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ANOTHER TOPOLOGICAL PROOF OF THE INFINITUDE OF PRIME NUMBERS

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Abstract

We present a new topological proof of the infinitude of prime numbers with a new topology. Furthermore, in this topology, we characterize the infinitude of any non-empty subset of prime numbers.

– I dedicate this work to my beloved wife, Alenka Calderón.

1. Introduction and Preliminaries

In [2], we introduces a new topology τ on the set of positive integers $\mathbf{N}$ generated by the base $\beta := \{\sigma_n : n \in \mathbf{N}\}$, where $\sigma_n := \{m \in \mathbf{N} : \gcd(n, m) = 1\}$. Indeed, below we present a proof of this fact.

Theorem 1 ([2])**.** *The set β is a base for some topology on $\mathbf{N}$.*

Proof. It is clear that $\bigcup_{n\in\mathbf{N}} \sigma_n = \sigma_1 = \mathbf{N}$. On the other hand, note that for all positive integers x, n and m, we have $\gcd(x, n) = \gcd(x, m) = 1$ if and only if $\gcd(x, nm) = 1$, and therefore $\sigma_{nm} = \sigma_n \cap \sigma_m$. Hence β is a base for some topology on $\mathbf{N}$. $\square$

Remark 1. Note that

$$\sigma_n = \bigcup_{\substack{1\leq m\leq n \\ \gcd(m,n)=1}} n\,(\mathbf{N} \cup \{0\}) + m,$$

since for every integer x we have that $\gcd(n, m) = \gcd(n, nx + m)$; see [12, Theorem 1.9]. It is easily deduced from here that σ_n is infinite for every positive integer n. Furthermore, it is deduced that the topology τ is strictly coarser than Golomb's topology.

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The topological space $\mathbf{X} := (\mathbf{N}, \tau)$ does not satisfy the T_0 axiom, is hyperconnected, and is ultraconnected. Among other properties, one that will be useful is presented in the following lemma.

Lemma 1 ([2])**.** *If p is a prime number, then $\mathbf{cl_X}(\{p\}) = p\mathbf{N}$. Here, $\mathbf{cl_X}(\{p\})$ is the closure in $\mathbf{X}$ of the singleton set $\{p\}$.*

Proof. Let $x \in \mathbf{cl_X}(\{p\})$. Now, if $x \notin p\mathbf{N}$, then $\gcd(x, p) = 1$. Consequently, $x \in \sigma_p$, which implies that $p \in \sigma_p$, a contradiction. Therefore, $x \in p\mathbf{N}$. On the other hand, suppose $x \in p\mathbf{N}$. Then, $x = np$ for some positive integer n. Now, take $\sigma_k \in \beta$ such that $x \in \sigma_k$. This implies that $\gcd(np, k) = \gcd(x, k) = 1$, so $\gcd(p, k) = 1$, and thus, $p \in \sigma_k$. Hence, $x \in \mathbf{cl_X}(\{p\})$. □

Remark 2. If $n = 1$, then $\mathbf{cl_X}(\{n\}) = \mathbf{N}$. Moreover, using Lemma 1, it can be proved that for every positive integer $n > 1$ we have $\mathbf{cl_X}(\{n\}) = \bigcap_{p|n} p\mathbf{N}$.

The objective of this short note is to provide a new topological proof of the infinitude of prime numbers, distinct from the topological proofs presented by Fürstenberg [6] and Golomb [7], which, in fact, are similar except for the topology they use. Furthermore, we topologically characterize the infinitude of any set of prime numbers; see Theorem 4.

2. The Topological Proof

Let $\mathbf{P}$ denote the set of prime numbers. The following theorem characterizes the infinitude of prime numbers in $\mathbf{X}$.

Theorem 2. *There are infinitely many prime numbers if and only if $\mathbf{P}$ is dense in $\mathbf{X}$.*

Proof. Suppose there are infinitely many prime numbers. Then, for any positive integer $n > 1$, we can choose a prime p such that $p > n$, and consequently, $p \in \sigma_n$ since $\gcd(n, p) = 1$. Therefore, $\mathbf{P}$ is dense in $\mathbf{X}$. On the other hand, assume that $\mathbf{P}$ is dense in $\mathbf{X}$. Let $\{p_1, p_2, \ldots, p_k\}$ be a finite collection of prime numbers and consider the non-empty basic element σ_x where $x = p_1 \cdot p_2 \cdots p_k$. Note that none of the p_i belong to σ_x, but since $\mathbf{P}$ is dense in $\mathbf{X}$, there must be another prime number q, different from each p_i, such that $q \in \sigma_x$. Consequently, there are infinitely many prime numbers. □

Theorem 2 indicates that we only need to prove the density of $\mathbf{P}$ in $\mathbf{X}$ to establish the infinitude of prime numbers. Precisely, that is what we will prove.

To achieve our goal, consider the set $\mathbf{N_1} := \mathbf{N} \setminus \{1\}$ and the subspace topology

$$\tau_1 := \{\mathbf{N_1} \cap \mathcal{O} : \mathcal{O} \in \tau\} \text{ generated by the base } \beta_1 := \{\mathbf{N_1} \cap \sigma_n : \sigma_n \in \beta\}.$$

Also, consider the topological subspace $\mathbf{X_1} := (\mathbf{N_1}, \tau_1)$ and the following lemma.

Lemma 2. *If* $\mathbf{P}$ *is dense in* $\mathbf{X_1}$*, then it is dense in* $\mathbf{X}$*.*

Proof. It is clear that $\mathbf{X_1}$ is dense in $\mathbf{X}$. So, by the transitive property of density, if $\mathbf{P}$ is dense in $\mathbf{X_1}$, then $\mathbf{P}$ is dense in $\mathbf{X}$. □

Now, let us prove that $\mathbf{P}$ is dense in $\mathbf{X_1}$.

Theorem 3. *The set of prime numbers is dense in* $\mathbf{X_1}$*.*

Proof. In any topological space, we have that the union of closures of subsets of that space is contained in the closure of the union of those sets. Therefore, $\bigcup_{p\in\mathbf{P}} \mathbf{cl}_{\mathbf{X_1}}(\{p\}) \subset \mathbf{cl}_{\mathbf{X_1}}\left(\bigcup_{p\in\mathbf{P}} \{p\}\right) = \mathbf{cl}_{\mathbf{X_1}}(\mathbf{P}) \subset \mathbf{N_1}$. On the other hand, from Lemma 1, it follows that $\bigcup_{p\in\mathbf{P}} \mathbf{cl}_{\mathbf{X_1}}(\{p\}) = \bigcup_{p\in\mathbf{P}} (\mathbf{cl}_{\mathbf{X}}(\{p\}) \cap \mathbf{N_1}) = \bigcup_{p\in\mathbf{P}} (p\mathbf{N} \cap \mathbf{N_1}) = \bigcup_{p\in\mathbf{P}} p\mathbf{N}$. However, using the fundamental theorem of arithmetic, it can be easily proved that $\bigcup_{p\in\mathbf{P}} p\mathbf{N} = \mathbf{N_1}$, and thus, we can conclude that $\mathbf{cl}_{\mathbf{X_1}}(\mathbf{P}) = \mathbf{N_1}$, that is, $\mathbf{P}$ is dense in $\mathbf{X_1}$. □

From Theorem 2, Lemma 2, and Theorem 3, we can conclude that there are infinitely many prime numbers.

3. Concluding Remarks

There are many proofs of the infinitude of prime numbers, such as Goldbach's Proof [1, p. 3], Elsholtz's Proof [3], Erdos's Proof [4], Euler's Proof [5], and more recent ones; see [13], [8], [10], and [9]. Moreover, more than 200 proofs of the infinitude of primes can be found in [11]. However, Fürstenberg's and Golomb's proofs are the only known a priori topological proofs, which, in essence, as mentioned earlier, are based on the same idea, except for the topology used. Despite being able to present a topological proof using the same idea with the topology τ (left as an exercise to the reader), we present a completely different proof, not only because of the topology used but also due to the underlying idea – proving that $\mathbf{P}$ is dense in $\mathbf{X}$.

Finally, we want to leave the reader with the following theorem.

Theorem 4. *Let* $A \subset \mathbf{P}$ *non-empty. Then,* A *is dense in* $\mathbf{X}$*, if and only if,* A *is infinite.*

Proof. Replace $\mathbf{P}$ with A in the proof of Theorem 2. □

Theorem 4 implies a new relationship between number theory and topology, at least we hope so. Indeed, to answer questions such as: are there infinitely many Mersenne primes? or, are there infinitely Fibonacci primes? it suffices to check the density of these sets on $\mathbf{X}$. Certainly, it may not be easy, but it is possible. The advantage of working with $\mathbf{X}$ is that this space is hyperconnected, so any subset is either dense or nowhere dense.

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