ON A MODEL OF $([k]; p; s)$-GENERALIZED PADOVAN–PERRIN AND FIBONACCI FUNDAMENTAL SYSTEM

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Abstract

In this study, we investigate a model of generalized $([k]; p; s)$-Padovan–Perrin sequences and deal with a special case, namely, the k-Padovan–Perrin sequence. To reach our goal, we consider some properties of the Fibonacci fundamental system, related to the elements of this model. Several results of this model are established. More specifically, we give some identities, as well as combinatorial and analytical representations, related to the sequences of this model.

1. Introduction

Over the past decades, there have been several research papers on notable families of number sequences defined by linear recursive relations, as well as their generalizations. In particular, the numbers of Fibonacci, Pell, Jacobsthal, Padovan, Perrin, and Van der Laan are studied in the literature. These sequences of numbers have been widely applied in various areas of mathematics and applied science, such as physics and engineering, and also in architecture, nature, and art (see for example [1, 11, 15] and references therein). The three classical families of integers of numbers of Padovan, Perrin, and Van der Laan are defined by the following classical linear

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recursive relation of order 3 :

$$v_n = v_{n-2} + v_{n-3}, \text{ for every } n \geq 3,$$

where the data of the initial conditions v_0, v_1, and v_2, contributes to characterizing the type of each sequence of these numbers. That is,

- for $v_0 = v_1 = v_2 = 1$, we get the Padovan numbers $v_n = P_n^{(1)}$;
- for $v_0 = 3$, $v_1 = 0$ and $v_2 = 2$, we find the Perrin numbers $v_n = P_n^{(2)}$;
- for $v_0 = 1$, $v_1 = 0$ and $v_2 = 1$, we obtain the Van der Laan numbers $v_n = V_n$.

Recently, abundant literature has been provided on generalizations of these three types of sequences (see, for instance, [12, 13] and references therein). In addition, several papers have established matrix representations of these sequences of numbers. In particular, the matrix representations of the sequences of Padovan and Perrin numbers have been considered in many research papers (see, for instance, [12]).

In the present study, we discuss the properties of a generalized model of Padovan and Perrin numbers, defined by a general linear recursive relation. More precisely, let k_1, k_2, $p \geq 2$, and s be positive integers, and set $[k] = (k_1, k_2)$. Consider the sequence $\{v_n([k]; p; s)\}_{n\geq 0}$ defined by the following linear recursive relation of order $s + p$:

$$v_n([k]; p; s) = k_1 v_{n-s}([k]; p; s) + k_2 v_{n-s-p}([k]; p; s), \text{ for } n \geq s + p, \tag{1}$$

with initial conditions $v_j([k]; p; s) = \alpha_j \in \mathbb{N}$, for $0 \leq j \leq s + p - 1$. As we will see, Equation (1) represents the general model of the Padovan–Perrin recursive relation called the $([k]; p; s)$-Padovan–Perrin model. Our main tools are based on the Fibonacci fundamental system related to the elements of the model defined by Equation (1) and the matrix formulation of this linear recursive relation of order $s + p$. Therefore, several new properties and identities of this general model are established, and many results from the literature are recovered. More precisely, we obtain the combinatorial expressions and the analytic formulas of sequences defined by (1). Finally, several special cases of the literature and applications are discussed.

The outline of this paper is as follows. Section 2 concerns the generalized $([k]; p; s)$-Padovan–Perrin model and its relation with the Fibonacci fundamental system. Sections 3 and 4 are devoted to the matrix formulation and combinatorial study of sequences defined by (1). In Section 5, we establish the analytical formulation for the generalized $([k]; p; s)$-Padovan–Perrin model in terms of the parameters $([k]; p; s)$. In Section 6, we apply our results to the special case of the k-Padovan–Perrin sequence.

2. The Generalized $([k]; p; s)$-Padovan–Perrin Model and Its Relation with the Fibonacci Fundamental System

The general fundamental Fibonacci system related to sequences defined by a linear recursive relation of the Fibonacci type represents a powerful tool for studying these types of sequences. It has been considered in its general form for generalized Fibonacci sequences by Rachidi et al. in [2, 8]. Subsequently, it was used in [16] to study generalized Fibonacci numbers, in [17] to provide results on the generalized Pell numbers, and in [5] to establish the properties of a generalized Pell model. Moreover, in [6] the generalized Cassini identities were provided using, as an approach, properties of the Fibonacci fundamental system. This section is devoted to the general fundamental Fibonacci system related to the $([k]; p; s)$-Padovan–Perrin model defined in (1).

2.1. Preliminary Considerations on the $([k]; p; s)$-Padovan–Perrin Model

As mentioned above, Equation (1) represents a general Padovan–Perrin model. Indeed, we have the following special cases.

- For $s = 1$, Equation (1) is reduced to a recursive relation of order p, defining a weighted p-generalized Fibonacci sequence.
- For $s = 2$, $p = k_1 = k_2 = 1$, Equation (1) represents the usual recursive relation defining the Padovan–Perrin numbers (for more details, see [9, 19, 15]).
- For $s = 2$, $p = k_1 = 1$ and $k = k_2 \geq 2$, Equation (1) represents the recursive relation defining the k-Padovan–Perrin numbers.
- For $s = 2$, $p \geq 2$ and $k_2 = k_1 = 1$, the recursive relation given by Equation (1) defines the p-generalized Padovan–Perrin numbers.
- For $s \geq 2$, $p \geq 2$ and $k_1 = k_2 = 1$, the recursive relation given by Equation (1) is nothing else but the (s, p)-generalized Padovan–Perrin numbers.
- For $s \geq 2$, $p \geq 2$, and $k_1 \geq 2$ or $k_2 \geq 2$, the recursive relation given by Equation (1) defines the $([k]; s; p)$-generalized Padovan–Perrin numbers, or just, $([k]; s; p)$-Padovan–Perrin numbers.

The previous special cases show that Equation (1) defines a generalized Padovan–Perrin model.

2.2. The Fibonacci Fundamental System Related to (1)

For every j such that $1 \leq j \leq p+s$, we consider the sequence of the $([k];s;p)$-Padovan–Perrin numbers $\{v_n^{(j)}([k];p;s)\}_{n\geq 0}$ defined by

$$\begin{cases} v_n^{(j)}([k];p;s) = k_1 v_{n-s}^{(j)}([k];p;s) + k_2 v_{n-s-p}^{(j)}([k];p;s), \text{ for } n \geq s+p \\ v_n^{(j)}([k];p;s) = \delta_{n+1}^{(j)}; \text{ for } n = 0,\, 1,\, \ldots,\, s+p-1, \end{cases} \tag{2}$$

where $\delta_i^{(j)} = 1$ if $i = j$,and $\delta_i^{(j)} = 0$ otherwise.

Example 1. For $s = 2$, $p = k_1 = k_2 = 1$, Equation (2) is given by

$$\begin{cases} v_n^{(j)}([k];1;2) = v_{n-2}^{(j)}([k];1;2) + v_{n-3}^{(j)}([k];1;2), \text{ for } n \geq s+p \\ v_n^{(j)}([k];1;2) = \delta_{n+1}^{(j)}; \text{ for } n = 0,\, 1,\, 2. \end{cases}$$

Table 1 describes the values of these sequences for $0 \leq n \leq 11$.

n	0	1	2	3	4	5	6	7	8	9	10	11
$v_n^{(1)}([k];1;2)$	1	0	0	1	0	1	1	1	2	2	3	4
$v_n^{(2)}([k];1;2)$	0	1	0	1	1	1	2	2	3	4	5	7
$v_n^{(3)}([k];1;2)$	0	0	1	0	1	1	1	2	2	3	4	5

Table 1: Sequences $\{v_n^{(j)}([k];1;2)\}_{n\geq 0}$ with parameter $[k] = (1,1)$ and $1 \leq j \leq 3$.

Consider the set $\mathcal{S}_{([k];p;s)} = \left\{\{v_n^{(j)}([k];p;s)\}_{n\geq 0};\ 1 \leq j \leq p+s\right\}$. Let us consider the sequence $\{v_n([k];p;s)\}_{n\geq 0}$ defined by (1) with initial conditions $\alpha_0, \ldots, \alpha_{p+s-1}$. Let $\{w_n([k];p;s)\}_{n\geq 0}$ be the sequence of general terms

$$w_n([k];p;s) = \sum_{j=0}^{p+s-1} \alpha_j v_n^{(j+1)}([k];p;s).$$

Then, using (2), we can derive that $w_j([k];p;s) = v_j([k];p;s) = \alpha_j$, for $j = 0, 1, \cdots, p+s-1$. We can show that the set $\mathcal{S}_{([k];p;s)}$ generates the sequences of the Padovan–Perrin model defined in (1), namely, $w_n([k];p;s) = v_n([k];p;s)$, for every $n \geq 0$. More precisely, we have the following result.

Proposition 1. *Let $\{v_n([k];p;s)\}_{n\geq 0}$ be the sequence defined by the recursive relation* (1) *and with initial conditions $\alpha_0, \alpha_1, \ldots, \alpha_{p+s-1}$. Then, for every $n \geq 0$, we have*

$$v_n([k];p;s) = \sum_{j=0}^{p+s-1} \alpha_j v_n^{(j+1)}([k];p;s).$$

Proof. Let $\{w_n([k];p;s)\}_{n\geq 0}$ be the sequence defined by

$$w_n([k];p;s) = \sum_{j=0}^{p+s-1} \alpha_j v_n^{(j+1)}([k];p;s), \text{ for every } n \geq 0,$$

with initial conditions $(\alpha_0, \alpha_1, \ldots, \alpha_{p+s-1})$. For every $n \geq p+s$, we have

$$\begin{aligned} w_n([k];p;s) &= \sum_{j=0}^{p+s-1} \alpha_j v_n^{(j+1)}([k];p;s) \\ &= \sum_{j=0}^{p+s-1} \alpha_j [k_1 v_{n-s}^{(j)}([k];p;s) + k_2 v_{n-s-p}^{(j)}([k];p;s)] \\ &= k_1 w_{n-s}([k];p;s) + k_2 w_{n-p-s}([k];p;s). \end{aligned}$$

Therefore, the sequence $\{w_n([k];p;s)\}_{n\geq 0}$ satisfies the recursive relation (1). In addition, for every d such that $0 \leq d \leq p+s-1$, we get

$$w_d([k];p;s) = \sum_{j=0}^{p+s-1} \alpha_j v_d^{(j+1)}([k];p;s) = \alpha_d,$$

because $v_d^{(j+1)}([k];p;s) = 1$ if $j = d$, and $v_d^{(j+1)}([k];p;s) = 0$ if $j \neq d$. Thus, the two sequences $\{v_n([k];p;s)\}_{n\geq 0}$ and $\{w_n([k];p;s)\}_{n\geq 0}$ satisfy the recursive relation (1) and own the same initial conditions $\alpha_0, \alpha_1, \ldots, \alpha_{p+s-1}$. Therefore, for every $n \geq 0$, we have $v_n([k];p;s) = w_n([k];p;s) = \sum_{j=0}^{p+s-1} \alpha_j v_n^{(j+1)}([k];p;s)$. □

The proof of Proposition 1 is based on the following lemma.

Lemma 1. *Consider two sequences $\{v_n\}_{n\geq 0}$ and $\{w_n\}_{n\geq 0}$ of the real vector space $\mathcal{E}([k];p;s)$ of sequences satisfying the recursive relation (1), whose initial conditions are $\alpha_0, \alpha_1, ..., \alpha_{r-1}$, and $\beta_0, \beta_1, ..., \beta_{r-1}$, respectively. Suppose that there exist n_0, m_0, N in $\mathbb{N}$ such that $v_{j+n_0} = w_{j+m_0}$, for $N \leq j \leq N+r-1$. Then, we have $v_{n+n_0} = w_{n+m_0}$, for every $n \geq N$.*

The proof of Lemma 1 is similar to the proof of Lemma 2.6 of [17].

Let $\mathcal{E}([k];p;s)$ be the set of sequences $\{v_n([k];p;s)\}_{n\geq 0}$ satisfying (1). Consider the sequences $\{v_n([k];p;s)\}_{n\geq 0}$ and $\{w_n([k];p;s)\}_{n\geq 0}$ satisfying (1), with initial conditions $\alpha_0, \alpha_1, \ldots, \alpha_{p+s-1}$, and $\beta_0, \beta_1, \ldots, \beta_{p+s-1}$, respectively. Applying Proposition 1, we define the addition and the multiplication by a scalar as follows:

$$\begin{aligned} v_n([k];p;s) + w_n([k];p;s) &= \sum_{j=0}^{p+s-1} (\alpha_j + \beta_j) v_n^{(j+1)}([k];p;s) \\ \gamma \cdot v_n([k];p;s) &= \sum_{j=0}^{p+s-1} \gamma \cdot \alpha_j v_n^{(j+1)}([k];p;s), \end{aligned} \tag{3}$$

for every $\gamma \in \mathbb{R}$. As a consequence of Proposition 1 we have the following corollary.

Corollary 1. *The set $\mathcal{E}([k];p;s)$ of sequences satisfying the recursive relation* (1)*, equipped with the addition and the multiplication by scalars* (3)*, is a real vector space. Moreover, the set $\mathcal{S}_{([k];p;s)} = \left\{ \{v_n^{(j)}([k];p;s)\}_{n\geq 0};\ 1 \leq j \leq p+s \right\}$ is a generator system of $\mathcal{E}([k];p;s)$.*

Therefore, we have the following definition.

Definition 1. The set $\mathcal{S}_{([k];p;s)} = \left\{ \{v_n^{(j)}([k];p;s)\}_{n\geq 0};\ 1 \leq j \leq p+s \right\}$ is called the *Fibonacci fundamental system* of the generalized $([k];p;s)$-Padovan–Perrin model defined in (1).

2.3. Fibonacci Fundamental Solution Related to Equation (1)

For the Fibonacci fundamental system (2) related to the generalized $([k];p;s)$-Padovan–Perrin model (1), namely, $\mathcal{S}_{([k];p;s)} = \left\{ \{v_n^{(j)}([k];p;s)\}_{n\geq 0};\ 1 \leq j \leq p+s \right\}$, the sequences $\{v_n^{(r)}([k];p;s)\}_{n\geq 0}$ play a central role, in the sense that any other element of this system can be expressed in terms of this solution. More precisely, we have the following result.

Proposition 2. *Let $\mathcal{S}_{([k];p;s)} = \left\{ \{v_n^{(j)}([k];p;s)\}_{n\geq 0};\ 1 \leq j \leq p+s \right\}$ be the Fibonacci fundamental system of the generalized $([K];p;s)$-Padovan–Perrin model* (1)*. Then, for every j such that $p+1 \leq j \leq p+s-1$, we have*

$$v_n^{(j)}([k];p;s) = k_1 v_{n-j+p}^{(s+p)}([k];p;s) + k_2 v_{n-j}^{(s+p)}([k];p;s) = v_{n+(s+p)-j}^{(s+p)}([k];p;s)$$

for $n \geq 0$. In particular, we have $v_n^{(j)}([k];p;s) = k_2 v_{n+j}^{(s+p)}([k];p;s)$, for every j such that $1 \leq j \leq p$.

The proof of Proposition 2 is based on Lemma 1. Indeed, using Lemma 1 and induction on j, we can prove Proposition 2 in a similar way to the proof of Theorem 2.1 of [17]. In addition, following Proposition 1, every $\{v_n([k];p;s)\}_{n\geq 0}$ defined by the recursive relation (1) and with initial conditions $(\alpha_0, \alpha_1, \ldots, \alpha_{p+s-1})$, can also be expressed in terms of the solution $\{v_n^{(s+p)}([k];p;s)\}_{n\geq 0}$. Indeed, we have

$$\begin{aligned} v_n([k];p;s) = & k_2 \sum_{j=0}^{p} \alpha_j v_{n+j}^{(s+p)}([k];p;s) \\ & + \sum_{j=p+1}^{s+p-1} \alpha_j v_{n+s+p-j}^{(s+p)}([k];p;s) + \alpha_{s+p-1} v_n^{(s+p)}([k];p;s). \end{aligned}$$

Taking into account the previous properties, we introduce the following definition.

Definition 2. The sequence $\{v_n^{(r)}([k];p;s)\}_{n\geq 0}$, given by Equation (2) (with $j = r$), is called the *the Fibonacci fundamental solution* of the generalized $([K];p;s)$-Padovan–Perrin model.

3. Matrix Formulation and Its Related Fibonacci Fundamental System

The matrix formulation related to sequences defined by a linear recursive relation of the Fibonacci type has been considered in several papers. In addition, the relation between matrices and the Fibonacci fundamental system provides interesting results (see [5, 16, 17]). This section is devoted to exploring the matrix formulation of Equation (1) and its close connection with the Fibonacci fundamental system.

3.1. Matrix Formulation of Equation (1)

The general setting of the matrix formulation of the recursive relation (1) is given by

$$\mathbb{V}_n([k];p;s) = \mathbb{A}(([k];p;s))\mathbb{V}_{n-1}([k];p;s), \text{ for every } n \geq s+p, \tag{4}$$

where

$$\mathbb{A}([k];p;s) = \begin{bmatrix} 0 & \dots & 0 & k_1 & 0 & \dots & 0 & k_2 \\ 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & \dots & 0 & 0 \\ \vdots & \dots & \ddots & 0 & \vdots & \vdots & \vdots & \vdots \\ \\ 0 & \dots & \dots & 0 & \dots & 1 & 0 & 0 \\ 0 & \dots & \dots & 0 & \dots & 0 & 1 & 0 \end{bmatrix}, \tag{5}$$

$$\mathbb{V}_{n-1}([k];p;s) = \begin{bmatrix} v_{n-1}([k];p;s) \\ \vdots \\ v_{n-s}([k];p;s) \\ \vdots \\ v_{n-s-p}([k];p;s) \end{bmatrix}.$$

Note that the matrix $\mathbb{A}([k];p;s)$ is of order $s+p$, the integer k_1 is located at the s-th column, and the integer k_2 is located at the $(s+p)$-th column.

As an example, consider $s = p = 2$. Then, Equation (1) takes the form $v_n([k];2;2) = k_1 v_{n-2}([k];2;2) + k_2 v_{n-4}([k];2;2)$ for $n \geq 4$, and its matrix formulation is given by $\mathbb{V}_n([k];2;2) = \mathbb{A}([k];2;2)\mathbb{V}_{n-1}([k];2;2)$, where

$$\mathbb{A}([k];2;2) = \begin{bmatrix} 0 & k_1 & 0 & k_2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } \mathbb{V}_{n-1}([k];2;2) = \begin{bmatrix} v_{n-1}([k];2;2) \\ v_{n-2}([k];2;2) \\ v_{n-3}([k];2;2) \\ v_{n-4}([k];2;2) \end{bmatrix}.$$

Note that the matrix $\mathbb{A}([k];2;2)$ is of order $p+s=4$.

3.2. Powers of the Companion Matrix (5) by the Fibonacci Fundamental System

A direct computation allows us to show that Equation (4) is equivalent to the following matrix formulation:

$$\mathbb{V}_n([k];p;s)=\mathbb{A}([k];p;s)^{n-p-s}\mathbb{V}_{p+s-1}([k];p;s),$$

where $\mathbb{V}_{p+s-1}([k];p;s)$ is as in (5), namely,

$$\mathbb{V}_{p+s-1}([k];p;s)=\begin{bmatrix} v_{p+s-1}([k];p;s) \\ \vdots \\ v_p([k];p;s) \\ \vdots \\ v_0([k];p;s) \end{bmatrix}.$$

In this subsection we will give an explicit form of the entries of the matrix powers $\mathbb{A}^n([k];p;s)$, in terms of the elements of the Fibonacci fundamental system (2). This will allow us to obtain properties of the $([k];p;s)$-Padovan–Perrin sequences. To reach our goal, we consider the well-known Casoratian matrix, related to the Fibonacci fundamental system (2).

The *Casoratian matrix* $\widehat{C}(n)$, associated to the family of sequences given by $\{v_n([k];p;s)^{(1)}\}_{\ n\in\mathbb{N}}, \ldots, \{v_n([k];p;s)^{(s+p)}\}_{n\in\mathbb{N}}$, is defined by $\widehat{C}(n)=(\widehat{V}^{(1)}(n), \ldots, \widehat{V}^{(s+p)}(n))$, where $\widehat{V}^{(j)}(n)$, with $1\leq j\leq s+p$, is the vector column $\widehat{V}^{(j)}(n)= (v_n^{(j)}([k];p;s), \cdots, v_{n+s+p-1}^{(j)}([k];p;s))^t$, namely,

$$\widehat{C}(n)=\begin{pmatrix} v_n^{(1)}([k];p;s) & \cdots & v_n^{(j)}([k];p;s) & \cdots & v_n^{(s+p)}([k];p;s) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ v_{n+s+p-1}^{(1)}([k];p;s) & \cdots & v_{n+s+p-1}^{(j)}([k];p;s) & \cdots & v_{n+s+p-1}^{(s+p)}([k];p;s) \end{pmatrix}$$

(see, for instance, [4]).

A direct verification shows that the Casoratian matrix $\widehat{C}(n)$ can be written in the form $\widehat{C}(n)=J\times\mathbb{M}_n([k];p;s)\times J$, where $\mathbb{M}_n([k];p;s)$ is in the form:

$$\begin{pmatrix} v_{n+s+p-1}^{(s+p)}([k];p;s) & \cdots & v_{n+s+p-1}^{(j)}([k];p;s) & \cdots & v_{n+s+p-1}^{(1)}([k];p;s) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ v_n^{(s+p)}([k];p;s) & \cdots & v_n^{(j)}([k];p;s) & \cdots & v_n^{(1)}([k];p;s) \end{pmatrix}, \quad (6)$$

where $J=(b_{i,j})_{1\leq i,j\leq p+s}$ is the anti-diagonal unit matrix, whose entries are given by $b_{i,j}=1$ for $i+j=p+s+1$ and $b_{i,j}=0$ otherwise.

Consider the sequence $\{v_n([k];p;s)\}_{n\geq 0}$ given in Equation (1) and the vector column $\mathbb{Y}_n = (v_{n+p+s-1}([k];p;s),\ \ldots,\ v_n([k];p;s))^t$. The sequences defined by (1) take the equivalent matrix form:

$$\mathbb{Y}_{n+1} = \mathbb{A}([k];p;s)\mathbb{Y}_n, \quad n \geq s+p-1, \tag{7}$$

where $\mathbb{A}([k];p;s)$ is the companion matrix (5). That is, Equation (7) can be written in the form:

$$\mathbb{Y}_{n+s+p-1} = \mathbb{A}^n([k];p;s)\mathbb{Y}_{s+p-1}, \quad n \geq s+p-1,$$

where $\mathbb{Y}_{s+p-1}$ is the vector column $\mathbb{Y}_{s+p-1} = (v_{p+s-1}([k];p;s),\ \ldots,\ v_0([k];p;s))^t$. By considering the generalized fundamental system, a direct computation shows that the vector $\mathbb{Y}_{n+s+p-1}$ can be written in the matrix form:

$$\mathbb{Y}_{n+s+p-1} = \mathbb{M}_n([k];p;s)\mathbb{Y}_{s+p-1}, \tag{8}$$

where the entries $m_{ij}^{(n)}$ of the matrix $\mathbb{M}_n([k];p;s) = \mathbb{A}^n([k];p;s)$ are stated, in terms of the elements of the generalized Fibonacci fundamental system, in the form $m_{ij}^{(n)} = v_{n+s+p-i}^{(s+p-j+1)}([k];p;s)$ (for more details, see [2, 8]). Equations (7) and (8) permit us to establish the following result.

Theorem 1. *Let* $\mathcal{S}_{([k];p;s)} = \left\{\{v_n^{(j)}([k];p;s)\}_{n\geq 0};\ 1 \leq j \leq p+s\right\}$ *be the Fibonacci fundamental system of the generalized* $([k];p;s)$*-Padovan–Perrin model defined in* (1)*. Then, for every* $n \geq 0$*, the entries of the powers* $\mathbb{A}^n([k];p;s) = (a_{ij}^{(n)})_{1\leq i,j\leq p+s}$ *are given in the form:*

$$a_{ij}^{(n)} = v_{n+s+p-i}^{(s+p-j+1)}([k];p;s). \tag{9}$$

In other terms, we have $\mathbb{A}^n([k];p;s) = \mathbb{M}_n([k];p;s)$*.*

It is worth noting that Equation (9) has been established by induction (see [2, 8], and references therein). Since the Casoratian matrix of the Fibonacci fundamental system of the generalized $([k];p;s)$-Padovan–Perrin model defined in (1), namely, $\mathcal{S}_{([k];p;s)} = \left\{\{v_n^{(j)}([k];p;s)\}_{n\geq 0};\ 1 \leq j \leq p+s\right\}$, is defined by $\widehat{C}(n)$ and Expression (6), the result of Theorem 1 allows us to connect the Casoratian matrix $\widehat{C}(n)$ of the Fibonacci fundamental system (1) with the powers $\mathbb{A}^n([k];p;s)$ of the matrix (5).

3.3. Some Identities of the Generalized ([k]; p; s)-Padovan–Perrin Model

In this subsection, we will emphasize some identities of the generalized $([k];p;s)$-Padovan–Perrin model. For this purpose we will start by talking about the entries of the companion matrix, then we will generalize the results. As a consequence of the previous propositions, we can exhibit some identities satisfied by the elements of the generalized Fibonacci fundamental system.

Equation (6) allows us to write $\widehat{C}(n+m) = J\mathbb{A}^{n+m}([k];p;s)J = J\mathbb{A}^{n}([k];p;s)$ $\mathbb{A}^{m}([k];p;s)J$. Since J is an anti-diagonal matrix, then it follows that $J \cdot J = diag(1,\ldots,1)$. Hence, we get $\widehat{C}(n+m) = (J\mathbb{A}^{n}([k];p;s)J) \cdot (J\mathbb{A}^{m}([k];p;s)J) = \widehat{C}(n) \cdot \widehat{C}(m)$. Then, for every positive integer n and m, we can verify the Casoratian matrix property, $\widehat{C}(n+m) = \widehat{C}(n) \cdot \widehat{C}(m)$. In general, given that $A^{m+n}([k];p;s) = \mathbb{A}^{m}([k];p;s) \cdot \mathbb{A}^{n}([k];p;s) = \mathbb{A}^{n}([k];p;s) \cdot \mathbb{A}^{m}([k];p;s) = (a_{ij}^{(m+n)})_{1\le i,j\le s+p}$, it follows

$$a_{ij}^{(m+n)} = \sum_{k=1}^{s+p} a_{ik}^{(m)} a_{kj}^{(n)} = \sum_{k=1}^{s+p} a_{ik}^{(n)} a_{kj}^{(m)}.$$

By Equation (9), the identity $a_{ij}^{(n)} = v_{n+s+p-i}^{(s+p-j+1)}([k];p;s)$ is verified, which implies that $a_{ik}^{(n)} = v_{n+s+p-i}^{(s+p-k+1)}([k];p;s)$ and $a_{kj}^{(n)} = v_{n+s+p-k}^{(s+p-j+1)}([k];p;s)$. Therefore, we have

$$a_{ij}^{(m+n)} = \sum_{k=1}^{r} a_{ik}^{(m)} a_{kj}^{(n)} = \sum_{k=1}^{s+p} v_{m+s+p-i}^{(s+p-k+1)}([k];p;s) v_{n+s+p-k}^{(s+p-j+1)}([k];p;s).$$

On the other side, we have $a_{ij}^{(m+n)} = v_{n+m+s+p-i}^{(s+p-j+1)}([k];p;s)$, which allows us to get the following formula:

$$v_{n+m+s+p-i}^{(s+p-j+1)}([k];p;s) = \sum_{k=1}^{s+p} v_{m+s+p-i}^{(s+p-k+1)}([k];p;s) v_{n+s+p-k}^{(s+p-j+1)}([k];p;s).$$

In summary, we have the following result.

Proposition 3. *Let* $\mathcal{S}_{([k];p;s)} = \Big\{\{v_n^{(j)}([k];p;s)\}_{n\ge0};\ 1 \le j \le p+s\Big\}$ *be the Fibonacci fundamental system of the generalized* $([k];p;s)$*-Padovan–Perrin model* (1)*. Then, for all integers* $m, n \ge 0$ *and* $1 \le i, j \le s+p$*, the following identity holds:*

$$v_{n+m+s+p-i}^{(s+p-j+1)}([k];p;s) = \sum_{k=1}^{s+p} v_{m+s+p-i}^{(s+p-k+1)}([k];p;s) v_{n+s+p-k}^{(s+p-j+1)}([k];p;s). \tag{10}$$

In general, we have the following theorem below.

Theorem 2. *Let* $\mathcal{S}_{([k];p;s)} = \Big\{\{v_n^{(j)}([k];p;s)\}_{n\ge0};\ 1 \le j \le p+s\Big\}$ *be the Fibonacci fundamental system of the generalized* $([k];p;s)$*-Padovan–Perrin model defined in* (1)*. Then, for all integers* $m, n \ge 0$ *and* t, q *with* $1 \le t, q \le s+p$*, the following identities are verified:*

$$\begin{aligned} v_{m+n+t}^{(q)}([k];p;s) =&(k_2)^2 \sum_{d=1}^{p} v_{m+t+d}^{(s+p)}([k];p;s) v_{n+d-1+q}^{(s+p)}([k];p;s) \\ &+ k_2 \sum_{d=p+1}^{s+p-1} v_{m+t+d}^{(s+p)}([k];p;s) v_{n+d-1+q}^{(s+p)}([k];p;s) \\ &+ k_2 v_{m+t}^{(s+p)}([k];p;s) v_{n+s+p-1+q}^{(s+p)}([k];p;s), \end{aligned}$$

for $1 \le q \le p$,

$$\begin{aligned} v^{(q)}_{n+m+t}([k];p;s) =& k_2 \sum_{d=1}^{p} v^{(s+p)}_{m+t+d}([k];p;s) v^{(s+p)}_{n+d-1+(s+p)-q}([k];p;s) \\ &+ \sum_{d=p+1}^{p+s-1} v^{(s+p)}_{m+t+d}([k];p;s) v^{(s+p)}_{n+d-1+(s+p)-q}([k];p;s) \\ &+ v^{(s+p)}_{m+t}([k];p;s) v^{(s+p)}_{n+s+p-1+(s+p)-q}([k];p;s), \end{aligned}$$

for $p+1 \le q \le s+p-1$, *and, for* $q = s+p$,

$$\begin{aligned} v^{(s+p)}_{n+m+t}([k];p;s) =& k_2 \sum_{d=1}^{p-1} v^{(s+p)}_{m+t+d}([k];p;s) v^{(s+p)}_{n+d-1}([k];p;s))) \\ &+ \sum_{d=p+1}^{s+p} v^{(d)}_{m+t}([k];p;s) v^{(s+p)}_{n+d-1}([k];p;s). \end{aligned}$$

Proof. Making the changes of variables $s+p-j+1 = q$, $t = s+p-i$, and $d = s+p-k+1$ in Equation (10), we get $v^{(q)}_{n+m+t}([k];p;s) = \sum_{d=1}^{s+p} v^{(d)}_{m+t}([k];p;s) v^{(q)}_{n+d-1}([k];p;s)$, for all integers $m, n \ge 0$ and t, q with $1 \le t, q \le p+s$. Applying Lemma 1 we show that $v^{(d)}_{n+t}([k];p;s) = v^{(d)}_{m+t}([k];p;s)$ and $v^{(q)}_{m+d-1}([k];p;s) = v^{(q)}_{n+d-1}([k];p;s)$ and consequently,

$$\begin{aligned} v^{(q)}_{n+m+t}([k];p;s) &= \sum_{d=1}^{s+p} v^{(d)}_{m+t}([k];p;s) v^{(q)}_{n+d-1}([k];p;s) \\ &= \sum_{d=1}^{s+p} v^{(d)}_{m+t}([k];p;s) v^{(q)}_{n+d-1}([k];p;s), \end{aligned}$$

for integers $m, n \ge 0$ and t, q such that $1 \le t, q \le s+p$.

Since Proposition 2 establishes $v^{(j)}_{n}([k];p;s) = v^{(s+p)}_{n+(s+p)-j}([k];p;s)$ for every j such that $p+1 \le j \le s+p-1$, and $v^{(j)}_{n}([k];p;s) = k_2 v^{(s+p)}_{n+j}([k];p;s)$ for every j such that $1 \le j \le p$, we get the following results:

$$\begin{aligned} v^{(q)}_{m+n+t}([k];p;s) =& (k_2)^2 \sum_{d=1}^{p} v^{(s+p)}_{m+t+d}([k];p;s) v^{(s+p)}_{n+d-1+q}([k];p;s) \\ &+ k_2 \sum_{d=p+1}^{s+p-1} v^{(s+p)}_{m+t+d}([k];p;s) v^{(s+p)}_{n+d-1+q}([k];p;s) \\ &+ k_2 v^{(s+p)}_{m+t}([k];p;s) v^{(s+p)}_{n+s+p-1+q}([k];p;s), \end{aligned}$$

for $1 \leq q \leq p$, and

$$\begin{aligned} v_{n+m+t}^{(q)}([k];p;s) =& k_2 \sum_{d=1}^{p} v_{m+t+d}^{(s+p)}([k];p;s) v_{n+d-1+(s+p)-q}^{(s+p)}([k];p;s) \\ =& \sum_{d=p+1}^{p+s-1} v_{m+t+d}^{(s+p)}([k];p;s) v_{n+d-1+(s+p)-q}^{(s+p)}([k];p;s) \\ & + v_{m+t}^{(s+p)}([k];p;s) v_{n+s+p-1+(s+p)-q}^{(s+p)}([k];p;s), \end{aligned}$$

for $p+1 \leq q \leq s+p-1$. Taking $q = s+p$ in Equation (10), we get

$$\begin{aligned} v_{n+m+t}^{(s+p)}([k];p;s) =& k_2 \sum_{d=1}^{p} v_{m+t+d}^{(s+p)}([k];p;s) v_{n+d-1}^{(s+p)}([k];p;s))) \\ & + \sum_{d=p+1}^{s+p} v_{m+t+s+p-d}^{(s+p)}([k];p;s) v_{n+d-1}^{(s+p)}([k];p;s), \end{aligned}$$

which concludes the proof. □

To better clarify our results, we apply Theorem 2, for the case $s = 2, p = k_1 = k_2 = 1$, associated with Padovan–Perrin numbers. Therefore, we arrive at the corollary below.

Corollary 2. *The terms of the Padovan–Perrin sequence* $\{v_n^{(3)}\}_{n\geq 0}$ *with initial conditions* $v_0 = 0, v_1 = 0$ *and* $v_2 = 1$*, verify the following identity:*

$$v_{n+r}^{(3)}([k];1;2) = v_{r+1}^{(3)}([k];1;2) v_n^{(3)}([k];1;2))) + \sum_{d=2}^{3} v_{r+3-d}^{(3)}([k];1;2) v_{n+d-1}^{(3)}([k];p;s).$$

4. Combinatorial Expression of the $([k];p;s)$-Padovan–Perrin Sequences

The combinatorial expressions of sequences $\{u_n\}_{n\geq 0}$ defined by the recurrence relation $u_{n+1} = \sum_{i=0}^{r-1} a_i u_{n-i-1}$, for $n \geq r$, have been largely studied in the literature (see, for example, [14, 18] and references therein). In [14], it was established that the combinatorial form for u_n is given by

$$u_n = \rho(n,r)A_0 + \rho(n-1,r)A_1 + \cdots + \rho(n-r+1,r)A_{r-1}, \text{ for every } n \geq r, \quad (11)$$

such that $A_m = a_{r-1}u_m + \cdots + a_m u_{r-1}$ and

$$\rho(n,r) = \sum_{t_0+2t_1+\cdots+rt_{r-1}=n-r} \frac{(t_0+\cdots+t_{r-1})!}{t_0!t_1!\ldots t_{r-1}!} a_0^{t_0} a_1^{t_1} \ldots a_{r-1}^{t_{r-1}}, \text{ for every } n \geq r, \quad (12)$$

where $\rho(j,r)=0$ for $0\le j\le r-1$ and $\rho(r,r)=1$.

By replacing the coefficients of the recurrence relation $u_{n+1}=\sum_{i=0}^{r-1}a_iu_{n-i-1}$ for $a_0=a_1=\cdots=a_{s-2}=0$; $a_{s-1}=k_1$, $a_s=\cdots=a_{s+p-2}=0$ and $a_{s+p-1}=k_2$, with the initial conditions $\alpha_0,\cdots,\alpha_{r-2},\alpha_{r-1}$, we get Equation (1) defining the model of $([k];p;s)$-generalized Padovan–Perrin sequences. Therefore, the construction that was done in [14] and Equation (12) imply the following result on the combinatorial aspect of the model of $([k];p;s)$-generalized Padovan–Perrin.

Proposition 4. *Consider the sequence of the* $([k];p;s)$*-generalized Padovan–Perrin numbers given by Equation* (2)*, with initial conditions* $\alpha_0=\ldots=\alpha_{s+p-2}=0$ *and* $\alpha_{s+p-1}=1$*. Then, the following identity is verified:*

$$v_n^{(s+p)}([k];p;s)=\rho(n+1,s+p),$$

for $n\ge s+p$*, with*

$$\rho(n,s+p)=\sum_{(s)t_{s-1}+(s+p)t_{s+p-1}=n-(s+p-1)}\frac{(t_{s-1}+t_{s+p-1})!}{t_{s-1}!t_{s+p-1}!}(k_1)^{t_{s-1}}(k_2)^{t_{s+p-1}},$$

for every $n\ge s+p$*,* $\rho(j,s+p)=0$ *for* $0\le j\le s+p-1$*, and* $\rho(s+p,s+p)=1$*.*

Proof. By the initial conditions $\alpha_0=\ldots=\alpha_{s+p-2}=0$ and $\alpha_{s+p-1}=1$, we have $A_0=0,A_1=0,\ldots,A_{s-1}=k_1$, $A_s=0,\ldots,A_{s+p-1}=k_2$. Then,

$$\begin{aligned}v_n^{(s+p)}([k];p;s)&=k_1\rho(s-2,s+p)+k_2\rho(n-(s+p)+1,s+p)\\&=k_1v_{n+s}^{(s+p)}([k];p;s)+k_2v_{n+s+p}^{(s+p)}([k];p;s),\end{aligned}$$

or $v_n^{(s+p)}([k];p;s)=\rho(n+1,s+p)$, where

$$\rho(n,s+p)=\sum_{(s)t_{s-1}+(s+p)t_{s+p-1}=n-(s+p-1)}\frac{(t_{s-1}+t_{s+p-1})!}{t_{s-1}!t_{s+p-1}!}(k_1)^{t_{s-1}}(k_2)^{t_{s+p-1}},$$

for every $n\ge s+p$, $\rho(j,s+p)=0$ for $0\le j\le s+p-1$, and $\rho(s+p,s+p)=1$. □

In general, we have the theorem below.

Theorem 3. *Consider the sequence of the* $([k];p;s)$*-generalized Padovan–Perrin numbers given by Equation* (1) *with arbitrary initial conditions* $\alpha_0,\ldots,\alpha_{s+p-1}$*. Then, we have*

$$v_n([k];p;s)=\rho(n,s+p)A_0+\rho(n-1,s+p)A_1+\cdots+\rho(n-(s+p)+1,s+p)A_{s+p-1},$$

for every $n\ge r$*, such that* $A_0=k_2\alpha_0+k_1\alpha_p, A_1=k_2\alpha_1+k_1\alpha_{p+1},\ldots,A_{s-1}=k_2\alpha_{s-1}+k_1\alpha_{s+p-1}$*,* $A_s=k_2\alpha_s,\ldots,A_{s+p-1}=k_2\alpha_{s+p-1}$*, and*

$$\rho(n,s+p)=\sum_{(s)t_{s-1}+(s+p)t_{s+p-1}=n-(s+p-1)}\frac{(t_{s-1}+t_{s+p-1})!}{t_{s-1}!t_{s+p-1}!}(k_1)^{t_{s-1}}(k_2)^{t_{s+p-1}},$$

for every $n\ge s+p$*,* $\rho(j,s+p)=0$ *for* $0\le j\le s+p-1$ *and* $\rho(s+p,s+p)=1$*.*

Proof. The proof is given by the direct application of Expressions (11) and (12) with arbitrary initial conditions $\alpha_0, \ldots, \alpha_{s+p-1}$. $\square$

Definition 1, Theorem 3 and Proposition 4 show us that $\rho(n+1, s+p)$ is a Fibonacci fundamental solution for the Fibonacci fundamental system of the generalized $([K];p;s)$-Padovan–Perrin model defined in (1).

Combining Propositions 4 and 2 we obtain the following combinatorial identities for the sequences of the Fibonacci fundamental system related to the generalized $([k];p;s)$-Padovan–Perrin model defined in (1).

Proposition 5. *Let* $\mathcal{S}_{([k];p;s)} = \left\{\{v_n^{(j)}([k];p;s)\}_{n\geq 0};\ 1 \leq j \leq p+s\right\}$ *be the Fibonacci fundamental system of the generalized* $([k];p;s)$*-Padovan–Perrin model given in* (1)*. Then, for every* $n \geq 0$*, we have:*

$$v_n^{(j)}([k];p;s) = v_{n+(s+p)-j}^{(s+p)}([k];p;s) = \rho(n+(s+p)-j+1, s+p),$$

for every j *such that* $p+1 \leq j \leq p+s-1$*, and*

$$v_n^{(j)}([k];p;s) = k_2\rho(n+j+1, s+p),$$

for every j *such that* $1 \leq j \leq p$*, where*

$$\rho(n, s+p) = \sum_{(s)t_{s-1}+(s+p)t_{s+p-1}=n-(s+p-1)} \frac{(t_{s-1}+t_{s+p-1})!}{t_{s-1}!t_{s+p-1}!}(k_1)^{t_{s-1}}(k_2)^{t_{s+p-1}},$$

for every $n \geq s+p$*,* $\rho(j, s+p) = 0$ *for* $0 \leq j \leq s+p-1$*, and* $\rho(s+p, s+p) = 1$*.*

Moreover, following Proposition 3 and Theorem 2, we obtain the combinatorial identities below.

Theorem 4. *Let* $\mathcal{S}_{([k];p;s)} = \left\{\{v_n^{(j)}([k];p;s)\}_{n\geq 0};\ 1 \leq j \leq p+s\right\}$ *be the Fibonacci fundamental system of the generalized* $([k];p;s)$*-Padovan–Perrin model* (1)*. Then, for all integers* $m, n \geq 0$ *and* t, q *such that* $1 \leq t, q \leq s+p$*, we have*

$$\begin{aligned} v_{m+n+t}^{(q)}([k];p;s) =& (k_2)^2 \sum_{d=1}^{p} \rho(m+t+d+1, s+p)\rho(n+d+q, s+p) \\ &+ k_2 \sum_{d=p+1}^{s+p-1} \rho(m+t+d+1, s+p)\rho(n+d+q, s+p) \\ &+ k_2\rho(m+t+1)\rho(n+s+p+q), \end{aligned}$$

for $1 \leq q \leq p$,

$$\begin{aligned} v^{(q)}_{n+m+t}([k];p;s) =& k_2 \sum_{d=1}^{p} \rho(m+t+d+1, s+p)\rho(n+d+(s+p)-q, s+p) \\ &+ \sum_{d=p+1}^{p+s-1} \rho(m+t+d+1, s+p)\rho(n+d+(s+p)-q, s+p) \\ &+ \rho(m+t+1, s+p)\rho(n+s+p+(s+p)-q, s+p), \end{aligned}$$

and for $p+1 \leq q \leq s+p-1$,

$$\begin{aligned} \rho(n+m+t+1, s+p) =& k_2 \sum_{d=1}^{p-1} \rho(m+t+d+1, s+p)\rho(n+d, s+p) \\ &+ \sum_{d=p+1}^{s+p} \rho(m+t+1, s+p)\rho(n+d, p+s), \end{aligned}$$

where

$$\rho(n, s+p) = \sum_{(s)t_{s-1}+(s+p)t_{s+p-1}=n-(s+p-1)} \frac{(t_{s-1}+t_{s+p-1})!}{t_{s-1}!t_{s+p-1}!}(k_1)^{t_{s-1}}(k_2)^{t_{s+p-1}},$$

for every $n \geq s+p$, $\rho(j, s+p) = 0$ *for* $0 \leq j \leq s+p-1$, *and* $\rho(s+p, s+p) = 1$.

Observe that the results of this section give us explicit formulas for the Fibonacci fundamental system of the generalized $([k];p;s)$-Padovan–Perrin model in (1). Moreover, it seems to us that the combinatorial identities presented in this section are new in the literature.

5. Analytic Expression of the $([k];p;s)$-Padovan–Perrin Sequences

Let $\{v_n([k];p;s)\}_{n\geq 0}$ be the sequence defined by the recursive relation (1) and with initial conditions $(\alpha_0, \alpha_1, \ldots, \alpha_{p+s-1})$. Its characteristic polynomial is given by

$$P(z) = z^{s+p} - k_1 z^p - k_2.$$

Suppose that λ is a double root of $P(z)$. Then, we have

$$P(\lambda) = 0 \text{ and } P'(\lambda) = 0.$$

where $P'(z) = \frac{dP}{dz}(z)$.

Consider the parameter $p = 1$, or $P(z) = z^{s+1} - k_1 z - k_2$. Then, we have the following algebraic equations,

$$\lambda^{s+1} = k_1\lambda + k_2 \text{ and } (s+1)\lambda^s = k_1.$$

Hence, we get $\lambda^s = \dfrac{k_1}{s+1}$. Since $\lambda^{s+1} = \lambda^s\lambda = k_1\lambda + k_2$, then we derive,

$$\lambda\left(\frac{1}{s+1} - 1\right) = \frac{k_2}{k_1}.$$

Thus, if λ a double root of $P(z)$ with $p = 1$, then $\lambda = \dfrac{-k_2(s+1)}{k_1 s}$. Hence,

$$\left(\frac{-k_2(s+1)}{k_1 s}\right)^s = \frac{k_1}{s+1}.$$

Therefore, we get the following result.

Lemma 2. *Let k_1, k_2 and s be positive integers, and $p = 1$. Suppose $\left(\dfrac{-k_2(s+1)}{k_1 s}\right)^s \neq \dfrac{k_1}{s+1}$. Then, the roots of the polynomial $P(z) = z^{s+1} - k_1 z - k_2$ are simple.*

Now suppose that $p \geq 2$. Hence, we have the following algebraic equations:

$$\lambda^{s+p} = k_1\lambda^p + k_2 \text{ and } (p+s)\lambda^{s+p-1} = pk_1\lambda^{p-1}.$$

Therefore, we get $\lambda^s = \dfrac{pk_1}{p+s}$. Since $\lambda^{s+p} = \lambda^p\lambda^s = k_1\lambda^p + k_2$, then we derive $\lambda^p\lambda^s - k_1\lambda^p = k_2$. Hence, we obtain

$$(\lambda^s - k_1)\lambda^p = (k_1\lambda^{p-1} - k_1)\lambda^p = \left(\frac{pk_1}{p+s} - 1\right)k_1\lambda^p = k_2.$$

Thus, we obtain $\lambda^p = -\dfrac{p+s}{s}\dfrac{k_2}{k_1}$. Therefore, we have

$$s\ln(|\lambda|) = \ln\left(\frac{pk_1}{p+s}\right) \text{ and } p\ln(|\lambda|) = \ln\left(\frac{p+s}{s}\frac{k_2}{k_1}\right).$$

Hence, we have

$$\left(\frac{pk_1}{p+s}\right)^{\frac{1}{s}} = \left(\frac{p+s}{s}\frac{k_2}{k_1}\right)^{\frac{1}{p}},$$

and a direct computation implies that

$$\left(\frac{p+s}{p}\right)^{s-p} = \frac{k_1^{s+p}}{k_2^s}.$$

In summary, we obtain the following lemma.

Lemma 3. *Let k_1, k_2, p and s be positive integers, with $p \geq 2$. Suppose that $\left(\frac{p+s}{p}\right)^{s-p} \neq \frac{k_1^{s+p}}{k_2^s}$. Then, the roots of the polynomial*

$$P(z) = z^{s+p} - k_1 z^p - k_2$$

are simple.

Moreover, as we have

$$|\lambda|^s = \frac{pk_1}{p+s} \text{ and } |\lambda|^p = \frac{p+s}{s}\frac{k_2}{k_1},$$

we show that $|\lambda|^s < k_1$ and $|\lambda|^p > \dfrac{k_2}{k_1}$. Suppose that $k_1^2 < k_2$ and $p < s$, then we have

$$|\lambda|^p > \frac{k_2}{k_1} = \frac{k_2}{k_1^2}k_1 > k_1 \geq 1.$$

Thus, we get $|\lambda| > 1$ and

$$|\lambda|^p > k_1 > |\lambda|^s,$$

which is impossible, because $|\lambda| > 1$ and $p < s$.

Lemma 4. *Let k_1, k_2, p and s be positive integers, with $p \geq 2$. Suppose that $k_1^2 < k_2$ and $p < s$. Then, the roots of the polynomial*

$$P(z) = z^{s+p} - k_1 z^p - k_2$$

are simple.

For a linear recursive sequence of Fibonacci type $\{u_n\}_{n\geq 0}$ defined by recurrence relation $u_{n+1} = \sum_{i=0}^{r-1} a_i u_{n-i-1}$, for $n \geq r$, the analytic expression is expressed in terms of the roots of the associated so-called characteristic polynomial and their multiplicities (see, instance, [3, 7, 18]). More precisely, the sequence $\{\rho(n,r)\}_{n\geq 0}$ defined by (12) is expressed in the analytical form given in the following lemma.

Lemma 5. *Let $\{\rho(n, s+p)\}_{n\geq 0}$ be the sequence defined by* (12)*. Suppose the roots $\lambda_1, \cdots, \lambda_{s+p}$ of its characteristic polynomial*

$$P(z) = z^{s+p} - a_0 z^{s+p-1} - \ldots - a_{s+p-2} z - a_{s+p-1}$$

($a_{s+p-1} \neq 0$) satisfy $\lambda_i \neq \lambda_j$ for $i \neq j$. Then, we have

$$\rho(n, s+p) = \sum_{i=1}^{s+p} \frac{\lambda_i^{n-1}}{P'(\lambda_i)} = \sum_{i=1}^{s+p} \frac{\lambda_i^{n-1}}{\prod\limits_{k\neq i}(\lambda_i - \lambda_k)} \text{ for every } n \geq s+p;$$

otherwise $\rho(s+p, s+p) = 1$, $\rho(i, s+p) = 0$ for $i \leq s+p-1$, where $P'(z) = \frac{dP}{dz}(z)$.

Combining the result of Theorem 3 and Lemmas 3, 4, and 5 we obtain the following analytical identities.

Proposition 6. *Consider k_1, k_2, p, and s in $\mathbb{N}^*$, with $p \geq 2$. Suppose that $k_1^2 < k_2$ and $p < s$. Consider the sequence of the $([k]; p; s)$-generalized Padovan–Perrin* *numbers given by Equation* (1)*, with arbitrary initial conditions* $\alpha_0, \ldots, \alpha_{s+p-2}$, α_{s+p-1}. *Suppose the roots* $\lambda_1, \cdots, \lambda_{s+p}$ *of its characteristic polynomial*

$$P(z) = z^{s+p} - a_0 z^{s+p-1} - \ldots - a_{s+p-2} z - a_{s+p-1}$$

$(a_{s+p-1} \neq 0)$ *satisfy* $\lambda_i \neq \lambda_j$ *for* $i \neq j$. *Then, it is verified that:*

$$v_n([k];p;s) = A_0 \sum_{i=1}^{s+p} \frac{\lambda_i^{n-1}}{\prod\limits_{k\neq i}(\lambda_i - \lambda_k)} + \cdots + A_{s+p-1} \sum_{i=1}^{s+p} \frac{\lambda_i^{n-(s+p)}}{\prod\limits_{k\neq i}(\lambda_i - \lambda_k)},$$

for every $n \geq s+p$, *such that* $A_0 = k_2\alpha_0 + k_1\alpha_p, A_1 = k_2\alpha_1 + k_1\alpha_{p+1}, \ldots, A_{s-1} = k_2\alpha_{s-1} + k_1\alpha_{s+p-1}$, $A_s = k_2\alpha_s, \ldots, A_{s+p-1} = k_2\alpha_{s+p-1}$.

6. A Special Case

In this section, we present the results of the previous sections applied to a special integer sequence, namely, the k-Padovan–Perrin numbers.

Recall that for $s = 2$, $p = 1$ and $[k] = (1, k), k \geq 2$, Equation (1) represents the recursive relation defining the k-Padovan–Perrin numbers, namely,

$$v_n([k];1;2) = v_{n-2}([k];1;2) + kv_{n-3}([k];1;2), \text{ for } n \geq 3, \tag{13}$$

with initial conditions $v_j([k];p;s) = \alpha_j$, for $0 \leq j \leq 2$, where $\alpha_j \in \mathbb{N}$.

The Fibonacci Fundamental System $\mathcal{S}_{([k];1;2)} = \Big\{\{v_n^{(j)}([k];1;2)\}_{n\geq 0};\ 1 \leq j \leq 3\Big\}$ associated to the k-Padovan–Perrin numbers is given by

$$\begin{cases} v_n^{(j)}([k];1;2) = v_{n-2}^{(j)}([k];1;2) + kv_{n-3}^{(j)}([k];1;2), \text{ for } n \geq s+p \\ v_n^{(j)}([k];1;2) = \delta_{n+1}^{(j)};\ \text{for } n = 0,\ 1,\ 2, \end{cases}$$

where $\delta_i^{(j)} = 1$ if $i = j$, and $\delta_i^{(j)} = 0$, otherwise. Thus, by Proposition 1, the k-Padovan–Perrin numbers are given in the form, for every $n \geq 0$,

$$v_n([k];1;2) = \alpha_0 v_n^{(1)}([k];1;2) + \alpha_1 v_n^{(2)}([k];1;2) + \alpha_2 v_n^{(3)}([k];1;2).$$

In addition, following Proposition 2, we have $v_n^{(1)}([k];1;2) = kv_{n+1}^{(3)}([k];1;2)$, and $v_n^{(2)}([k];1;2) = v_{n+1}^{(3)}([k];1;2)$ for $n \geq 0$. Therefore,

$$v_n([k];1;2) = (k\alpha_0 + \alpha_1)v_{n+1}^{(3)}([k];1;2) + \alpha_2 v_n^{(3)}([k];1;2).$$

The matrix formulation for Equation (13) is given by

$$\mathbb{V}_n([k];1;2) = \mathbb{A}([k];1;2)\mathbb{V}_{n-1}([k];1;2),$$

where

$$\mathbb{A}([k];1;2)=\begin{bmatrix}0 & k_1 & k_2\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} \text{ and } \mathbb{V}_{n-1}([k];1;2)=\begin{bmatrix}v_{n-1}([k];1;2)\\ v_{n-2}([k];1;2)\\ v_{n-3}([k];1;2)\end{bmatrix}.$$

By direct application of Theorem 1, the entries of the powers $\mathbb{A}^n([k];1;2) = (a_{ij}^{(n)})_{1\le i,j\le 3}$ are given by $a_{ij}^{(n)} = v_{n+3-i}^{(3-j+1)}([k];1;2)$, for every $n \geq 0$. Then, using Proposition 3 and Theorem 2, we derive the following proposition.

Proposition 7. *Let* $\mathcal{S}_{([k];1;2)} = \left\{\{v_n^{(j)}([k];1;2)\}_{n\geq 0};\ 1\leq j\leq 3\right\}$ *be the Fibonacci fundamental system of the generalized* $([k];1;2)$*-Padovan–Perrin model* (13)*. Then, for all integers* $m,n \geq 0$ *and* i,t *such that* $1 \leq i,t \leq 3$*, the following identities are verified:*

$$v_{n+m+3-i}^{(3-j+1)}([k];1;2)=\sum_{k=1}^{3} v_{m+3-i}^{(3-k+1)}([k];1;2)v_{n+3-k}^{(3-j+1)}([k];1;2),$$

$$\begin{aligned} v_{m+n+t}^{(1)}([k];1;2) =&(k)^2 v_{m+t+1}^{(3)}([k];1;2)v_{n+1}^{(3)}([k];1;2)\\ &+ k v_{m+t+2}^{(3)}([k];1;2)v_{n+2}^{(3)}([k];1;2)\\ &+ k v_{m+t}^{(3)}([k];1;2)v_{n+3}^{(3)}([k];1;2),\end{aligned}$$

$$\begin{aligned} v_{n+m+t}^{(2)}([k];1;2) =&k v_{m+t+1}^{(3)}([k];1;2)v_{n+1}^{(3)}([k];1;2) + v_{m+t+2}^{(3)}([k];1;2)v_{n+2}^{(3)}([k];1;2)\\ &+ v_{m+t}^{(3)}([k];1;2)v_{n+3}^{(3)}([k];1;2),\end{aligned}$$

and

$$\begin{aligned} v_{n+m+t}^{(3)}([k];1;2) =&k v_{m+t+1}^{(3)}([k];1;2)v_{n}^{(3)}([k];1;2)))\\ &+\sum_{d=2}^{3} v_{m+t+3-d}^{(3)}([k];1;2)v_{n+d-1}^{(3)}([k];1;2).\end{aligned}$$

Next, we will give explicit combinatorial formulas for k-Padovan–Perrin numbers. By replacing the parameters $s=2$, $p=1$, and $[k]=(1,k), k\geq 2$ in Equation (12) we get

$$\rho(n,3)=\sum_{2t_1+3t_2=n-2}\frac{(t_1+t_2)!}{t_1!t_2!}(k)^{t_2}, \tag{14}$$

for every $n \geq 3$, $\rho(j,3)=0$ for $0\leq j\leq 2$, and $\rho(3,3)=1$. Then, a direct application of Theorem 3 allows us to obtain the following proposition.

Proposition 8. *Consider the sequence* $\{v_n([k];1;2)\}_{n\geq 0}$ *of the k-Padovan–Perrin numbers defined by Equation* (13)*, with arbitrary initial conditions* $\alpha_0, \alpha_1, \alpha_2$*. Then, we have*

$$v_n([k];1;2) = \rho(n,3)(k\alpha_0 + \alpha_1) + k\alpha_1\rho(n-1,3) + k\alpha_2\rho(n-2,3),$$

for every $n \geq r$*, such that* $\rho(n,3)$ *is given in the form* (14)*.*

Similarly as in Section 4, as a consequence of Propositions 5 and 8, we get the explicit combinatorial formulas to the generalized $([k];1;2)$-Padovan–Perrin model (13).

Proposition 9. *Let* $\mathcal{S}_{([k];1;2)} = \left\{\{v_n^{(j)}([k];1;2)\}_{n\geq 0};\ 1 \leq j \leq 3\right\}$ *be the Fibonacci fundamental system of the generalized* $([k];1;2)$*-Padovan–Perrin model given in* (13)*. Then, for* $n \geq 0$*, we have*

$$\begin{aligned}
v_n^{(3)}([k];1;2) &= \rho(n+1,3),\\
v_n^{(2)}([k];1;2) &= \rho(n+2,3),\\
v_n^{(1)}([k];1;2) &= k\rho(n+2,3),
\end{aligned}$$

where $\rho(n,3)$ *is given in the form* (14)*.*

In addition, the combinatorial identities are provided in the result below

Theorem 5. *Let* $\mathcal{S}_{([k];1;2)} = \left\{\{v_n^{(j)}([k];1;2)\}_{n\geq 0};\ 1 \leq j \leq 3\right\}$ *be the Fibonacci fundamental system of the generalized* $([k];1;2)$*-Padovan–Perrin model defined in* (13)*. Then, for all integers* $m, n \geq 0$*, and* t *such that* $1 \leq t \leq 3$*, it is verified that we have*

$$\begin{aligned}
v^{(1)}_{m+n+t}([k];1;2) =& (k)^2\rho(m+t+2,3)\rho(n+1+q,3)\\
&+ k\rho(m+t+3,3)\rho(n+3,3)\\
&+ k\rho(m+t+1,3)\rho(n+s+p+1,3),
\end{aligned}$$

$$\begin{aligned}
v^{(2)}_{n+m+t}([k];p;s) =& k\rho(m+t+2,3)\rho(n+2,3) + \rho(m+t+3,3)\rho(n+3,3)\\
&+ \rho(m+t+1,3)\rho(n+4,3),
\end{aligned}$$

and

$$\rho(n+m+t+1,3) = k\rho(m+t+2,3)\rho(n+1,3) + \sum_{d=2}^{3}\rho(m+t+1,3)\rho(n+d,3),$$

where $\rho(n,3)$ *is given in the form* (14)*.*

Now we will study the analytical formulas for $v_n([k];1;2)$ using the determinant approach. The *Sylvester matrix* is a matrix associated with two univariate polynomials $P(z)$ and $Q(z)$, whose entries are given by coefficients of these two polynomials [10]. When the determinant of the Sylvester matrix $S_{P,Q}$, called the resultant, is zero, then the two polynomials have a common root (in the case of coefficients in a field) or a non-constant common divisor (in the case of coefficients in an integral domain). Considering the polynomial $P(z)$ and its derivative $P'(z)$, if the determinant of the Sylvester matrix $S_{P,P'}$ is different from 0, then the polynomials $P(z)$ and $P'(z)$ do not have common roots. This means that if $det(S_{P,P'}) \neq 0$, then the roots of $P(z)$ are simple. In this special case the associated characteristic polynomial is $P(z) = z^3 - z - k$, with derivative $P'(z) = 3z^2 - 1$, and its Sylvester matrix associated to P and P' is given by

$$S_{P,P'} = \begin{pmatrix} 1 & 0 & -1 & -k & 0 \\ 0 & 1 & 0 & -1 & -k \\ 3 & 0 & -1 & 0 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & 0 & 3 & 0 & -1 \end{pmatrix}.$$

Hence, $det(S_{P,P'}) = 27k^2 - 4$. Then, $det(S_{P,P'}) \neq 0$ if $k \neq \pm\sqrt{\frac{4}{27}}$, which permits us to get the following analytical property.

Proposition 10. *Consider the sequence* $\{v_n([k];1;2)\}_{n\geq 0}$ *of the k-Padovan–Perrin numbers defined by Equation* (13) *with arbitrary initial conditions* $\alpha_0, \alpha_1, \alpha_2$. *Suppose that* $k \neq \pm\sqrt{\frac{4}{27}}$ *and the roots* λ_1, λ_2 *and* λ_3 *of its characteristic polynomial* $P(z) = z^3 - z - k$, *satisfy* $\lambda_i \neq \lambda_j$ *for* $i \neq j$. *Then, the following is true:*

$$v_n([k];1;2) = \sum_{i=1}^{3} \frac{(k\alpha_0 + \alpha_1)\lambda_i^{n-1} + k\alpha_1\lambda_i^{n-2} + k\alpha_2\lambda_i^{n-3}}{\prod_{k\neq i}(\lambda_i - \lambda_k)}$$

for every $n \geq 3$.

7. Concluding Remarks and Perspectives

In this paper, we have studied the model of generalized $([k];p;s)$-Padovan–Perrin sequences. Moreover, some identities and combinatorial identities for the model of generalized $([k];p;s)$-Padovan–Perrin are provided. On the other hand, we presented a study of the characteristic polynomial associated with the generalized $([k];p;s)$-Padovan–Perrin sequence and provided analytic formulas, without using the usual method of determinant. Also, in the context of the special case of $([k];2;1)$-Padovan–Perrin sequences, we present several properties, namely, we give some identities and combinatorial identities related to this sequence. In addition, the use of the determinants of the Sylvester matrix allows us to obtain a new analytic representation. It seems to us that several results of our study are new in the literature.

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