NEW SERIES REPRESENTATIONS FOR THE RIEMANN ZETA FUNCTION AND ITS ASSOCIATED FUNCTIONS

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Abstract

This paper aims to introduce novel series representations for the Riemann Zeta function, along with its associated functions like the Dirichlet eta (alternating zeta) function and the Dirichlet lambda function. These series representations involve an independent parameter and are derived through generalized series incorporating psi functions. Furthermore, we present even more generalized series representations, encompassing the aforementioned ones as specific instances.

1. Introduction and Preliminaries

The well-known *Riemann Zeta function* is defined by (see, for example, [31, p. 164, Equation (1)])

$$\zeta(t) = \begin{cases} \sum\limits_{n\geqslant 1} \frac{1}{n^t} = \frac{\lambda(t)}{1-2^{-t}} & (\Re(t) > 1) \\ \frac{\eta(t)}{1-2^{1-t}} & (\Re(t) > 0;\ t \neq 1). \end{cases}$$

Here $\eta\,(t)$ is the *Dirichlet eta* (or *alternating*) *function* given by

$$\eta\,(t) := \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n^t} = \left(1 - 2^{1-t}\right)\zeta\,(t) = \Phi\,(-1, t, 1)\,, \tag{1}$$

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where $\Phi(\cdot,\cdot,\cdot)$ represents the *Hurwitz–Lerch Zeta function*. $\lambda(t)$ denotes the *Dirichlet lambda function* defined by

$$\lambda(t) := \sum_{n\geqslant 1} \frac{1}{(2n-1)^t} = \left(1-2^{-t}\right)\zeta(t) = \frac{\zeta(t)+\eta(t)}{2}, \tag{2}$$

whose rightmost expression signifies termwise arithmetic mean. Among a variety of identities about these three functions, we recall their integral representations (see, for example, [31, Section 2.3]):

$$\zeta(t) = \frac{1}{\Gamma(t)} \int_0^\infty \frac{z^{t-1}}{e^z-1}\,\mathrm{d}z \quad (\Re(t) > 1),$$

$\Gamma(t)$ being the Gamma function,

$$\eta(t) = \frac{1}{\Gamma(t)} \int_0^\infty \frac{z^{t-1}}{e^z+1}\,\mathrm{d}z \quad (\Re(t) > 1),$$

and

$$\lambda(t) = \frac{1}{\Gamma(t)} \int_0^\infty \frac{e^z\ z^{t-1}}{e^{2z}-1}\,\mathrm{d}z \quad (\Re(t) > 1).$$

For more integral representations for the Riemann zeta function, one can consult [2], [5], [8], [9], [10, p.32-p.34], [12], [13], [14], [16], [17], [19], [23], [31, Section 2.3], [32], and [37].

Since Euler solved the Basel Problem

$$\zeta(2) = \sum_{n\geqslant 1} \frac{1}{n^2} = \frac{\pi^2}{6}$$

in 1768 (see [17]), a number of intriguing and ingenious proofs for the Basel Problem have been published (see, for example, [2], [9], [12], [15], [20], [21], [22], [23], [24], [25], [26], [28], [33], [34], [35], [36], and [37]). In 1978, Apéry [6] accomplished a significant breakthrough by proving the irrationality of $\zeta(3)$. From that point forward, the constant $\zeta(3)$ has been referred to as the Apery constant. His proof relied on an expediently converging series for $\zeta(3)$:

$$\zeta(3) = \frac{5}{2} \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n^3 \binom{2n}{n}}.$$

Subsequently, a multitude of series representations for $\zeta(n)$ ($n \in \mathbb{Z}_{\geqslant 2}$) have been introduced in the literature (see, for example, [3], [5], [7], [9], [11], [12], [14], [19], and [27]). Here and elsewhere, let $\mathbb{Z}$ be the set of integers, and $\mathbb{Z}_{\geqslant \ell}$ (or, $\mathbb{Z}_{\leqslant \ell}$) denote the set of all integers that are greater than or equal to (or, less than or equal to) some integer ℓ. Also let $\mathbb{N} := \mathbb{Z}_{\geqslant 1}$. In this paper, we aim to provide new series representations for the Riemann zeta, Dirichlet eta and Dirichlet lambda functions. To do this, the subsequent definitions and notations are recalled. The *generalized harmonic numbers* $H_n^{(t)}(b)$ *of order* t are defined by

$$H_n^{(t)}(b) := \sum_{j=1}^{n} \frac{1}{(j+b)^t} \quad \left(t \in \mathbb{C},\ b \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1},\ n \in \mathbb{N}\right),$$

where $H_n^{(t)} := H_n^{(t)}(0)$ are the harmonic numbers of order t. Here and in the following, let $\mathbb{C}$ denote the set of complex numbers. The *harmonic numbers* $H_n := H_n^{(1)}$ are given by

$$H_n = \sum_{j=1}^{n} \frac{1}{j} = \gamma + \psi(n+1) \quad (n \in \mathbb{N}) \quad \text{and} \quad H_0 := 0,$$

where γ is the Euler-Mascheroni constant (see, for example, [31, Section 1.2]) and $\psi(b)$ denotes the *digamma* (or *psi*) *function* defined by

$$\psi(b) := \frac{d}{db}(\log \Gamma(b)) = \frac{\Gamma'(b)}{\Gamma(b)} \quad \left(b \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}\right).$$

The following properties for the psi function, among numerous others, are recalled:

$$\psi(b+1) = \psi(b) + \frac{1}{b}, \tag{3}$$

$$\psi(b) - \psi(1-b) = -\pi \cot(\pi b) \quad (b \in \mathbb{C} \setminus \mathbb{Z}),$$

and

$$\psi\left(b + \tfrac{1}{2}\right) = 2\psi(2b) - \psi(b) - 2\log 2 \quad \left(b,\ b + \tfrac{1}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}\right).$$

The *polygamma function* $\psi^{(k)}(z)$ is defined by

$$\psi^{(k)}(z) := \frac{d^k}{dz^k}\{\psi(z)\} = (-1)^{k+1}\, k! \sum_{r=0}^{\infty} \frac{1}{(r+z)^{k+1}} = (-1)^{k+1}\, k!\, \zeta(k+1, z), \tag{4}$$

$$\left(k \in \mathbb{N};\ z \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}\right),$$

where $\zeta(s,z)$ is the *generalized* (or *Hurwitz*) *zeta function* defined by

$$\zeta(s,z) = \sum_{m=0}^{\infty} \frac{1}{(m+z)^s} \quad \left(\Re(s) > 1,\ z \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}\right).$$

The polygamma function $\psi^{(k)}(z)$ has the recurrence:

$$\psi^{(k)}(z+1) = \psi^{(k)}(z) + \frac{(-1)^k\, k!}{z^{k+1}}. \tag{5}$$

The *alternating*, or *skew*, *harmonic numbers* $A_n^{(t)}$ *of order* t are defined by

$$A_n^{(t)} := \sum_{j=1}^{n} \frac{(-1)^{j+1}}{j^t} \quad (t \in \mathbb{C},\ n \in \mathbb{N})$$

and $A_n := A_n^{(1)}$. The alternating harmonic numbers and the harmonic numbers have the following relation:

$$A_n^{(t)} = H_n^{(t)} - 2^{1-t} H_{[n/2]}^{(t)}.$$

The symbol $[x]$ indicates the greatest integer less than or equal to an $x \in \mathbb{R}$. Elsewhere in this context, let $\mathbb{R}$ denote the set of real numbers. The *generalized alternating harmonic numbers* $A_n^{(t)}(b)$ are defined by

$$A_n^{(t)}(b) := \sum_{j=1}^{n} \frac{(-1)^{j+1}}{(j+b)^t} \quad (t \in \mathbb{C},\, b \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1},\, n \in \mathbb{N}).$$

The *Hurwitz–Lerch Zeta function* $\Phi(z, s, a)$ is defined by (see, for example, [31, Section 2.5])

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{6}$$
$$(a \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0};\ s \in \mathbb{C} \ \text{ when } \ |z| < 1;\ \Re(s) > 1 \ \text{ when } \ |z| = 1),$$

which satisfies the obvious functional relation:

$$\Phi(z, s, a) = z^n\, \Phi(z, s, n + a) + \sum_{k=0}^{n-1} \frac{z^k}{(k+a)^s} \quad (n \in \mathbb{N};\ a \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}).$$

Clearly

$$\zeta(s) = \Phi(1, s, 1) \quad \text{and} \quad \zeta(s, a) = \Phi(1, s, a).$$

The function $\mathbf{b}(z)$, which incorporates the digamma function, is defined by (cf. [10, p. 20])

$$\mathbf{b}(z) := \frac{1}{2} \left\{ \psi\left(\frac{z+1}{2}\right) - \psi\left(\frac{z}{2}\right) \right\}$$

and has a number of useful properties regarding its integral representation given in (see, for instance, [10, p. 20], [13]). Some examples are

$$\mathbf{b}(z) = \int_0^1 \frac{t^{z-1}}{1+t}\, \mathrm{d}t \quad (\Re(z) > 0),$$

$$\mathbf{b}\left(\frac{z}{p}\right) = p \int_0^1 \frac{t^{z-1}}{1+t^p}\, \mathrm{d}t \quad (\Re(z) > 0,\, p \in \mathbb{R}_{>0}), \tag{7}$$

$$\int_0^1 \frac{t^z}{(1+t)^2}\,\mathrm{d}t = z\mathbf{b}(z) - \frac{1}{2} \quad (\Re(z) > -1)\,,$$

and

$$\int_0^1 \frac{t^z}{(1+t^p)^2}\,\mathrm{d}t = \frac{z+1-p}{p^2}\mathbf{b}\left(\frac{z+1-p}{p}\right) - \frac{1}{2p} \quad (\Re(z) > -1,\, p \in \mathbb{R}_{>0})\,.$$

Here $\mathbb{R}_{>0} := \{x \in \mathbb{R} : x > 0\}$.

The *polylogarithm function* $\mathrm{Li}_t(z)$ *of order* t is defined by

$$\mathrm{Li}_t(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^t} = \int_0^z \frac{\mathrm{Li}_{t-1}(t)}{t}\,\mathrm{d}t \quad (|z| \leqslant 1;\ t \in \mathbb{Z}_{\geqslant 2})\,.$$

In particular, the *dilogarithm function* $\mathrm{Li}_2(z)$ is given by

$$\mathrm{Li}_2(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^2} = -\int_0^z \frac{\log(1-t)}{t}\,\mathrm{d}t \quad (|z| \leqslant 1)\,.$$

2. Main Results

In pursuit of deriving a family of new series representations for the Riemann Zeta function $\zeta(t)$, the Dirichlet eta (alternating) function $\eta(t)$, and the Dirichlet lambda function $\lambda(t)$, which serves as the primary focus of this article, we initiate our investigation with the following theorem.

Theorem 1. *Let $q,\, t \in \mathbb{N}$ and $-1 \leqslant a < 1$. Then the following identity holds:*

$$\begin{aligned}
&\frac{1}{2q}\sum_{n\geqslant 1}\frac{1}{n^t}\left\{\psi\left(\tfrac{n+a+1+q}{2q}\right) - \psi\left(\tfrac{n+a+1}{2q}\right)\right\} \\
&\quad + \frac{1}{2q}\sum_{n\geqslant 1}\frac{1}{n^t}\left\{\psi\left(\tfrac{n-a-1+2q}{2q}\right) - \psi\left(\tfrac{n-a-1+q}{2q}\right)\right\} \\
&= \frac{(-1)^t}{q^t}\sum_{n\geqslant 1}\frac{2n+q}{(n+a+1)(n-a-1+q)}A^{(t)}_{\left[\frac{n+q-1}{q}\right]}\left(-\frac{n+q}{q}\right).
\end{aligned} \tag{8}$$

In particular, for $a = -1$, we have a new representation of the Riemann zeta function: for $q \in \mathbb{N}$,

$$\begin{aligned}
\zeta(t+1) &= \sum_{n\geqslant 1}\frac{2n+q}{n(n+q)}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{(-1)^{j+1}}{(n+q-qj)^t} \\
&= \frac{(-1)^t}{q^t}\sum_{n\geqslant 1}\frac{2n+q}{n(n+q)}A^{(t)}_{\left[\frac{n+q-1}{q}\right]}\left(-\frac{n+q}{q}\right).
\end{aligned} \tag{9}$$

Proof. Let q, $t \in \mathbb{N}$ and $-1 \leqslant a < 1$ and let

$$\Lambda(a,q,t) := \int_0^1 \frac{x^a \ \mathrm{Li}_t(x)}{1+x^q}\,\mathrm{d}x + \int_0^1 \frac{x^{q-a-2} \ \ \mathrm{Li}_t(x)}{1+x^q}\,\mathrm{d}x. \tag{10}$$

Utilizing the definition of the polylogarithm function, upon interchanging the order of summation and integral, which is verifiable under the constraints, we express $\Lambda(a,q,t)$ as

$$\begin{aligned}\Lambda(a,q,t) &= \int_0^1 \frac{x^a}{1+x^q}\sum_{n\geqslant 1}\frac{x^n}{n^t}\,\mathrm{d}x + \int_0^1 \frac{x^{q-a-2}}{1+x^q}\sum_{n\geqslant 1}\frac{x^n}{n^t}\,\mathrm{d}x \\ &= \sum_{n\geqslant 1}\frac{1}{n^t}\int_0^1 \frac{x^{n+a}}{1+x^q}\,\mathrm{d}x + \sum_{n\geqslant 1}\frac{1}{n^t}\int_0^1 \frac{x^{n+q-a-2}}{1+x^q}\,\mathrm{d}x.\end{aligned} \tag{11}$$

The integrals in the right-most side of Equation (11) are associated with Equation (7) to give

$$\Lambda(a,q,t) = \frac{1}{2q}\sum_{n\geqslant 1}\frac{1}{n^t}\left\{\begin{aligned}&\psi\left(\tfrac{n+a+1+q}{2q}\right) - \psi\left(\tfrac{n+a+1}{2q}\right) \\ &+\psi\left(\tfrac{n-a-1+2q}{2q}\right) - \psi\left(\tfrac{n-a-1+q}{2q}\right)\end{aligned}\right\}. \tag{12}$$

By using a series manipulation of double series, we find that, for $0 < x < 1$,

$$\begin{aligned}\frac{x^a \ \mathrm{Li}_t(x)}{1+x^q} &= x^a \sum_{n\geqslant 1}\frac{x^n}{n^t}\sum_{r\geqslant 0}(-1)^r x^{qr} \\ &= \sum_{n\geqslant 1} x^{n+a}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{(-1)^{j+1}}{(n+q-qj)^t} \\ &= \frac{(-1)^t}{q^t}\sum_{n\geqslant 1} x^{n+a} \ A^{(t)}_{\left[\frac{n+q-1}{q}\right]}\left(-\frac{n+q}{q}\right).\end{aligned} \tag{13}$$

Integrating both sides of Equation (13) over $x \in (0,1)$ and interchanging the order of integral and summation, which is guaranteed under the restrictions, we get

$$\int_0^1 \frac{x^a \ \mathrm{Li}_t(x)}{1+x^q}dx = \sum_{n\geqslant 1}\frac{1}{n+a+1}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{(-1)^{j+1}}{(n+q-qj)^t}. \tag{14}$$

Similarly

$$\int_0^1 \frac{x^{q-a-2} \ \mathrm{Li}_t(x)}{1+x^q}dx = \sum_{n\geqslant 1}\frac{1}{n+q-a-1}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{(-1)^{j+1}}{(n+q-qj)^t}. \tag{15}$$

Putting Equation (14) and Equation (15) in Equation (10) gives

$$\begin{aligned}\Lambda(a,q,t) &= \sum_{n\geqslant 1}\frac{2n+q}{(n+a+1)(n-a-1+q)}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{(-1)^{j+1}}{(n+q-qj)^{t}} \\ &= \frac{(-1)^t}{q^t}\sum_{n\geqslant 1}\frac{2n+q}{(n+a+1)(n-a-1+q)}A^{(t)}_{\left[\frac{n+q-1}{q}\right]}\left(-\frac{n+q}{q}\right).\end{aligned} \tag{16}$$

Finally, matching Equation (12) and Equation (16) provides the desired identity Equation (8).

In particular, setting $a=-1$ in (8) offers

$$\begin{aligned}&\frac{1}{2q}\sum_{n\geq 1}\frac{1}{n^t}\left\{\psi\left(\frac{n+q}{2q}\right)-\psi\left(\frac{n}{2q}\right)\right\} \\ &\quad+\frac{1}{2q}\sum_{n\geq 1}\frac{1}{n^t}\left\{\psi\left(\frac{n}{2q}+1\right)-\psi\left(\frac{n+q}{2q}\right)\right\} \\ &=\frac{(-1)^t}{q^t}\sum_{n\geq 1}\frac{2n+q}{n(n+q)}A^{(t)}_{\left[\frac{n+q-1}{q}\right]}\left(-\frac{n+q}{q}\right),\end{aligned}$$

which, upon using Equation (3), leads to an alternative desired identity (9). □

We see that Equation (9) provides a family of series representations for $\zeta(t+1)$ which are distinct for each $q\in\mathbb{N}$. There are many representations of the Riemann zeta function; see, for example, [7], [8], and [11]. The representation (9) bears resemblance to the Hasse series [14]:

$$\zeta(t)=\frac{1}{t-1}\sum_{n\geqslant 0}\sum_{j\geqslant 0}\frac{(-1)^j}{(n+1)(j+1)^{t-1}}\binom{n}{j} \tag{17}$$

as well as a prior finding by Ser [7]:

$$\zeta(t)=\frac{1}{t-1}\sum_{n\geqslant 0}\frac{1}{n+2}\sum_{j=0}^{n}\frac{(-1)^j\binom{n}{j}}{(j+1)^t}. \tag{18}$$

Indeed, Blagouchine [7] demonstrates the equivalence between Equation (17) and Equation (18). The series in Equation (17) converges at a slower rate compared to the series (9). Notably, within the representation (9), we encounter two intriguing instances: the renowned Basel problem for $\zeta(2)$ when $t=1$ and Apery's constant $\zeta(3)$ when $t=2$. The Riemann zeta function $\zeta(2)$ has many different series representations such as

$$\zeta(2)=3\sum_{n\geqslant 1}\frac{1}{n^2\binom{2n}{n}},\quad \zeta(2)=\frac{5}{3}\sum_{n\geqslant 1}\frac{(-1)^n\binom{2n}{n}}{2^{4n}(2n+1)^2},$$

and the notable BBP type formula:

$$\zeta(2) = \frac{3}{16}\sum_{n\geqslant 0}\frac{1}{2^{2n}}\left(\frac{16}{(6n+1)^2} - \frac{24}{(6n+2)^2} - \frac{8}{(6n+3)^2} - \frac{6}{(6n+4)^2} + \frac{6}{(6n+5)^2}\right).$$

In terms of a double series, Cloitre [36] gives

$$\zeta(2) = \sum_{n\geqslant 1}\sum_{j\geqslant 1}\frac{1}{jn\binom{n+j}{j}}.$$

Afanasyev and Solovyeva [1] also obtain some interesting series related to the $\zeta(2)$ function.

From Equation (9) with $t = 1, q = 2$ we have

$$\zeta(2) = \frac{4}{3} + \frac{3}{8} - \frac{16}{45} - \frac{5}{48} + \frac{52}{175} + \frac{35}{288} - \frac{1216}{6615} - \frac{21}{30} + \frac{1052}{6237}\cdots.$$

Apery's constant $\zeta(3)$, when the value of t in Equation (9) is 2, provides a series representation for each $q \in \mathbb{N}$. For further series representations for $\zeta(3)$, one can consult [8], [11], [18], and [19]. Recently, Amderberhan, Moll, and Straub [5] discovered the pleasing identity:

$$\zeta(3) = \frac{24}{5}\int_1^{\infty}\int_0^1\int_0^1 \frac{\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x}{x(x+y)(x+y+z)}$$

and the equivalent double sum:

$$\zeta(3) = \frac{24}{13}\sum_{n\geqslant 1}\sum_{j\geqslant n}\frac{(-1)^{n+1}}{n^2 j 2^j}.$$

Srivastava et al. [32] evaluated the integral:

$$\int_0^{\pi/w} x^{p-1}\cot x\ \mathrm{d}x \quad (p,\, w \in \mathbb{Z}_{\geqslant 2})$$

in two different ways, one involving the generalized Clausen function, to obtain

$$\zeta(3) = \frac{4\,\zeta(2)}{3}\left(\log 2 + 2\sum_{n\geqslant 0}\frac{\zeta(2n)}{(2n+3)\,4^n}\right).$$

Setting $t+1=2p$ ($p\in\mathbb{N}$) in Equation (9) when $q=1$ and employing Euler's evaluation:

$$\zeta(2t)=\frac{(-1)^{1+t}\,2^{2t-1}\pi^{2t}B_{2t}}{(2t)!}\quad(t\in\mathbb{N}),$$

B_t being the Bernoulli numbers (see, for instance, [31, Section 1.7]), gives

$$\sum_{n\geqslant 1}\frac{(-1)^{n+1}(2n+1)}{n(n+1)}\;A_n^{(2p-1)}=\frac{(-1)^p\,2^{2p-1}\,\pi^{2p}\,B_{2p}}{(2p)!}\quad(p\in\mathbb{N}).$$

Setting $q=1$ in Equation (9) provides another interesting result:

$$\begin{aligned}\zeta(t+1)&=\sum_{n\geqslant 1}\frac{(-1)^{n+1}(2n+1)}{n(n+1)}\;\sum_{j=1}^{n}\frac{(-1)^{j+1}}{j^t}\\&=\eta(t)\sum_{n\geqslant 1}\frac{(-1)^{n+1}(2n+1)}{n(n+1)}+\frac{1}{2^t}\sum_{n\geqslant 1}\frac{2n+1}{n(n+1)}\;\left\{\zeta\left(t,\frac{n+1}{2}\right)-\zeta\left(t,\frac{n+2}{2}\right)\right\},\end{aligned}$$

which, upon using the identity

$$\sum_{n\geqslant 1}\frac{(-1)^{n+1}(2n+1)}{n(n+1)}=1,$$

yields

$$\zeta(t+1)-\eta(t)=\frac{1}{2^t}\sum_{n\geqslant 1}\frac{2n+1}{n(n+1)}\;\left\{\zeta\left(t,\frac{n+1}{2}\right)-\zeta\left(t,\frac{n+2}{2}\right)\right\}.\tag{19}$$

Specific values of the parameter a in Equation (8) will produce a wide variety of sums of psi function identities. The next corollary highlights the case of $a=-1/2$.

Corollary 1. *Let q, $t\in\mathbb{N}$. Then*

$$\begin{aligned}&\frac{1}{2q}\sum_{n\geqslant 1}\frac{1}{n^t}\left\{\psi\left(\frac{n+1/2+q}{2q}\right)-\psi\left(\frac{n+1/2}{2q}\right)\right\}\\&\quad+\frac{1}{2q}\sum_{n\geqslant 1}\frac{1}{n^t}\left\{\psi\left(\frac{n-1/2+2q}{2q}\right)-\psi\left(\frac{n-1/2+q}{2q}\right)\right\}\\&=\sum_{n\geqslant 1}\frac{2n+q}{(n+1/2)(n-1/2+q)}\;\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{(-1)^{j+1}}{(n+q-qj)^t}.\end{aligned}\tag{20}$$

The case $q = 1$ of Equation (20) gives

$$\sum_{n\geqslant 1}\frac{1}{n^t}\left\{\psi\left(\frac{n}{2}-\frac{1}{4}\right)-\psi\left(\frac{n}{2}+\frac{1}{4}\right)\right\} = 4\sum_{n\geqslant 1}\frac{(-1)^{n+1}A_n^{(t)}}{2n+1} - 4\sum_{n\geqslant 1}\frac{1}{n^t\,(2n-1)}. \tag{21}$$

Here,

$$\sum_{n\geqslant 1}\frac{1}{n^t\,(2n-1)} = 2^t\,\log 2 - \sum_{j=1}^{t-1} 2^{t-1-j}\,\zeta(j+1) \quad (t\in\mathbb{N}), \tag{22}$$

and

$$\sum_{n\geqslant 1}\frac{(-1)^{n+1}A_n^{(t)}}{2n+1} = -\frac{\pi}{2}\eta\,(t) - \pi 2^{t-2}\beta\,(t)\,\mathbf{i} - \frac{2^{t-1}}{t!}\sum_{j=0}^{t} j!\binom{t}{j}\pi^{t-j}\,B_{t-j}\,\beta\,(j+1)\,\mathbf{i}^{t+j}, \tag{23}$$

where t is an odd positive integer, $\mathbf{i} = \sqrt{-1}$, and $\beta\,(t)$ is the Dirichlet beta function defined by

$$\beta\,(t) := \sum_{n\geqslant 1}\frac{(-1)^{n+1}}{(2n-1)^t}.$$

The case $t = 5$ of Equation (21), with the aid of Equation (22) and Equation (23), provides

$$\sum_{n\geqslant 1}\frac{1}{n^5}\left\{\psi\left(\frac{n}{2}-\frac{1}{4}\right)-\psi\left(\frac{n}{2}+\frac{1}{4}\right)\right\} = 16\,\beta\,(6) - 2G\,\zeta\,(4) - 8\,\zeta\,(2)\,\beta\,(4) - \frac{15}{64}\,\pi\zeta\,(5)\,,$$

where G is the Catalan constant defined by

$$G := \sum_{k\geqslant 1}\frac{(-1)^{k+1}}{(2k-1)^2} = \beta\,(2) \approx 0.915\,965\,594\cdots.$$

Proof. For Equation (21), we used Equation (3). Identity (22) can be given as a particular case of the general identity in Equation (3.3) in [4] with the help of special values of the psi function and Equation (4). For Equation (23), see [29]. □

The next theorem deals with series representation of the Dirichlet eta function. Evidently, a ready-made representation of $\eta\,(t+1)$ can be derived from Equations

(1) and (9):

$$\eta\left(t+1\right)=\left(1-2^{-t}\right)\sum_{n\geqslant 1}\frac{2n+q}{n\left(n+q\right)}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{\left(-1\right)^{j+1}}{\left(n+q-qj\right)^{t}}.$$

Theorem 2. *Let q, $t\in\mathbb{N}$ and $-1\leqslant a<1$. Then the following identity holds:*

$$\begin{aligned}&\frac{1}{2q}\sum_{n\geqslant 1}\frac{\left(-1\right)^{n+1}}{n^{t}}\left\{\psi\left(\frac{n+a+1+q}{2q}\right)-\psi\left(\frac{n+a+1}{2q}\right)\right\}\\&\quad+\frac{1}{2q}\sum_{n\geqslant 1}\frac{\left(-1\right)^{n+1}}{n^{t}}\left\{\psi\left(\frac{n-a-1+2q}{2q}\right)-\psi\left(\frac{n-a-1+q}{2q}\right)\right\}\\&=\sum_{n\geqslant 1}\frac{\left(-1\right)^{n+1}\left(2n+q\right)}{\left(n+a+1\right)\left(n-a-1+q\right)}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{\left(-1\right)^{(q+1)(j+1)}}{\left(n+q-qj\right)^{t}}.\end{aligned}\tag{24}$$

In particular, setting $a=-1$ in Equation (24) *offers a new series representation of the Dirichlet eta function: for $q\in\mathbb{N}$,*

$$\eta\left(t+1\right)=\sum_{n\geqslant 1}\frac{\left(-1\right)^{n+1}\left(2n+q\right)}{n\left(n+q\right)}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{\left(-1\right)^{(q+1)(j+1)}}{\left(n+q-qj\right)^{t}}.\tag{25}$$

Proof. The proof begins with the integrals:

$$\Omega\left(a,q,t\right):=\int_{0}^{1}\frac{x^{a}\ \mathrm{Li}_{t}\left(-x\right)}{1+x^{q}}\,\mathrm{d}x+\int_{0}^{1}\frac{x^{q-a-2}\ \mathrm{Li}_{t}\left(-x\right)}{1+x^{q}}\,\mathrm{d}x.\tag{26}$$

Then, a similar process of the proof of Theorem 1 verifies Equation (24). Setting $a=-1$ in Equation (24), with the aid of Equation (3), establishes Equation (25). □

The upcoming theorem addresses a new series representation of the Dirichlet lambda function, and it is evident that a readily available representation of $\lambda\left(t+1\right)$ can be obtained from Equations (2) and (9):

$$\lambda\left(t+1\right)=\left(1-2^{-1-t}\right)\sum_{n\geqslant 1}\frac{2n+q}{n\left(n+q\right)}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{\left(-1\right)^{j+1}}{\left(n+q-qj\right)^{t}}.$$

Theorem 3. *Let $q,\, t \in \mathbb{N}$ and $-1 \leqslant a < 1$. Then the following identity holds:*

$$\begin{aligned}
&\frac{1}{2q}\sum_{n\geqslant 1}\frac{1}{(2n-1)^t}\left\{\psi\left(\frac{2n+a+q}{2q}\right)-\psi\left(\frac{2n+a}{2q}\right)\right\}\\
&\quad+\frac{1}{2q}\sum_{n\geqslant 1}\frac{1}{(2n-1)^t}\left\{\psi\left(\frac{2n-a-2+2q}{2q}\right)-\psi\left(\frac{2n-a-2+q}{2q}\right)\right\}\\
&=\sum_{n\geqslant 1}\frac{2\,(2n+q)}{(n+a+1)\,(n-a-1+q)}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{(-1)^{j+1}}{(n+q-qj)^t}\\
&\quad+\sum_{n\geqslant 1}\frac{2\,(-1)^{n+1}\,(2n+q)}{(n+a+1)\,(n-a-1+q)}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{(-1)^{(q+1)(j+1)}}{(n+q-qj)^t}.
\end{aligned} \tag{27}$$

Specifically, setting $a=-1$ gives a new series representation of the Dirichlet lambda function: for $q \in \mathbb{N}$,

$$\begin{aligned}
\lambda\,(t+1)=&\sum_{n\geqslant 1}\frac{2\,(2n+q)}{n\,(n+q)}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{(-1)^{j+1}}{(n+q-qj)^t}\\
&+\sum_{n\geqslant 1}\frac{(-1)^{n+1}\,2\,(2n+q)}{n\,(n+q)}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{(-1)^{(q+1)(j+1)}}{(n+q-qj)^t}.
\end{aligned} \tag{28}$$

Proof. The proof begins with the integrals:

$$\Psi\,(a,q,t):=\int_0^1\frac{x^a\ \{\mathrm{Li}_t\,(x)-\mathrm{Li}_t\,(-x)\}}{1+x^q}\,\mathrm{d}x+\int_0^1\frac{x^{q-a-2}\ \{\mathrm{Li}_t\,(x)-\mathrm{Li}_t\,(-x)\}}{1+x^q}\,\mathrm{d}x. \tag{29}$$

A similar process in the proof of Theorem 1 offers

$$\begin{aligned}
\Psi\,(a,q,t)&=\frac{1}{q}\sum_{n\geqslant 1}\frac{1}{(2n-1)^t}\left\{\Phi\left(-1,1,\frac{2n+q}{q}\right)+\Phi\left(-1,1,\frac{2n+q-a-2}{q}\right)\right\}\\
&=\frac{1}{2q}\sum_{n\geqslant 1}\frac{1}{(2n-1)^t}\left\{\begin{aligned}&\psi\left(\frac{2n-a-2+2q}{2q}\right)-\psi\left(\frac{2n-a-2+q}{2q}\right)\\&+\psi\left(\frac{2n+a+q}{2q}\right)-\psi\left(\frac{2n+a}{2q}\right)\end{aligned}\right\}.
\end{aligned} \tag{30}$$

From Equations (8) and (24), we derive

$$\begin{aligned}\Psi(a,q,t)=&\sum_{n\geqslant 1}\frac{2(2n+q)}{(n+a+1)(n-a-1+q)}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{(-1)^{j+1}}{(n+q-qj)^{t}}\\&+\sum_{n\geqslant 1}\frac{2(-1)^{n+1}(2n+q)}{(n+a+1)(n-a-1+q)}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{(-1)^{(q+1)(j+1)}}{(n+q-qj)^{t}}.\end{aligned}\tag{31}$$

Finally, identifying Equations (30) and (31) justifies the desired Identity (27). Putting $a=-1$ in Equation (27) gives Equation (28). □

Remark 1. Replacing q by $2q$ in Equation (28) gives

$$\begin{aligned}\lambda(t+1)&=\sum_{n\geqslant 1}\frac{8(2n-1+q)}{(2n-1)(2n-1+2q)}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{(-1)^{j+1}}{(2n-1+2q-2qj)^{t}}\\&=\frac{2^{3-t}(-1)^{t}}{q^{t}}\sum_{n\geqslant 1}\frac{2(2n-1+q)}{(2n-1)(2n-1+2q)}\,A^{(t)}_{\left[\frac{n+q-1}{q}\right]}\left(\frac{1-2n-2q}{2q}\right).\end{aligned}$$

3. General Version

By introducing a logarithmic term into the integrands of Equations (10), (26), and (29), it becomes possible to extend the findings presented in Theorems 1, 2, and 3.

Theorem 4. *Let q, $t\in\mathbb{N}$ and p be a nonnegative even integer. Then the following identity holds:*

$$\begin{aligned}\zeta(p+t+1)&=\sum_{n\geqslant 1}\frac{n^{p+1}+(n+q)^{p+1}}{\{n(n+q)\}^{p+1}}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{(-1)^{j+1}}{(n+q-qj)^{t}}\\&=\frac{(-1)^{t}}{q^{t}}\sum_{n\geqslant 1}\frac{n^{p+1}+(n+q)^{p+1}}{\{n(n+q)\}^{p+1}}A^{(t)}_{\left[\frac{n+q-1}{q}\right]}\left(-\frac{n+q}{q}\right).\end{aligned}\tag{32}$$

Proof. Differentiating both sides of Equations (11) and (12) with respect to the parameter a, p-times, we have

$$\frac{\partial^{p}}{\partial a^{p}}\Lambda(a,q,t)=\sum_{n\geqslant 1}\frac{1}{n^{t}}\left(\int_{0}^{1}\frac{x^{n+a}\log^{p}(x)}{1+x^{q}}\,\mathrm{d}x+\int_{0}^{1}\frac{x^{n+q-a-2}\ \log^{p}(x)}{1+x^{q}}\,\mathrm{d}x\right)\tag{33}$$

and

$$\frac{\partial^p}{\partial a^p}\Lambda\left(a,q,t\right)=\frac{1}{\left(2q\right)^{p+1}}\sum_{n\geqslant 1}\frac{1}{n^t}\left\{\begin{array}{c}\psi^{(p)}\left(\frac{n+a+1+q}{2q}\right)-\psi^{(p)}\left(\frac{n+a+1}{2q}\right)\\ \\ +\psi^{(p)}\left(\frac{n-a-1+2q}{2q}\right)-\psi^{(p)}\left(\frac{n-a-1+q}{2q}\right)\end{array}\right\}. \quad (34)$$

As in the proof of Theorem 1, we get

$$\int_0^1 \frac{x^a\ \log^p\left(x\right)\ \mathrm{Li}_t\left(x\right)}{1+x^q}dx=\sum_{n\geqslant 1}\frac{p!}{\left(n+a+1\right)^{p+1}}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{\left(-1\right)^{j+1}}{\left(n+q-qj\right)^t} \quad (35)$$

and

$$\int_0^1 \frac{x^{q-a-2}\ \log^p\left(x\right)\ \mathrm{Li}_t\left(x\right)}{1+x^q}dx=\sum_{n\geqslant 1}\frac{p!}{\left(n+q-a-1\right)^{p+1}}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{\left(-1\right)^{j+1}}{\left(n+q-qj\right)^t}. \quad (36)$$

Putting Equations (35) and (36) in Equation (33) gives

$$\begin{aligned}\frac{\partial^p}{\partial a^p}\Lambda\left(a,q,t\right)&=p!\sum_{n\geqslant 1}\frac{\left(n+a+1\right)^{p+1}+\left(n+q-a-1\right)^{p+1}}{\left(\left(n+a+1\right)\left(n-a-1+q\right)\right)^{p+1}}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{\left(-1\right)^{j+1}}{\left(n+q-qj\right)^t}\\ &=\frac{\left(-1\right)^t p!}{q^t}\sum_{n\geqslant 1}\frac{\left(n+a+1\right)^{p+1}+\left(n+q-a-1\right)^{p+1}}{\left(\left(n+a+1\right)\left(n-a-1+q\right)\right)^{p+1}}A^{(t)}_{\left[\frac{n+q-1}{q}\right]}\left(-\frac{n+q}{q}\right).\end{aligned} \quad (37)$$

Then matching Equation (37) and Equation (34) provides

$$\begin{aligned}&\frac{1}{\left(2q\right)^{p+1}}\sum_{n\geqslant 1}\frac{1}{n^t}\left\{\begin{array}{c}\psi^{(p)}\left(\frac{n+a+1+q}{2q}\right)-\psi^{(p)}\psi\left(\frac{n+a+1}{2q}\right)\\ \\ +\psi^{(p)}\left(\frac{n-a-1+2q}{2q}\right)-\psi^{(p)}\left(\frac{n-a-1+q}{2q}\right)\end{array}\right\} \quad (38)\\ &=\frac{\left(-1\right)^t p!}{q^t}\sum_{n\geqslant 1}\frac{\left(n+a+1\right)^{p+1}+\left(n+q-a-1\right)^{p+1}}{\left\{\left(n+a+1\right)\left(n-a-1+q\right)\right\}^{p+1}}A^{(t)}_{\left[\frac{n+q-1}{q}\right]}\left(-\frac{n+q}{q}\right).\end{aligned}$$

Setting $a=-1$ in Equation (38) offers

$$\begin{aligned}&\frac{1}{\left(2q\right)^{p+1}}\sum_{n\geqslant 1}\frac{1}{n^t}\left\{\psi^{(p)}\left(\frac{n+2q}{2q}\right)-\psi^{(p)}\left(\frac{n}{2q}\right)\right\}\\ &\quad+\frac{1}{\left(2q\right)^{p+1}}\sum_{n\geqslant 1}\frac{1}{n^t}\left\{\psi^{(p)}\left(\frac{n}{2q}+1\right)-\psi^{(p)}\left(\frac{n+q}{2q}\right)\right\}\\ &=\frac{\left(-1\right)^t p!}{q^t}\sum_{n\geqslant 1}\frac{\left(n+a+1\right)^{p+1}+\left(n+q-a-1\right)^{p+1}}{\left(\left(n+a+1\right)\left(n-a-1+q\right)\right)^{p+1}}A^{(t)}_{\left[\frac{n+q-1}{q}\right]}\left(-\frac{n+q}{q}\right),\end{aligned}$$

which, upon using Recurrence (5), yields the desired Identity (32). $\square$

Remark 2. The case $p=0$ in Equation (32) leads to the result in Theorem 1.

Setting $q=1$ in Equation (32) gives

$$\begin{aligned}\zeta(p+t+1)-\eta(t)\\ =\frac{1}{2^t}\sum_{n\geqslant 1}\left(\frac{1}{n^{p+1}}+\frac{1}{(n+1)^{p+1}}\right)\left\{\zeta\left(t,\frac{n+1}{2}\right)-\zeta\left(t,\frac{n+2}{2}\right)\right\}.\end{aligned} \tag{39}$$

The case $p=0$ of Equation (39) gives Equation (19).

Furthermore, the theorem presented below encapsulates the generalized series representations of those stated in Theorems 2 and 3.

Theorem 5. *Let q, $t\in\mathbb{N}$ and p be a nonnegative even integer. Then the following identities hold:*

$$\eta(p+t+1)=\sum_{n\geqslant 1}\frac{(-1)^{n+1}\left(n^{p+1}+(n+q)^{p+1}\right)}{\{n(n+q)\}^{p+1}}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{(-1)^{(q+1)(j+1)}}{(n+q-qj)^t}, \tag{40}$$

and

$$\begin{aligned}\lambda(p+t+1)=&\sum_{n\geqslant 1}\frac{2\left(n^{p+1}+(n+q)^{p+1}\right)}{\{n(n+q)\}^{p+1}}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{(-1)^{j+1}}{(n+q-qj)^t}\\ &+\sum_{n\geqslant 1}\frac{2(-1)^{n+1}\left(n^{p+1}+(n+q)^{p+1}\right)}{\{n(n+q)\}^{p+1}}\sum_{j=1}^{\left[\frac{n+q-1}{q}\right]}\frac{(-1)^{(q+1)(j+1)}}{(n+q-qj)^t}.\end{aligned} \tag{41}$$

Setting $p=0$ in Equation (40) and Equation (41) leads to the identities in Theorems 2 *and* 3, *respectively.*

Proof. The proof is left as an exercise for the interested reader. □

4. Concluding Remarks

We have introduced novel families of series representations for the Riemann zeta function $\zeta(t)$ in Equation (9), along with its related functions: the Dirichlet eta (or alternating zeta) function $\eta(t)$ in Equation (25), and the Dirichlet lambda function $\lambda(t)$ in Equation (28). These series representations incorporate an independent parameter and have been derived from more generalized series involving psi functions. Furthermore, we have presented expanded and more inclusive series representations (32), (40), and (41) compared to the previously mentioned Equations (9), (25), and (28).

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