For any positive integer j, the linear coefficient $\gamma_1^{[j]}$ in the Laurent expansion of $\zeta(s,\{1\}^{j-1})$ at $s=1$ has the integral expression*

$$\gamma_1^{[j]} = \int_0^1 ((1-r)\zeta_r(j+2) - \zeta_r(j+1))\, dr - [s^{j+1}](\Gamma(s+1)^{-1}).$$

Example 5. For height $j=1$ we have the expression

$$\gamma_1 = \int_0^1 ((1-r)\zeta_r(3) - \zeta_r(2))\, dr + \frac{\pi^2}{12} - \frac{\gamma^2}{2}. \tag{4.4}$$

for the usual first Stieltjes constant.

The constant $\nu_{-1,0}$ also arises in several other contexts in number theory. It appears [10] in the asymptotic formula for the number of divisors of $n!$, several series [3, 8] relate it to the Riemann zeta function and to harmonic numbers, and it occurs in certain Ramanujan summations [6, 7] involving harmonic numbers. Among its many series expressions we find

$$\nu_{-1,0} = -\sum_{n=2}^{\infty} \zeta'(n) \tag{4.5}$$

[4, p. 142], which leads to the double series expression

$$\nu_{-1,0} = -\sum_{k=3}^{\infty}\sum_{j=0}^{\infty} \zeta'(k, \{1\}^j) \tag{4.6}$$

by means of (2.14).

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