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CLASSIFICATION FOR EXISTENCE OF THE CONTINUED FRACTION EXPANSIONS OF $\sqrt{\mathrm{D}}$ AND $(1+\sqrt{\mathrm{D}})/2$

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Abstract

The continued fraction expansions of quadratic irrational numbers are closely related to the fundamental units of the real quadratic fields. It is well known that the continued fraction expansion of $\sqrt{d}$ (respectively $(1+\sqrt{d})/2$) has the form

$$[a_0;\overline{a_1,\ldots,a_{l_d-1},2a_0}]\ \text{(respectively } [a'_0;\overline{a'_1,\ldots,a'_{l'_d-1},2a'_0-1}]\text{)},$$

where $a_1,\ldots,a_{l_d-1}$ (respectively $a'_1,\ldots,a'_{l'_d-1}$) is a palindromic sequence of positive integers. In this paper, for any given positive integer l_d (respectively l'_d) and a palindromic sequence of positive integers $a_1,\ldots,a_{l_d-1}$ (respectively $a'_1,\ldots,a'_{l'_d-1}$), we completely classify when a value of d exists that conforms to the form of the continued fraction

$$\sqrt{d}=[a_0;\overline{a_1,\ldots,a_{l_d-1},2a_0}]\ \text{(respectively } (1+\sqrt{d})/2=[a'_0;\overline{a'_1,\ldots,a'_{l'_d-1},2a'_0-1}]\text{)}.$$

1. Introduction

Let d be a non-square positive integer. We denote the continued fraction of $\sqrt{d}$ by

$$\sqrt{d}=a_0+\cfrac{1}{a_1+\cfrac{1}{\cdots+\cfrac{1}{a_{l_d}+\cfrac{1}{a_1+\cdots}}}}=[a_0;\overline{a_1,\ldots,a_{l_d}}],$$

where l_d is the length of the period of the continued fraction expansion. Then the period is *palindromic*, that is, $a_{l_d-t}=a_t$ for $1\le t<l_d$ and $a_{l_d}=2a_0$ (see [15], Satz

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3.29). Similarly, we denote the continued fraction of $(1+\sqrt{d})/2$ by

$$\frac{1+\sqrt{d}}{2} = a_0' + \cfrac{1}{a_1' + \cfrac{1}{\cdots + \cfrac{1}{a_{l_d'}' + \cfrac{1}{a_1' + \cdots}}}} = [a_0'; \overline{a_1', \ldots, a_{l_d'}'}],$$

where l_d' is the length of the period of the continued fraction expansion. Then the continued fraction of $(1+\sqrt{d})/2$ has a similar property with the continued fraction of $\sqrt{d}$. In fact, the period is also palindromic and $a_{l_d'}' = 2a_0' - 1$ (see [15], Satz 3.30).

There exist many results for the relation between the continued fraction of $\sqrt{d}$ and the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{d})$ [1, 2, 3, 4, 11, 12, 17, 18]. In particular, for an explicit form of the fundamental unit of $\mathbb{Q}(\sqrt{d})$, the following theorem is well known.

Theorem 1 ([4, 12])**.** *Let d be a positive square-free integer and ϵ_d the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{d})$. Let l_d be the length of the period of the continued fraction of $\sqrt{d}$ and p_{l_d-1}/q_{l_d-1} the (l_d-1)-th convergent of it. Then*

$$\epsilon_d = p_{l_d-1} + q_{l_d-1}\sqrt{d}$$

or

$$\epsilon_d^3 = p_{l_d-1} + q_{l_d-1}\sqrt{d},$$

and the latter can only occur if $d \equiv 5 \pmod 8$.

Theorem 1 says that except for the case in which $d \equiv 5 \pmod 8$, the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{d})$ is $\epsilon_d = p_{l_d-1} + q_{l_d-1}\sqrt{d}$. If d is a positive square-free integer congruent to 5 modulo 8, then $\epsilon_d = p_{l_d-1} + q_{l_d-1}\sqrt{d}$ or $\epsilon_d^3 = p_{l_d-1} + q_{l_d-1}\sqrt{d}$.

In this paper, for any given positive integer l_d (respectively l_d') and a palindromic sequence of positive integers $a_1, \ldots, a_{l_d-1}$ (respectively $a_1', \ldots, a_{l_d'-1}'$), we completely classify when a value of d exists that conforms to the form of the continued fraction

$$\sqrt{d} = [a_0; \overline{a_1, \ldots, a_{l_d-1}, 2a_0}] \text{ (respectively } (1+\sqrt{d})/2 = [a_0'; \overline{a_1', \ldots, a_{l_d'-1}', 2a_0'-1}]).$$

2. Preliminaries

In this section, we start with the basic properties of a continued fraction. Similar discussion and the proofs for the properties can be seen in many excellent books and papers such as [5, 6, 8, 9, 13, 14, 16]. First, we obtain positive integers p_n, q_n from partial quotients $a_0, a_1, \ldots, a_n$ of the continued fraction of $\sqrt{d}$ by using recurrence relations:

$$\begin{aligned} p_{-1} &= 1, \quad p_0 = a_0, \quad p_n = a_n p_{n-1} + p_{n-2} \ (n \geq 1), \\ q_{-1} &= 0, \quad q_0 = 1, \quad q_n = a_n q_{n-1} + q_{n-2} \ (n \geq 1), \\ r_{-1} &= 1, \quad r_0 = 0, \quad r_n = a_n r_{n-1} + r_{n-2} \ (n \geq 1). \end{aligned} \tag{1}$$

Note that

$$\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n], \quad \lim_{n\to\infty} \frac{p_n}{q_n} = \sqrt{d},$$

and

$$\frac{q_n}{r_n} = [a_1, a_2, \ldots, a_n].$$

We can easily prove the following recurrence relations for the sequences $\{p_n\}$, $\{q_n\}$, and $\{r_n\}$:

$$q_n r_{n-1} - r_n q_{n-1} = (-1)^n, \tag{2}$$

$$q_n r_{n-2} - r_n q_{n-2} = (-1)^{n-1} a_n, \tag{3}$$

$$p_n - a_0 q_n = r_n. \tag{4}$$

We can also give a similar expression for the continued fraction expansion of $(1 + \sqrt{d})/2$. For any positive integer l_d (respectively l'_d) and a palindromic sequence of positive integers $a_1, \ldots, a_{l_d - 1}$ (respectively $a'_1, \ldots, a'_{l'_d - 1}$), the necessary and sufficient conditions for the existence of d having the form of the continued fraction expansion

$$\sqrt{d} = [a_0; \overline{a_1, \ldots, a_{l_d-1}, 2a_0}] \text{ (respectively } (1+\sqrt{d})/2 = [a'_0; \overline{a'_1, \ldots, a'_{l'_d-1}, 2a'_0 - 1}])$$

are known as follows.

Proposition 1 ([7])**.** *There exists d having the form of the continued fraction* $\sqrt{d} = [a_0; \overline{a_1, \ldots, a_{l_d-1}, 2a_0}]$ *if and only if one of the following two cases holds*:

(i) q_{l_d-1} *is odd;*

(ii) *both* q_{l_d-1} *and* r_{l_d-2} *are even, and* q_{l_d-2} *is odd.*

Proposition 2 ([10])**.** *We define* p'_i/q'_i *by the i-th convergent of the continued fraction of* $[a'_0; a'_1, \ldots, a'_n]$ *and* q'_i/r'_i *the i-th convergent of the continued fraction of* $[a'_1, \ldots, a'_n]$. *Then there exists d having the form of the continued fraction* $(1 + \sqrt{d})/2 = [a'_0; \overline{a'_1, \ldots, a'_{l'_d-1}, 2a'_0 - 1}]$ *if and only if one of the following two cases holds*:

(i) $q'_{l'_d-1}$ *is odd;*

(ii) *both* $q'_{l'_d-2}$ *and* $r'_{l'_d-2}$ *are odd, and* $q'_{l'_d-1}$ *is even.*

Propositions 1 and 2 give the necessary and sufficient conditions for the continued fractions of $\sqrt{d}$ (respectively $(1+\sqrt{d})/2$) to exist. We are interested in the conditions to immediately check from a given sequence $a_1, \ldots, a_{l_d-1}$ (respectively $a'_1, \ldots, a'_{l'_d-1}$) whether the continued fraction of $\sqrt{d}$ (respectively $(1+\sqrt{d})/2$) with that sequence as partial quotients exists.

3. Main Theorems

First, we consider the recurrence relation q_i and r_i in Equation (1).

Proposition 3. *For* $0 \le i \le l-2$, *we have the following recurrence relations:*

(i) $q_{l-1} = q_i q_{l-1-i} + q_{i-1} q_{l-2-i}$,

(ii) $r_{l-2} = r_i r_{l-1-i} + r_{i-1} r_{l-2-i}$.

Proof. If $i = 0$, then $q_{l-1} = q_0 q_{l-1} + q_{-1} q_{l-2}$ since $q_{-1} = 0$ and $q_0 = 1$ by Equation (1). Suppose that $q_{l-1} = q_i q_{l-1-i} + q_{i-1} q_{l-2-i}$ for $1 \le i \le l-3$. Then

$$\begin{aligned} q_{l-1} &= q_i q_{l-1-i} + q_{i-1} q_{l-2-i} \\ &= q_i(a_{l-1-i} q_{l-2-i} + q_{l-3-i}) + q_{i-1} q_{l-2-i} \\ &= q_i(a_{i+1} q_{l-2-i} + q_{l-3-i}) + q_{i-1} q_{l-2-i} \\ &= (a_{i+1} q_i + q_{i-1}) q_{l-2-i} + q_i q_{l-3-i} \\ &= q_{i+1} q_{l-2-i} + q_i q_{l-3-i} \\ &= q_{i+1} q_{l-1-(i+1)} + q_{(i+1)-1} q_{l-2-(i+1)}. \end{aligned}$$

In a similar way, one can also prove the recurrence relation (ii) for r_i. □

In particular, if l is even, we derive the following results for parity, which remain relevant for understanding the structure of these sequences.

Corollary 1. *For* $0 \le i \le l-2$, *we have the following equations for parity:*

(i) $q_{l-1} \equiv a_{l/2} q_{l/2-1} (\text{mod } 2)$,

(ii) $r_{l-2} \equiv a_{l/2} r_{l/2-1} (\text{mod } 2)$.

Proof. Substituting $i = l/2$ in the recurrence relation (i) for q_i of Proposition 3, we have

$$\begin{aligned} q_{l-1} &= q_{l/2} q_{l/2-1} + q_{l/2-1} q_{l/2-2} \\ &= q_{l/2-1}(q_{l/2} + q_{l/2-2}) \\ &= q_{l/2-1}(a_{l/2} q_{l/2-1} + 2q_{l/2-2}) \\ &\equiv a_{l/2} q_{l/2-1} (\text{mod } 2). \end{aligned}$$

In a similar way, one can also obtain the second part of Corollary 1 by using the recurrence relation for r_i of Proposition 3. $\square$

Now, we are ready to state our main theorems.

Theorem 2. (i) *If l_d is odd, there exists d having the form of the continued fraction expansion $\sqrt{d} = [a_0; \overline{a_1, \ldots, a_{l_d-1}, 2a_0}]$ if and only if q_{l_d-1} is odd.*

(ii) *If l_d is even, there exists d having the form of the continued fraction expansion $\sqrt{d} = [a_0; \overline{a_1, \ldots, a_{l_d-1}, 2a_0}]$ if and only if both $a_{l_d/2}$ and q_{l_d-1} are odd or $a_{l_d/2}$ is even.*

Proof. First, note that q_{l_d-1} and q_{l_d-2} are relatively prime by the recurrence relation for q_i in Equation (1). Therefore, if l_d is odd and q_{l_d-1} is even, then q_{l_d-2} is odd. On the other hand, $q_{l_d-1}r_{l_d-2} - r_{l_d-1}q_{l_d-2} = (-1)^{l_d-1} = 1$, which means that $q_{l_d-1}r_{l_d-2} - q^2_{l_d-2} = 1$ since $r_{l_d-1} = q_{l_d-2}$ (see (2.7) of [10]). But, if r_{l_d-2} is even, then $q_{l_d-1}r_{l_d-2} - q^2_{l_d-2} \equiv 3 \pmod 4$, which is a contradiction. It means that r_{l_d-2} should be odd. Therefore, by Proposition 1, there exists d having the form of the continued fraction expansion $\sqrt{d} = [a_0; \overline{a_1, \ldots, a_{l_d-1}, 2a_0}]$ if and only if q_{l_d-1} is odd. Next, suppose that l_d is even and $a_{l_d/2}$ is even. Then, by Corollary 1, q_{l_d-1} and r_{l_d-2} are even. It means that there exists d having the form of the continued fraction expansion $\sqrt{d} = [a_0; \overline{a_1, \ldots, a_{l_d-1}, 2a_0}]$ by Proposition 1. Suppose l_d is even and $a_{l_d/2}$ is odd. If q_{l_d-1} is even, then $q_{l_d/2-1}$ is even by part (i) of Corollary 1, which means that $r_{l_d/2-1}$ is odd because $q_{l_d/2-1}$ and $r_{l_d/2-1}$ are relatively prime. Therefore, $r_{l_d-2} \equiv a_{l_d/2}r_{l_d/2-1} \equiv 1 (\mathrm{mod}\ 2)$ and there does not exist d having the form of the continued fraction expansion $\sqrt{d} = [a_0; \overline{a_1, \ldots, a_{l_d-1}, 2a_0}]$ by Proposition 1. This completes the proof of Theorem 2. $\square$

Let us move to the case for $(1 + \sqrt{d})/2$.

Theorem 3. (i) *If l'_d is odd, there always exists d having the form of the continued fraction expansion $(1 + \sqrt{d})/2 = [a'_0; \overline{a'_1, \ldots, a'_{l'_d-1}, 2a'_0 - 1}]$.*

(ii) *If l'_d is even, there exists d having the form of the continued fraction expansion $(1 + \sqrt{d})/2 = [a'_0; \overline{a'_1, \ldots, a'_{l'_d-1}, 2a'_0 - 1}]$ if and only if $a'_{l'_d/2}$ is odd.*

Proof. If l'_d is odd and $q'_{l'_d-1}$ is odd, there always exists d having the form of the continued fraction expansion $(1+\sqrt{d})/2 = [a'_0; \overline{a'_1, \ldots, a'_{l'_d-1}, 2a'_0 - 1}]$ by Proposition 2. If l'_d is odd and $q'_{l'_d-1}$ is even, by an argument similar to the case where l_d is odd, we can check that both $q'_{l'_d-2}$ are $r'_{l'_d-2}$ are odd. Therefore, by Proposition 2, there always exists d having the form of the continued fraction expansion $(1 + \sqrt{d})/2 = [a'_0; \overline{a'_1, \ldots, a'_{l'_d-1}, 2a'_0 - 1}]$ in the case that l'_d is odd. Next, suppose l'_d is even and $a'_{l'_d/2}$ is even. Then by Corollary 1, both $q'_{l'_d-1}$ and $r'_{l'_d-2}$ are even, which means that there does not exist d having the form of the continued fraction expansion

$(1+\sqrt{d})/2 = [a'_0; \overline{a'_1, \ldots, a'_{l'_d-1}, 2a'_0 - 1}]$ by Proposition 2. If l'_d is even and $a'_{l'_d/2}$ is odd, we consider the two cases separately, that is, $q'_{l'_d/2-1}$ is odd or $q'_{l'_d/2-1}$ is even. If $q'_{l'_d/2-1}$ is odd, then $q'_{l'_d-1}$ is odd and there exists d having the form of the continued fraction expansion $(1+\sqrt{d})/2 = [a'_0; \overline{a'_1, \ldots, a'_{l'_d-1}, 2a'_0 - 1}]$ by part (i) of Corollary 1 and Proposition 2. Finally, if $q'_{l'_d/2-1}$ is even, then $q'_{l'_d-1}$ is even by part (i) of Corollary 1. But then $r'_{l'_d/2-1}$ is odd since $q'_{l'_d/2-1}$ and $r'_{l'_d/2-1}$ are relatively prime. Therefore, $r'_{l'_d-2}$ is odd and there exists d having the form of the continued fraction expansion $(1+\sqrt{d})/2 = [a'_0; \overline{a'_1, \ldots, a'_{l'_d-1}, 2a'_0 - 1}]$ by part (ii) of Corollary 1 and Proposition 2, which completes the proof of Theorem 3. □

Remark 1. The statement (i) of Theorem 3 gives the answer for the presented problem of the continued fraction expansion of $(1+\sqrt{d})/2$ in Remark 3.9 of [10]. Also, if l_d or l'_d is even, the central terms $a_{l_d/2}$ and $a'_{l'_d/2}$ of palindromic sequences have a crucial role in our conditions.

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