#A48 INTEGERS 24 (2024)

UNDER WHAT CONDITIONS ON THE EXPONENTS DOES $x^m + y^n = z^r$ HAVE SOLUTIONS?

Richard F. Ryan[1]
Department of Natural and Health Sciences, Marymount California University, Rancho Palos Verdes, California
rryan_99@yahoo.com

Received: 1/24/24, Accepted: 4/26/24, Published: 5/20/24

Abstract

The equation $x^m + y^n = z^r$ is considered for integer values of m, n, and r that are greater than one; x, y, and z represent nonzero integers. Let $g_1 = \gcd(m, n)$, $g_2 = \gcd(m, r)$, and $g_3 = \gcd(n, r)$. If $\gcd(m, n, r) = 1$ and at least two of the integers g_1, g_2, and g_3 are greater than one, we show that $x^m + y^n = z^r$ has no solutions, unless (m, n, r) is of the form $(3s, 6t, 2w)$, $(6t, 3s, 2w)$, $(2w, 3s, 6t)$, or $(3s, 2w, 6t)$ for some pairwise relatively prime, positive integers s, t, and w such that s is odd and w is not divisible by three. This result allows us to completely answer the question in the title of this note. We do not restrict our study to primitive solutions.

1. Introduction

Throughout this note, we assume that the variables x, y, and z represent nonzero integers. Thus, when we state that a given equation involving these variables has no solutions, we are, in fact, saying that there is no solution with integer values for x, y, and z such that $xyz \neq 0$. We are mainly concerned with the generalized Fermat equation of the form

$$x^m + y^n = z^r \tag{1}$$

such that m, n, and r are given integers that are greater than one. As usual, $\gcd(x, y, z)$ represents the greatest common divisor of (the values of) x, y, and z. A solution to Equation (1) is said to be *primitive* if $\gcd(x, y, z) = 1$, and is called *non-primitive* otherwise. The following lemma is a summary of some results, for a special case of Equation (1), which were previously presented by the present author.

DOI: 10.5281/zenodo.11221659

[1] Professor Emeritus of Mathematics

Lemma 1 ([4]). *Consider the equation*

$$x^{3s} + y^{6t} = z^{2w} \tag{2}$$

such that s, t, and w are given positive integers. If s, t, and w are pairwise relatively prime such that s is odd and w is not divisible by three, then there are infinitely many solutions to Equation (2)*. Otherwise, Equation* (2) *has no solutions.*

For the case in which solutions to Equation (2) exist, formulas that generate all solutions have been provided [4]. We continue to assume that s, t, and w are given positive integers. Consider the equation

$$x^{2w} + y^{3s} = z^{6t}. \tag{3}$$

When s is odd, Equation (3) can be rewritten as $(-y)^{3s} + z^{6t} = x^{2w}$. It follows from Lemma 1 that Equation (3) has infinitely many solutions when s is odd, s, t, and w are pairwise relatively prime, and w is not divisible by three, but this equation has no solutions for other cases in which s is odd. When s is even, Equation (3) has no solutions, due to items 1 and 2(a) in the existence criteria (below).

The following theorem is a key result. Throughout this note, we let $g_1 = \gcd(m, n)$, $g_2 = \gcd(m, r)$, and $g_3 = \gcd(n, r)$. Remember that m, n, and r are assumed to be given integers that are greater than one.

Theorem 1. *Suppose that $\gcd(m, n, r) = 1$ and at least two of the integers g_1, g_2, and g_3 are greater than one. Then Equation* (1) *has no solutions, unless (m, n, r) is of the form $(3s, 6t, 2w)$, $(6t, 3s, 2w)$, $(2w, 3s, 6t)$, or $(3s, 2w, 6t)$ for some pairwise relatively prime, positive integers s, t, and w such that s is odd and w is not divisible by three.*

The proof of Theorem 1 will be given in the next section of this note. In light of this theorem, we are now able to complete the following classification.

Conditions under which solutions to Equation (1) do, or do not, exist:

1. When $\gcd(m, n, r) > 2$, due to the validation of Fermat's last theorem by Wiles [5], Equation (1) has no solutions.

2. When $\gcd(m, n, r) = 2$:

   (a) If $m/2$, $n/2$, and $r/2$ are not pairwise relatively prime, then Equation (1) has no solutions [3].

   (b) If $m/2$, $n/2$, and $r/2$ are pairwise relatively prime, then Equation (1) has infinitely many solutions [3].

3. When $\gcd(m, n, r) = 1$:

(a) If at least two of the integers g_1, g_2, and g_3 are equal to one, then Equation (1) has infinitely many solutions [2], [3].

(b) As a result of Lemma 1, if (m, n, r) is of the form $(3s, 6t, 2w)$, $(6t, 3s, 2w)$, $(2w, 3s, 6t)$, or $(3s, 2w, 6t)$ for some pairwise relatively prime, positive integers s, t, and w such that s is odd and w is not divisible by three, then there are infinitely many solutions to Equation (1). Otherwise, Equation (1) has no solutions when at least two of the integers g_1, g_2, and g_3 are greater than one, due to Theorem 1.

For case 2(b), formulas that generate infinitely many solutions to Equation (1) for each triple of allowable values (m, n, r) do exist [3]. However, we cannot presently prove that these formulas yield all solutions to this equation in case 2(b). For case 3(a), procedures that generate all solutions to Equation (1) have been found [2]. In case 3(b), it obviously follows that Equation (1) has no solutions when g_1, g_2, and g_3 are greater than one. For the instances of case 3(b) in which solutions exist, the formulas that were alluded to in the paragraph immediately after Lemma 1 can easily be applied, or adapted, to yield all solutions to Equation (1).

2. Proof of Theorem 1

Before we give the proof of Theorem 1, we remind the reader of results that we make use of in this proof. Bartolomé and Mihăilescu [1] established a theorem for the strong Fermat-Catalan equation, and proved a corollary for the equation

$$x^p + y^p = z^q. \tag{4}$$

We need the following portion of that corollary.

Corollary 1 ([1])**.** *Equation* (4) *has no solutions in mutually coprime integers* (x, y, z) *if the prime exponents* p *and* q *satisfy* $5 \le p \le q$.

The following lemma lists several results from a previous article.

Lemma 2 ([4])**.** *Consider the equation*

$$x^u + y^{uv} = z^v \tag{5}$$

such that u *and* v *are integers greater than one.*

(A) *For given values of* u *and* v*, Equation* (5) *has solutions if and only if it has at least one primitive solution.*

(B) *If* u *or* v *is even and* $(u, v) \notin \{(2, 2), (3, 2)\}$*, then Equation* (5) *has no solutions.*

(C) *If u or v is divisible by three and $(u, v) \neq (3, 2)$, then Equation* (5) *has no solutions.*

Remember that $g_1 = \gcd(m, n)$, $g_2 = \gcd(m, r)$, and $g_3 = \gcd(n, r)$.

Proof of Theorem 1. For instances in which (m, n, r) is of the form $(3s, 6t, 2w)$, $(6t, 3s, 2w)$, $(2w, 3s, 6t)$, or $(3s, 2w, 6t)$ for some positive integers s, t, and w, we know that Equation (1) has solutions when s, t, and w are pairwise relatively prime such that s is odd and w is not divisible by three; otherwise, Equation (1) has no solutions when (m, n, r) is in one of these four forms (due to Lemma 1 and the paragraph that immediately follows it). Throughout the remainder of this proof, we assume that (m, n, r) is not in one of these four forms. Suppose that $\gcd(m, n, r) = 1$, at least two of the integers g_1, g_2, and g_3 are greater than one, and that Equation (1) has at least one solution. Let $(x, y, z) = (a_0, b_0, c_0)$ be a solution to Equation (1). There are three cases to consider.

`Case 1.` For the case in which g_1 and g_3 are greater than one, there exist prime numbers p_1 and q_1 such that p_1 divides g_1 and q_1 divides g_3. Because $\gcd(m, n, r) = 1$, we know that $p_1 \neq q_1$. In this case, there exist positive integers m_1, n_1, and r_1 such that $m = m_1 p_1$, $n = n_1 p_1 q_1$, and $r = q_1 r_1$. Substituting into Equation (1), we see that

$$a_0{}^{m_1 p_1} + b_0{}^{n_1 p_1 q_1} = c_0{}^{q_1 r_1}. \tag{6}$$

Due to the assumption that (m, n, r) is not of the form $(3s, 6t, 2w)$, we know that $(p_1, q_1) \neq (3, 2)$. Note that $(x, y, z) = (a_0{}^{m_1}, b_0{}^{n_1}, c_0{}^{r_1})$ is a solution to

$$x^{p_1} + y^{p_1 q_1} = z^{q_1}. \tag{7}$$

Because $(p_1, q_1) \notin \{(2, 2), (3, 2)\}$, there are no solutions to Equation (7) when $p_1 = 2$ or $q_1 = 2$, due to Lemma 2(B). Likewise, there are no solutions to Equation (7) when $p_1 = 3$ or $q_1 = 3$, due to Lemma 2(C). Thus, $p_1 \geq 5$ and $q_1 \geq 5$. Due to Lemma 2(A), Equation (7) has a primitive solution. Let $(x, y, z) = (a, b, c)$ be a primitive solution to Equation (7). Then (a, b^{q_1}, c) is a primitive solution to

$$x^{p_1} + y^{p_1} = z^{q_1}.$$

As a result of Corollary 1, $q_1 < p_1$. Because (a, b, c) is a primitive solution to Equation (7), it follows that $(x, y, z) = (a, -b^{p_1}, c)$ is primitive solution to

$$z^{q_1} + y^{q_1} = x^{p_1}$$

with $5 \leq q_1 < p_1$, which contradicts Corollary 1. Therefore, there are no solutions to Equation (1) in this case.

`Case 2.` When g_1 and g_2 are greater than one, there exist prime numbers p_2 and q_2 such that p_2 divides g_1, q_2 divides g_2, and $p_2 \neq q_2$. Thus, there are positive integers

m_2, n_2, and r_2 such that $m = m_2p_2q_2$, $n = n_2p_2$, and $r = q_2r_2$. Substituting into Equation (1) and interchanging the first two terms, we see that

$$b_0{}^{n_2p_2} + a_0{}^{m_2p_2q_2} = c_0{}^{q_2r_2},$$

which has the same form as Equation (6). Proceeding in the same fashion as we did in case 1, we determine that there are no solutions to Equation (1) in Case 2.

Case 3. When g_2 and g_3 are greater than one, there exist prime numbers p_3 and q_3 such that p_3 divides g_2, q_3 divides g_3, and $p_3 \neq q_3$. In this case, there exist positive integers m_3, n_3, and r_3 such that $m = m_3p_3$, $n = n_3q_3$, and $r = p_3q_3r_3$. Substituting into Equation (1), we see that

$$a_0{}^{m_3p_3} + b_0{}^{n_3q_3} = c_0{}^{p_3q_3r_3}.$$

Because (m, n, r) is neither of the form $(2w, 3s, 6t)$ nor $(3s, 2w, 6t)$, we know that $(p_3, q_3) \notin \{(2, 2), (2, 3), (3, 2)\}$. Note that $(x, y, z) = (a_0{}^{m_3}, b_0{}^{n_3}, c_0{}^{r_3})$ is a solution to

$$x^{p_3} + y^{q_3} = z^{p_3q_3}. \tag{8}$$

If $p_3 = 2$, then q_3 is odd and Equation (8) can be rewritten as

$$(-y)^{q_3} + z^{2q_3} = x^2.$$

It follows from Lemma 2(B) that there are no solutions to Equation (8) when $p_3 = 2$. Similarly, there are no solutions to Equation (8) when $q_3 = 2$. Thus, p_3 and q_3 are odd, and Equation (8) can expressed as

$$(-x)^{p_3} + z^{p_3q_3} = y^{q_3}. \tag{9}$$

Note that Equation (9) has the same form as Equation (7). Utilizing similar steps to those we used in case 1, starting with the second sentence after Equation (7), it is easy to show that there are no solutions to Equation (1) in this case.

Therefore, Equation (1) has no solutions under the conditions stated in this theorem, unless (m, n, r) can be written in the form $(3s, 6t, 2w)$, $(6t, 3s, 2w)$, $(2w, 3s, 6t)$, or $(3s, 2w, 6t)$ for some pairwise relatively prime, positive integers s, t, and w such that s is odd and w is not divisible by three. $\square$

3. Additional Comments

One of the fundamental questions in mathematics is, "Under what conditions do solutions exist?" When the present author began the journey that resulted in an answer to this question for Equation (1), he was surprised to learn that the quest for primitive solutions took precedence over the basic existence question for this equation. At this point, we briefly discuss the places where primitive solutions are located in the "Conditions under which solutions to Equation (1) do, or do not, exist" from the first section of this note. It was shown, in a previous article [4], that the only primitive solutions to Equation (2) are $(x, y, z) = (2, \pm 1, \pm 3)$. These solutions occur when $(s, t, w) = (1, k, 1)$ for each positive integer k. Due to this result and Theorem 1, it follows that the only primitive solutions to Equation (1) in case 3(b) are $(m, n, r, x, y, z) = (3, 6k, 2, 2, \pm 1, \pm 3)$, $(6k, 3, 2, \pm 1, 2, \pm 3)$, $(2, 3, 6k, \pm 3, -2, \pm 1)$, or $(3, 2, 6k, -2, \pm 3, \pm 1)$, with k being any positive integer. Otherwise, when a primitive solution exists, it can only occur in case 2(b) or 3(a). Like many number theorists, the author is fascinated by the search for primitive solutions to Equation (1). However, the importance of resolving the fundamental existence question for this equation should not be overlooked.

References

[1] B. Bartolomé and P. Mihăilescu, Semilocal approximation for the Fermat-Catalan and further popular Diophantine norm equations, preprint, `arXiv: 2108.08572`.

[2] R. Ryan, A generalized Fermat equation with an emphasis on non-primitive solutions, *Int. Math. Forum* **12** (2017), 835-840.

[3] R. Ryan, Solutions to a generalized Fermat equation that has even exponents, *Int. Math. Forum* **14** (2019), 237-246.

[4] R. Ryan, Special forms of the generalized Fermat equation, *Int. Math. Forum* **17** (2022), 201-213.

[5] A. Wiles, Modular elliptic curves and Fermat's last theorem, *Ann. of Math. (2)* **141** (1995), 443-551.