LEXICOGRAPHICALLY MINIMAL EXTENSION OF A FINITE BINOMID INDEX

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Abstract

Let $f = (f_n)_n$ be a sequence of positive integers. If the f-nomid coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_f := \frac{f_n f_{n-1} \cdots f_{n-k+1}}{f_k f_{k-1} \cdots f_1}$$

is an integer for all $n, k \in \mathbb{N}$ with $n \geq k$, then f is called a *binomid sequence*. A binomid sequence can be expressed as a product of p-factors $(p^{e_n})_n$ where p is a prime number. The sequence $(e_n)_n$ of nonnegative exponents or any of its finite prefixes is called a binomid index. In this paper, we discuss the problem of extending a finite binomid index into an infinite one that is lexicographically minimal. We show that these extensions are eventually periodic and form a monoid with respect to componentwise addition.

1. Introduction

Let $\mathbb{N} := \{1, 2, 3, \dots\}$. Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be a sequence given by $f = (f_n)_n = (f_1, f_2, \dots)$. For $n, k \in \mathbb{N}$ with $n \geq k$, define the *f-nomid coefficient*

$$\begin{bmatrix} n \\ k \end{bmatrix}_f = \frac{f_n f_{n-1} \cdots f_{n-k+1}}{f_k f_{k-1} \cdots f_1}.$$

We define $\begin{bmatrix} n \\ 0 \end{bmatrix}_f := 1$ for $n \geq 0$. If $\begin{bmatrix} n \\ k \end{bmatrix}_f \in \mathbb{N}$ for all $n, k \in \mathbb{N}$, then we say that f is a *binomid sequence* [6]. Binomid sequences and f-nomid coefficients have been

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studied by several authors using various terms to refer to them including Raney sequences [2] and, as mentioned in [6], Fontenè-Ward coefficients. Observe that setting $f = (n)_n = (1, 2, 3, \dots)$ recovers the usual binomial coefficients. Hence, $(n)_n$ is binomid. The following integer sequences are also binomid:

1. $F = (F_n)_n$ where F_n is the n-th Fibonacci number (see [3], where the F-nomid coefficients are called Fibonomial coefficients),
2. $G_q = \left(\sum_{i=0}^{n-1} q^i\right)_n$ where $q \in \mathbb{N}$ (see [6]),
3. any column of Pascal's triangle (see [6]),
4. a strong divisibility sequence (SDS) $(f_n)_n$ i.e., $\gcd(f_m, f_n) = f_{\gcd(m,n)}$ for all $m, n \in \mathbb{N}$ (see [1]).

If f and g are binomid, then the componentwise product fg is also binomid. On the other hand, if fg is binomid, then f and g are not necessarily binomid. For instance, if $f = (1, 1, 3, 4, 5, \dots)$ and $g = (1, 2, 1, 1, 1, \dots)$, then $fg = (1, 2, 3, 4, 5, 6, \dots)$ is binomid. However, g is not binomid since $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_g = \frac{1}{2}$.

Let $\mathbb{P}$ be the set of prime numbers. For $p \in \mathbb{P}$ and a sequence $f = (f_1, f_2, \dots)$ of positive integers, we define the *p-factor* of f as the sequence $f(p) = (p^{\nu_p(f_n)})_n$ where $\nu_p(f_n)$ is the p-adic valuation of f_n. If $f = (f_1, f_2, \dots)$ is binomid, then f has the following factorization into its p-factors:

$$f = \prod_{p \in \mathbb{P}} f(p).$$

Moreover, for each $p \in \mathbb{P}$, $f(p)$ is binomid and

$$\begin{bmatrix} n \\ k \end{bmatrix}_f = \prod_{p \in \mathbb{P}} \begin{bmatrix} n \\ k \end{bmatrix}_{f(p)}.$$

Note that if p is not prime, then $f(p)$ may not be binomid, even if f is binomid. For instance, let $f = (1, 36, 4, 9, \overline{36})$. Then $f(6) = (1, 36, 1, 1, \overline{36})$ is not binomid. Suppose $f(p) = (p^{e_1}, p^{e_2}, \dots)$ where $e_i \in \mathbb{N} \cup \{0\}$. Then

$$\begin{bmatrix} n \\ k \end{bmatrix}_{f(p)} = \frac{p^{e_n} p^{e_{n-1}} \dots p^{e_{n-k+1}}}{p^{e_k} p^{e_{k-1}} \dots p^{e_1}}$$

for all $n, k \in \mathbb{N}$ with $n \geq k$. It is clear that $f(p)$ is a binomid sequence if and only if

$$\sum_{i=n-k+1}^{n} e_i \geq \sum_{i=1}^{k} e_i.$$

Note that $(p^{e_1}, p^{e_2}, \dots)$ is binomid if and only if $(q^{e_1}, q^{e_2}, \dots)$ is also binomid for $1 \neq p, q \in \mathbb{N}$.

Definition 1. Let $\eta = (e_i)_{i<m}$ be an infinite or finite sequence of nonnegative integers, where $m \in \mathbb{N} \cup \{\infty\}$. We call η a *binomid index* if, for all k, n such that $1 \leq k \leq n < m$,

$$\sum_{i=n-k+1}^{n} e_i \geq \sum_{i=1}^{k} e_i.$$

Note that the componentwise sum of two binomid indices is a binomid index. Consider $f = (p^{e_n})_n$ with $p \in \mathbb{P}$ and $e_i \in \mathbb{N} \cup \{0\}$ such that $e_{n+1} \geq e_n$ for all $n \in \mathbb{N}$. Then $f = (p^{e_n})_n$ is a divisor-chain, i.e., $f_n \mid f_{n+1}$. By Lemma 3 (2) of [6], f is binomid.

Proposition 1. *Let $(e_i)_i$ be a monotonic increasing sequence of nonnegative integers, i.e., $e_{n+1} \geq e_n$ for all $n \in \mathbb{N}$. Then $(e_i)_i$ is a binomid index.*

In the next section, we consider a class of binomid indices that falls outside of Proposition 1 by imposing a certain property of minimality. In particular, given a finite binomid index, we show that this can be extended into an infinite binomid index that is lexicographically minimal. In Section 3, we show that this lex-minimal extension is eventually periodic. This result, interestingly, is similar to the following: an SDS (automatically, a binomid sequence) $\{u_n\}_{n\geq 0}$ with $u_0 \neq 0$ satisfying a k-th order linear recurrence is necessarily purely periodic [4]. Meanwhile, lex-minimality, in some sense, is a form an 'infinite' order linear recurrence (see Lemma 1). Moreover, we show that lex-minimality is closed under componentwise addition. In Section 4, using the framework of lex-minimal extensions, we give an alternative proof that, given a prime number p, Pascal's triangle contains infinitely many rows of zeroes (excluding the boundaries of the rows) when reduced modulo p. We also deduce a similar result for the 'triangle' arising from the Fibonacci numbers. In Section 5, we consider the reverse problem of determining whether an eventually periodic binomid sequence is a lex-minimal extension of some finite binomid index. Note that the answer is affirmative for purely periodic binomid indices. We pose a question on the structure of the space of eventually periodic binomid indices.

2. Space of Binomid Indices

Let $(\mathbb{N} \cup \{0\})^{\mathbb{N}}$ be the space of all infinite sequences of nonnegative integers. For a finite word w, let w^k denote the concatenation of k copies of w where $k \in \mathbb{N} \cup \{\infty\}$. In particular, for $n \in \mathbb{N} \cup \{0\}$, we have $n^{\infty} = (n, n, n, \dots)$ is an element of $(\mathbb{N} \cup \{0\})^{\mathbb{N}}$. Under componentwise addition, $(\mathbb{N} \cup \{0\})^{\mathbb{N}}$ is a monoid with $0^{\infty} = (0, 0, 0, \dots)$ as the identity element. With respect to the lexicographic order, $(\mathbb{N} \cup \{0\})^{\mathbb{N}}$ is totally ordered. Let $\mathbb{I}$ be the set of all infinite binomid indices. Then $\mathbb{I}$ is a totally ordered submonoid of $(\mathbb{N} \cup \{0\})^{\mathbb{N}}$. Let $\eta = (e_i)_i$ be an element of $\mathbb{I}$. Since $e_1 \leq e_i$ for all $i \in \mathbb{N}$, it follows that $\eta' = (e_i - e_1)_i$ is a binomid index. Indeed,

$$\sum_{i=n-k+1}^{n} (e_i - e_1) = \sum_{i=n-k+1}^{n} e_i - ke_1 \geq \sum_{i=1}^{k} e_i - ke_1 = \sum_{i=1}^{k} (e_i - e_1).$$

Thus, every $\eta = (e_i)_i \in \mathbb{I}$ has a decomposition $\eta = e_1^{\infty} + \eta'$, where the first coordinate of η' is zero. Let $\mathbb{I}_0$ be the subset of all elements in $\mathbb{I}$ whose first coordinate is zero. If we identify n^{∞} with $n \in \mathbb{N} \cup \{0\}$, then $\mathbb{I} = (\mathbb{N} \cup \{0\}) \oplus \mathbb{I}_0$.

For finite or infinite sequences $\eta = (e_i)_{i<m}$ and $\tau = (t_i)_{i<n}$ such that $m \leq n$ and $e_i = t_i$ for all $i < m$, we call η a prefix of τ and τ an extension of η.

Definition 2. Let η be a finite binomid index. Let $\mathbb{I}_{\eta} = \{\tau \in \mathbb{I} : \eta \text{ is a prefix of } \tau\}$. If $\tilde{\eta} \in \mathbb{I}_{\eta}$ is the minimum element of $\mathbb{I}_{\eta}$ lexicographically, then we call $\tilde{\eta}$ the *lex-minimal extension* of η.

First, we show that the lex-minimal extension of a finite binomid index always exists.

Proposition 2. *Let $\eta = (e_i)_{i<m}$ be a finite binomid index. The lex-minimal extension of η is the binomid index $\tilde{\eta} = (f_n)$ where if $n < m$, then $f_n = e_n$, and if $n \geq m$, then*

$$f_n = \max_{1 \leq k < n} \left(\sum_{i=1}^{k} f_i - \sum_{i=n-k+1}^{n-1} f_i \right),$$

with $\sum_{i=n}^{n-1} f_i = 0$ for $k = 1$.

Proof. Let $\tilde{\eta} = (f_n)$ be defined as above. Then, for $n \geq m$, we have

$$\begin{aligned}
\sum_{i=n-k+1}^{n} f_i &= f_n + \sum_{i=n-k+1}^{n-1} f_i \\
&= \max_{1 \leq k < n} \left(\sum_{i=1}^{k} f_i - \sum_{i=n-k+1}^{n-1} f_i \right) + \sum_{i=n-k+1}^{n-1} f_i \\
&\geq \left(\sum_{i=1}^{k} f_i - \sum_{i=n-k+1}^{n-1} f_i \right) + \sum_{i=n-k+1}^{n-1} f_i \\
&= \sum_{i=1}^{k} f_i.
\end{aligned}$$

Thus, the sequence $\tilde{\eta} \in \mathbb{I}_{\eta}$. The minimality of $\tilde{\eta}$ is trivial. $\square$

η	lex-minimal extension $\tilde{\eta}$	pre-period	period
(0,6,1,18,0)	$(0,6,1,18)^\infty$	0	4
(0,6,1,18,1)	$(0,6)(1,18,1,5)^\infty$	2	4
(0,6,1,18,2)	$(0,6,1,18)(2,4,2,17)^\infty$	4	4
(0,6,1,18,3)	$(0,6,1,18)(3,3,3,16)^\infty$	4	4
(0,6,1,18,4)	$(0,6,1,18,4,2,4,15,4,4,2,15)(4,4,4,13)^\infty$	12	4
(0,6,1,18,5)	$(0,6,1,18,5,1,5,14,5,5,1,14)(5,5,5,10)^\infty$	12	4
(0,6,1,18,6)	$(0,6,1,18,6,0,6,13,6,6,0,13)(6,6,6,7)^\infty$	12	4
(0,6,1,18,7)	$(0,6,1,18,7,0,6,12,7,7,0,11)(7,7,7,4)^\infty$	12	4
(0,6,1,18,8)	$(0,6,1,18,8,0,6,11,8,8,0,9)(8,8,8,1,8)^\infty$	12	5
(0,6,1,18,9)	$(0,6,1,18,9,0,6,10,9,9)(0,7,9,9,9)^\infty$	8	5
(0,6,1,18,10)	$(0,6,1,18)(10,0,6,9,10)^\infty$	4	5

Table 1: The lex-minimal extension of $(0,6,1,18,i)$ where $0 \le i \le 10$.

Remark 1.

1. For the length-one index $\eta = (n)$ where $n \in \mathbb{N} \cup \{0\}$, the lex-minimal extension is $\tilde{\eta} = n^\infty$. Meanwhile, if $\eta = (e_1, e_2)$ where $e_2 \ge e_1$, then $\tilde{\eta} = (e_1, e_2)^\infty$ (see Proposition 3). Note that both n^∞ and $(e_1, e_2)^\infty$ are purely periodic.

2. In Table 1, we give the lex-minimal extension of the finite word $\eta = (0,6,1,18,i)$ where $0 \le i \le 10$, which is lifted from an experiment that computes the lex-minimal extension of all length 5 finite word $(0, e_2, e_3, e_4, e_5)$ with $0 \le e_i \le 20$. Observe that all of the computed extensions are eventually periodic. The length of the periodic part is bounded above by 5, which is the length of η (see the proof of Theorem 1). The pre-period seems to be bounded above by 12.

   Question. In general, if η is finite of length m and its lex-minimal extension is eventually periodic, does there exist an integer constant, depending on m, that bounds the pre-period?

For $\eta = (e_i)_i \in (\mathbb{N} \cup \{0\})^\infty$, define $S(\eta) = (\sum_{j=1}^{i} e_j)_i \in (\mathbb{N} \cup \{0\})^\infty$. Then S is a lexicographic order-preserving monoid homomorphism. Moreover, the following simple characterization follows immediately from Proposition 2.

Lemma 1. *Let $\eta \in (\mathbb{N} \cup \{0\})^\infty$ and $S(\eta) = (s_n)_n$. Then η is a binomid index if and only if $s_n \ge s_{n-k} + s_k$ for $1 \le k < n$. In particular, if η is the lex-minimal extension of a finite binomid index of length m, then*

$$s_n = \max_{1 \le k < n} (s_{n-k} + s_k)$$

for all $n > m$.

3. Properties of the Lex-Minimal Extension

We begin this section by providing a sufficient condition for pure periodicity of the lex-minimal extension of a finite binomid index.

Proposition 3. *Let η be a finite binomid index. If η is monotonic increasing, then $\tilde{\eta} = \eta^\infty$.*

Proof. Let $\eta = (e_1, \dots, e_m)$ such that $e_1 \leq \cdots \leq e_m$. Let $S(\eta^\infty) = (s_n)_n$. We first show that, for $i, j < m$, we have $s_{i+j} \geq s_i + s_j$ by considering two cases.

`Case 1.` $i + j \leq m$. By the monotonicity of η, we have

$$s_i + s_j = s_i + e_1 + \cdots + e_j \leq s_i + e_{i+1} + \cdots + e_{i+j} = s_{i+j}.$$

`Case 2.` $i + j > m$. By the monotonicity of η and the pure periodicity of η^∞, we have

$$\begin{aligned} s_i + s_j &= s_i + e_1 + \cdots + e_{i+j-m} + e_{i+j-m+1} + \cdots + e_j \\ &\leq s_i + s_{i+j-m} + e_{i+1} + \cdots + e_m \\ &= s_m + s_{i+j-m} \\ &= s_{i+j}. \end{aligned}$$

Since η^∞ is purely periodic, if $q \in \mathbb{N}$ and $r \in \{0, 1, \dots, m-1\}$, then $s_{qm+r} = qs_m + s_r$. Hence, $s_{i+j} \geq s_i + s_j$ for all $i, j \geq 1$. By Lemma 2.2, η^∞ is a binomid index. Moreover, for any $k > m$, write $k = qm + r$ where $q \in \mathbb{N}$ and $0 \leq r < m$. If $r \neq 0$, then $s_k = s_{qm} + s_r$. If $r = 0$, then $s_k = s_{(q-1)m} + s_q$. Thus,

$$s_k = \max\{s_i + s_j \mid i + j = k\}.$$

Therefore, η^∞ is the lexicographically minimal extension of η. $\square$

To illustrate Proposition 3, if $\eta = (0, 2, 3, 12, 20)$, then $\tilde{\eta} = \eta^\infty$. Meanwhile, if $\eta = (0, 2, 3, 0, 2)$, then $\tilde{\eta} = (0, 2, 3)^\infty$. On the other hand, if $\eta = (0, 5, 3, 4, 5)$, then $\tilde{\eta} = \eta^\infty$. Hence, the converse of Proposition 3 does not hold.

Question. What are the solutions to $\tilde{\eta} = \eta^\infty$? (cf. Proposition 5)

Let $\eta = (e_i)_i$ be a binomid index and $S(\eta) = (s_i)_i$ be its sequence of partial sums. For $i, l \geq 1$, we define the following averages associated with η:

$$\begin{aligned} A_{i,l}(\eta) &:= \frac{s_{i+l} - s_l}{i} \\ A_{i,0}(\eta) &:= \frac{s_i}{i} \\ A_l(\eta) &:= \limsup_i A_{i,l}(\eta). \end{aligned}$$

We drop the argument η when the context is clear.

Lemma 2. *Let η be a binomid index. For $i, l \geq 1$, we have*

1. $A_{i,l} \geq A_{i,0}$
2. $A_{ki,0} \geq A_{i,0}$
3. $A_{i+l,0} = \frac{i}{i+l} A_{i,l} + \frac{l}{l+i} A_{l,0}$
4. $A_0(\eta) = \sup_i A_{i,0}$.

Proof. By Lemma 1, $s_{i+l} \geq s_l + s_i$. Then (1) follows immediately. For (2), using (1), we obtain

$$s_{ki} = s_i + \sum_{j=2}^{k} (s_{ji} - s_{(j-1)i}) = iA_{i,0} + i \sum_{j=2}^{k} A_{i,(j-1)i} \geq kiA_{i,0}.$$

Note that (3) is straightforward. For (4), first fix $i \in \mathbb{N}$. For $k \in \mathbb{N}$, we have

$$A_{i,0} \leq A_{ki,0} \leq \sup_{j \geq k} A_{j,0}.$$

Since k is arbitrary, we have $A_{i,0} \leq \limsup_k A_{k,0} = A_0(\eta)$. Thus, $\sup_i A_{i,0} \leq A_0(\eta)$. The reverse inequality is clear. □

Lemma 3. *Let $\tilde{\eta}$ be the lex-minimal extension of a finite binomid index η. Then $\tilde{\eta}$ satisfies the following Average Condition: there exists $i \in \mathbb{N}$ such that $A_0(\tilde{\eta}) = A_{i,0}(\tilde{\eta})$.*

Proof. Let $\eta = (e_j)_{j \leq l}$. Take a number $i \leq l$ such that

$$A_{i,0}(\tilde{\eta}) = \max_{k \leq l} A_{k,0}(\tilde{\eta}).$$

We show $A_{i,0}(\tilde{\eta}) = A_0(\tilde{\eta})$. Let $S(\tilde{\eta}) = (s_j)_j$. By Lemma 1, for any $m > l$, there exist $m' < m$ and $m'' < m$ such that $s_m = s_{m'} + s_{m''}$. By applying this argument repeatedly, we can show that there is a decomposition $m = \sum_{k=1}^{t} j(k)$ where $1 \leq j(k) \leq l$ and $s_m = \sum_{k=1}^{t} s_{j(k)}$. It follows that

$$s_m = \sum_{k=1}^{t} s_{j(k)} = \sum_{k=1}^{t} j(k) A_{j(k),0}(\tilde{\eta}) \leq \sum_{k=1}^{t} j(k) A_{i,0}(\tilde{\eta}) = mA_{i,0}(\tilde{\eta}).$$

Thus, $A_{m,0}(\tilde{\eta}) \leq A_{i,0}(\tilde{\eta})$. Since $m > l$ is arbitrary, it follows that

$$A_0(\tilde{\eta}) = \sup_m A_{m,0}((\tilde{\eta})) \leq A_{i,0}(\tilde{\eta}).$$ □

Theorem 1. *Let η be an infinite binomid index satisfying the Average Condition. Then η is the lex-minimal extension of some finite binomid index. Moreover, η is eventually periodic.*

Proof. Take $i \in \mathbb{N}$ satisfying $A_0(\eta) = A_{i,0}(\eta)$. Since $A_{i,0} \leq A_{ki,0} \leq A_0(\eta) = A_{i,0}$, it follows that, for any $k \in \mathbb{N}$, $s_{ki} = ks_i$. For $j < i$ and $k \in \mathbb{N}$,

$$\begin{aligned} jA_{j,(k+1)i} &= s_{(k+1)i+j} - s_{(k+1)i} \\ &\geq s_{ki+j} + s_i - (k+1)s_i \\ &= s_{ki+j} - ks_i \\ &= s_{ki+j} - s_{ki} = jA_{j,ki}. \end{aligned}$$

Moreover, since $(s_k)_k$ is increasing,

$$jA_{j,ki} = s_{ki+j} - s_{ki} \leq s_{(k+1)i} - s_{ki} = (k+1)s_i - ks_i = s_i.$$

Thus, $(jA_{j,ki})_k$ is an increasing and bounded sequence of nonnegative integers. Take a sufficiently large $\tilde{k}$ such that $A_{j,ki} = A_{j,\tilde{k}i}$ for $1 \leq j < i$ and $k \geq \tilde{k}$. It follows that

$$\begin{aligned} s_{ki+j} &= jA_{j,ki} + s_{ki} \\ &= jA_{j,\tilde{k}i} + s_{ki} \\ &= s_{\tilde{k}i+j} + (k - \tilde{k})s_i \\ &= s_{\tilde{k}i+j} + s_{(k-\tilde{k})i}. \end{aligned}$$

Clearly, $s_{ki} = s_i + s_{(k-1)i}$. Thus, for $n > \tilde{k}i$, we can find an integer k such that $s_n = s_k + s_{n-k}$. This implies that η is the lex-minimal extension of $(e_1, e_2, \ldots, e_{\tilde{k}i})$ by Lemma 1.

Finally, if $1 \leq j < i$ and $k \geq \tilde{k}$, we have

$$e_{ki+j} = s_{ki+j} - s_{ki+j-1} = s_{\tilde{k}i+j} - s_{\tilde{k}i+j-1}.$$

If $k > \tilde{k}$, we have

$$e_{ki} = s_{ki} - s_{(k-1)i+(i-1)} = s_{ki} - s_{\tilde{k}i+(i-1)} - s_{(k-1-\tilde{k})i} = s_{(\tilde{k}+1)i} - s_{\tilde{k}i+(i-1)}.$$

In the above calculation, we see that e_{ki+j} depends only on j and that

$$\eta = (e_1, e_2, \ldots, e_{\tilde{k}i})(e_{\tilde{k}i+1}, \ldots, e_{(\tilde{k}+1)i})^{\infty}. \qquad \square$$

Remark 2. Combining Lemma 3 and Theorem 1, the following are clear.

1. Let η be a finite binomid index. Then its lex-minimal extension $\tilde{\eta}$ is eventually periodic.

2. For an infinite binomid index, its lex-minimality as an extension of a finite binomid index is equivalent to the Average Condition.

For the next result, we first remark that, in general settings, the minimality (*or maximality*) of summands does not usually extend to the sum. For readers familiar with beta expansions, which generalize the decimal expansions, consider the following. Let $\beta = \frac{1+\sqrt{5}}{2}$ be the base of the beta expansion. Under a greedy process (akin to the Euclidean algorithm), the digits of the beta expansion can be generated for any real number between 0 and 1. For example, the following are (greedy) beta expansions: $\frac{3-\sqrt{5}}{2} = (0,0)(1,0)^\infty$ and $\frac{\sqrt{5}-1}{2} = (0,1)^\infty$. However, the componentwise sum $(0,0)(1,0)^\infty + (0,1)^\infty = (0)(1)^\infty$ cannot be a beta expansion because it contains the illegal digit sequence $(1,1)$ (see [5]).

This is not the case for lex-minimality. The corollary below tells us that the lex-minimal extensions form a monoid under componentwise addition.

Corollary 1. *If η_1 and η_2 are lex-minimal extensions of some binomid indices, then so is the componentwise sum $\eta_1 + \eta_2$. In other words, the space of lex-minimal extensions is a monoid under componentwise addition.*

Proof. Since binomid indices are closed under componentwise addition, $\eta_1 + \eta_2$ is a binomid index. By Lemma 3, let $j(i)$ be an index such that $A_0(\eta_i) = A_{j(i),0}(\eta_i)$ for $i = 1, 2$. Since $A_0(\eta_i) \geq A_{kl,0}(\eta_i) \geq A_{l,0}(\eta_i)$, we have $A_{j(i),0}(\eta_i) = A_{j(1)j(2),0}(\eta_i)$. Moreover, $A_{j,0}(\eta_1 + \eta_2) = A_{j,0}(\eta_1) + A_{j,0}(\eta_2)$ for all $j \in \mathbb{N}$. Thus,

$$\begin{aligned} A_0(\eta_1 + \eta_2) &\leq A_0(\eta_1) + A_0(\eta_2) \\ &= A_{j(1)j(2),0}(\eta_1) + A_{j(1)j(2),0}(\eta_2) \\ &= A_{j(1)j(2),0}(\eta_1 + \eta_2). \end{aligned}$$

This implies that $A_0(\eta_1 + \eta_2) = A_{j(1)j(2),0}(\eta_1 + \eta_2)$. Therefore, $\eta_1 + \eta_2$ is a lex-minimal extension by Theorem 1. □

Using the previous results, it is possible to obtain a more detailed description of the lex-minimal extensions, as in the next proposition.

Proposition 4. *The periodic part of a lex-minimal extension does not begin at the second coordinate.*

Proof. Let $\eta = (e_i)_{i<m}$ be a finite binomid index. Recall that $\mathbb{I} = (\mathbb{N} \cup 0) \oplus \mathbb{I}_0$. Without loss of generality, we assume that $\tilde{\eta} \in \mathbb{I}_0$, that is, the first component of η is zero. By way of contradiction, we assume that $\tilde{\eta} = 0(p_1, \ldots, p_n)^\infty$. Let $S(\tilde{\eta}) = (s_i)_i$. Set $P_l := p_1 + \cdots + p_l$ for $1 \leq l \leq n$ and set $P_0 := 0$. By periodicity, $s_{kn+l+1} = kP_n + P_l$ for $0 \leq l < n$ and $k \in \mathbb{N}$. Thus,

$$A_{kn+l+1,0} = \frac{kP_n + P_l}{kn + l + 1}.$$

This implies that $A_0(\eta) = \frac{P_n}{n}$. By Lemma 3, there is an integer i such that $A_0(\eta) = A_{i,0}$. This means that $s_i = \frac{iP_n}{n}$. Moreover, $A_{ni,0} = A_{i,0}$. This leads to a contradiction. Indeed,

$$s_{ni} = (i-1)P_n + P_{n-1} = \frac{niP_n}{n}.$$

Thus, $P_{n-1} = P_n$. In other words, $p_n = 0$. So,

$$\tilde{\eta} = 0(p_1, \ldots, p_{n-1}, 0)^\infty = (0, p_1, \ldots, p_{n-1})^\infty. \qquad \square$$

4. Pascal's Triangles of Binomial and Fibonomial Coefficients

Let $p \in \mathbb{P}$ and $\eta = (e_i)_i$ where $e_i \in \mathbb{N} \cup \{0\}$. Define $p^\eta := (p^{e_i})_i$. Let $S(\eta) = (s_i)_i$ be the sequence of partial sums of η. Then

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p^\eta} = p^{s_n - s_k - s_{n-k}}.$$

By Lemma 1, the following holds.

Corollary 2. *Let $\eta = (e_i)_{i<m}$ be a finite binomid index with $\tilde{\eta}$ as its lex-minimal extension. Then, for all $n > m$, there exists an integer k such that $1 \leq k < n$ and*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p^{\tilde{\eta}}} = 1.$$

On the other hand, suppose μ is not a lex-minimal extension of any binomid index. Then there are infinitely many n such that, for all k with $1 \leq k < n$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p^\mu} \equiv 0 \pmod{p}.$$

The above corollary gives an alternative proof of a result concerning the rows of Pascal's triangle of the usual binomial coefficients when viewed modulo a prime number.

Corollary 3. *For any prime p, there are infinitely many n such that for all k with $1 \leq k < n$,*

$$\begin{bmatrix} n \\ k \end{bmatrix} \equiv 0 \pmod{p}.$$

Proof. Let $I = (1, 2, 3, \ldots)$ be the sequence generating the usual binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}$. Let $E_i := (0, 0, \ldots, 0, 1)$ where the only nonzero element lies on the i-th position. For $p \in \mathbb{P}$, let $v_p(I)$ be the p-adic valuation of I. Then

$$v_p(I) = \sum_{i=1}^{\infty} (E_{p^i})^{\infty},$$

where $(E_{p^i})^{\infty} = E_{p^i} E_{p^i} \dots$. Clearly, $v_p(I)$ is not eventually periodic. By Lemma 3 and Theorem 1, $v_p(I)$ is not a lex-minimal extension of any binomid index. The result now follows from Corollary 2. □

We can apply the above arguments to the sequence of Fibonacci numbers.

Corollary 4. *Let $F = (F_n)_n$ be the Fibonacci sequence. For any prime p, there are infinitely many n such that for all k with $1 \le k < n$,*

$$\begin{bmatrix} n \\ k \end{bmatrix}_F \equiv 0 \pmod{p}.$$

Proof. As usual, define $F_0 = 0$. For $m \in \mathbb{N}$, the sequence $(F_n \mod m)_{n \in \mathbb{N} \cup \{0\}}$ of Fibonacci numbers modulo m is purely periodic with a period no bigger than m^2 (Theorem 1 of [7]). Let $p \in \mathbb{P}$ be fixed. For $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $p^n | F_m$ because $F_0 = 0$. This implies that the sequence $(v_p(F_n))_n$ of p-adic valuations contains an increasing infinite subsequence. As a result, the sequence $(v_p(F_n))_n$ is not eventually periodic. Thus, it cannot be the lex-minimal extension of any binomid index. □

5. Eventually Periodic Binomid Index

In this section, we consider the problem of characterizing the space of all eventually periodic binomid indices. We begin by looking at purely periodic binomid indices.

Proposition 5. *Let η be a purely periodic infinite binomid index. Then η is the lex-minimal extension of some finite binomid index.*

Proof. Let $\eta = (e_1, e_2, \dots, e_m)^{\infty}$ and let $S(\eta) = (s_i)_i$. By periodicity,

$$s_{km+j} = ks_m + s_j = kmA_{m,0} + jA_{j,0}$$

for any $j, k \ge 1$. By Lemma 2 (2), $A_{m!,0} \ge A_{j,0}$ for all $j \le m$ where $m!$ is the factorial of m. Thus,

$$(km + j)A_{km+j,0} = s_{km+j} \le (km + j)A_{m!,0}.$$

This implies that $A_0(\eta) = A_{m!,0}(\eta)$. Thus, η satisfies the Average Condition. The result now follows from Theorem 1. □

Let $\mathbb{I}$ be the set of all infinite binomid indices and $\mathbb{L}$ be the set of all lex-minimal extensions. For $l \in \mathbb{N}$, let $\sigma^l : \mathbb{I} \longrightarrow \mathbb{I}$ be the shift map given by $\sigma^l(\eta) = 0^l\eta$, where 0^l is the concatenated l copies of 0. It is easy to see that σ^l is monoid homomorphism of $\mathbb{I}$ with respect to componentwise addition. By Corollary 1, the image $\sigma^l(\mathbb{L})$ of $\mathbb{L}$ is a submonoid of $\mathbb{I}$. The space $\sigma^l(\mathbb{L})$ naturally inherits the results for $\mathbb{L}$. In particular, the sequences in $\sigma^l(\mathbb{L})$ are eventually periodic. The next example illustrates that $\mathbb{L} \neq \sigma(\mathbb{L})$.

Example 1. Since $1^\infty = (1, 1, \dots) \in \mathbb{L}$, we have $\sigma(1^\infty) = (0, 1, 1, \dots) \in \sigma(\mathbb{L})$. But $\sigma(1^\infty) \notin \mathbb{L}$. Indeed, take $\eta = 01^n$ where $n \in \mathbb{N}$. The lex-minimal extension of η is $\tilde{\eta} = \eta^\infty = (0, 1, 1, \dots, 1, 0, 1, \dots)$. Clearly, $\tilde{\eta} \neq \sigma(1^\infty)$. Thus, any finite prefix of $\sigma(1^\infty)$ cannot be extended lex-minimally to $\sigma(1^\infty)$. Thus $\sigma(1^\infty) \notin \mathbb{L}$.

Proposition 6. *If $l, m \in \mathbb{N} \cup \{0\}$ with $l \neq m$, then $\sigma^l(\mathbb{L}) \cap \sigma^m(\mathbb{L}) = \{0^\infty\}$.*

Proof. Since σ is injective, we only need to consider the case $m = 0$ and $l \geq 1$. Let $\eta \in \mathbb{L} \cap \sigma^l(\mathbb{L})$ with $\mu \in \mathbb{L}$ such that $\eta = \sigma^l(\mu)$. By Lemma 3, there are integers j and k such that

$$A_0(\eta) = A_{j,0}(\eta) \quad \text{and} \quad A_0(\mu) = A_{k,0}(\mu).$$

Let $S(\eta) = (s_i)_i$ be the sequence of partial sums of η. Then $s_1 = s_2 = \cdots = s_l = 0$ and $S(\mu) = (s_{i+l})_i$. By Lemma 2 (3), for all $i \in \mathbb{N}$, we have

$$A_0(\eta) = A_{ij,0}(\eta) \quad \text{and} \quad A_0(\mu) = A_{ik,0}(\mu).$$

Thus,

$$s_{ij} = ijA_0(\eta) \quad \text{and} \quad s_{ik+l} = ikA_0(\mu).$$

By the definition of $A_0(\eta)$, it follows that

$$A_0(\eta) \geq \limsup_i \frac{s_{ik+l}}{ik+l} = \limsup_i \frac{ikA_0(\mu)}{ik+l} = A_0(\mu).$$

Using the definition of $A_0(\mu)$, we obtain $A_0(\eta) \leq A_0(\mu)$. Therefore, $A_0(\mu) = A_0(\eta)$. By way of contradiction, assume $A_0(\eta) \neq 0$. By Lemma 2 (4), we have

$$A_0(\mu) = \sup_m A_{m,0}(\mu) \geq \frac{s_{ij}}{ij-l} = \frac{ijA_0(\eta)}{ij-l} > \frac{ijA_0(\eta)}{ij} = A_0(\eta) = A_0(\mu)$$

for any i satisfying $ij > l$. This is a contradiction. Thus, $A_0(\eta) = 0$, implying that $\eta = 0^\infty$. □

Since $\sigma^l(\mathbb{L})$ is a submonoid of the set of all eventually periodic binomid indices, we ask the following natural question.

Question. Is the set of all eventually periodic binomid indices the sum

$$\sum_{l\in\mathbb{N}\cup\{0\}} \sigma^l(\mathbb{L})?$$

Note that $\sum_{l\in\mathbb{N}\cup\{0\}} \sigma^l(\mathbb{L})$ contains many eventually periodic binomid indices. Consider $\eta = 0^{l-1}(e_1,\ldots,e_l)^\infty$. We claim that $\eta \in \sum_{l\in\mathbb{N}\cup\{0\}} \sigma^l(\mathbb{L})$; and, in particular, η is a binomid index. Let $E_i := 0^{i-1}1$. Then,

$$\begin{aligned}
\eta &= 0^{l-1}(e_1,\ldots,e_l)^\infty \\
&= 0^{l-1}(e_1,0,\ldots,0)^\infty + 0^{l-1}(0,e_2,0,\ldots,0)^\infty + \cdots + 0^{l-1}(0,\ldots,0,e_l)^\infty \\
&= (0^{l-1},e_1)^\infty + \sigma((0^{l-1},e_2)^\infty) + \cdots + \sigma^{l-1}((0^{l-1},e_l)^\infty) \\
&= e_1E_l^\infty + e_2\sigma(E_l^\infty) + \cdots + e_l\sigma^{l-1}(E_l^\infty).
\end{aligned}$$

The fact that E_l^∞ is the lex-minimal extension of E_l proves the claim.

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References

[1] S. Ando and D. Sato, On the proof of GCD and LCM equalities concerning the generalized binomial and multinomial coefficients, in *Applications of Fibonacci Numbers: Volume 4 Proceedings of The Fourth International Conference on Fibonacci Numbers and Their Applications*, Kluwer Acad. Publ., Dordrecht, Netherlands, 1990.

[2] S. Ando and D. Sato, On the generalized binomial coefficients defined by strong divisibility sequences, in *Applications of Fibonacci Numbers: Volume 8 Proceedings of The Eighth International Research Conference on Fibonacci Numbers and Their Applications*, Kluwer Acad. Publ., Dordrecht, Netherlands, 1999.

[3] V. E. Hoggatt Jr, Fibonacci numbers and generalized binomial coefficients, *Fibonacci Quart.* **5** (1967), 383-400.

[4] C. Kimberling, Strong divisibility sequences with nonzero initial term, *Fibonacci Quart.* **6** (1978), 541-544.

[5] W. Parry, On the β-expansions of real numbers, *Acta Math. Hungar.* **11** (3-4) (1960), 401-416.

[6] D. Shapiro, Divisibility properties for integer sequences, *Integers* **23** (2023), #A57.

[7] D. Wall, Fibonacci series modulo m, *Amer. Math. Monthly* **67** (6) (1960), 525-532.