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THE DISTRIBUTION OF THE NUMBER OF FIXED POINTS IN k-PERMUTATIONS

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Abstract

Let S_n be the set of all permutations of an n-element set. We investigate the sequence $a(n,k,j)$, which counts the number of permutations, $\sigma \in S_n$, that have k cycles and j fixed points. We prove that the polynomial associated with $a(n,k,j),\ 1 \leq k \leq n$, has only real zeros for each $k \in \{1, 2, \ldots, n\}$. We improve an asymptotic expansion for Stirling numbers of the first kind; then, we use it to prove the asymptotic normality of the sequence $a(n,k,j)$ in a certain range of the integer k.

1. Introduction

The set of all permutations of n objects is denoted as S_n. Let

$$\hat{s}(n,k) = \{f \in S_n\},$$

where f has k cycles in its decomposition. Then,

$$|\hat{s}(n,k)| = c(n,k) = (-1)^{n+k} s(n,k),$$

where $c(n,k)$ is the signless Stirling number of the first kind, and $s(n,k)$ is the Stirling number of the first kind.

In this paper, the number of permutations $\sigma \in \hat{s}(n,k)$ having j fixed points, $0 \leq j \leq k$, denoted $a(n,k,j)$, is investigated. First, we determine the sequence $a(n,k,j)$, as well as its generating function.

We define the random variable $X_{n,k}$ associated with the sequence $a(n,k,j)$ as

$$\Pr(X_{n,k} = j) = \frac{a(n,k,j)}{c(n,k)},\ 0 \leq j \leq k \leq n.$$

The generating function of the sequence $a(n,k,j)$ will help us to find the parameters of the random variable $X_{n,k}$.

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Usually, when the generating polynomial, $P_n(x) = \sum_{k=0}^{n} a_{n,k}x^k$, associated with a positive sequence, $(a_{n,k})_{k=0}^{n}$ (where $\sum_{k=0}^{n} a_{n,k} > 0$), is known to have only negative zeros, a central limit theorem is obtained by proving that the variance

$$\sigma_n^2 = \left(\frac{P_n^{''}(1)}{P_n(1)} + \frac{P_n^{'}(1)}{P_n(1)} - \left(\frac{P_n^{'}(1)}{P_n(1)}\right)^2\right)$$

grows infinitely with n.

In Section 3, we prove that the polynomial $P(x) = \sum_{j=0}^{k} a(n,k,j)x^j$ has only real zeros. In Theorem 8, we show that the sequence $a(n,k,j)$ is asymptotically normal by proving that

$$\lim_{n\longrightarrow\infty} \sigma_n^2 = +\infty,$$

and k satisfies $\frac{k}{\ln n} \longrightarrow +\infty,\ k < n - O(n^\alpha)$, with $0 < \alpha < 1$.

Finally, we note that the asymptotic normality of the sequence $a(n,k,j)$ can not be deduced via the theory developed in [14, 15]. This is due to the fact that there is no convenient recursion formula relating the $a(n,k,j)$; hence, the real-rootedness of the polynomial $P(x)$ is obtained by brute force, that is to say, by applying (many times) a classical result due to Schur.

2. Preliminaries

In this section, we give all the results that will be needed in this paper. All of this material may be found in [9].

2.1. Permutations

Definition 1. Let $c_1, c_2, \ldots, c_n$ be positive integers such that $\sum_{i=1}^{n} ic_i = n$. A permutation is of type $(c_1, c_2, \ldots, c_n)$ if it contains c_i cycles of length i.

The number of permutations of type $(c_1, c_2, \ldots, c_n)$ is given by the following proposition.

Proposition 1 (Cauchy). *The number of permutations of type $c = (c_1, c_2, \ldots, c_n)$ is given by*

$$P(n, c_1, c_2, \ldots, c_n) = \frac{n!}{c_1! \cdot c_2! \cdot c_3! \cdots c_n! 1^{c_1} \cdot 2^{c_2} \cdot 3^{c_3} \cdots n^{c_n}}.$$

In the next proposition, we give an infinite generating function for the sequence $a(n,k,j)$.

Proposition 2 (Cauchy)**.** *Let $P(n,k,c_1,c_2,\dots,c_n)$ be the number of permutations of type c with k cycles $\left(\sum_{i=1}^{n} c_i \ = k\right)$. The infinite generating function of $P(n,k,c_1,c_2,\dots,c_n)$ is given by*

$$\begin{aligned}\Phi(z,u,x_1,x_2,\ \dots,\ ) &= \sum_{n,k,c_1,c_2,\dots\geq 0} \frac{P(n,k,c_1,c_2,\dots,c_n)}{n!} z^n u^k x_1^{c_1} x_2^{c_2}\cdots \\ &= \sum_{n,k,c_1,c_2,\dots\geq 0} \frac{1}{c_1!\cdot c_2!\cdot c_3!\cdots 1^{c_1}\cdot 2^{c_2}\cdot 3^{c_3}\cdots} z^n u^k x_1^{c_1} x_2^{c_2}\cdots \\ &= \exp u\left(x_1 z + x_2\frac{z^2}{2}+\cdots\right),\end{aligned}$$

where $\sum_{n,k,c_1,c_2,\dots\geq 0}$ *means the multiple summation* $\sum_{n=0}^{+\infty}\sum_{k=0}^{+\infty}\sum_{c_1=0}^{+\infty}\cdots$.

Proof. Remembering that

$$P(n,c_1,c_2,\dots,c_n) = \frac{n!}{c_1!\cdot c_2!\cdot c_3!\cdots c_n! 1^{c_1}\cdot 2^{c_2}\cdot 3^{c_3}\cdots n^{c_n}},$$

$\sum_{i=1}^{n} c_i = k,$ and $\sum_{i=1}^{n} ic_i \ = n$, we obtain

$$\begin{aligned}\Phi(z,u,x_1,x_2,\ \dots,\ ) &= \sum_{n,k,c_1,c_2,\dots\geq 0} \frac{P(n,k,c_1,c_2,\dots,c_n)}{n!} z^n u^k x_1^{c_1} x_2^{c_2}\cdots \\ &= \sum_{n,k,c_1,c_2,\dots\geq 0} \frac{z^n u^k x_1^{c_1} x_2^{c_2}\cdots}{c_1!\cdot c_2!\cdot c_3!\cdots c_n! 1^{c_1}\cdot 2^{c_2}\cdot 3^{c_3}\cdots n^{c_n}} \\ &= \sum_{c_1,c_2,\dots\geq 0} \frac{z^{c_1+2c_2+3c_3\cdots} u^{c_1+c_2+c_3+\cdots} x_1^{c_1} x_2^{c_2}\cdots}{c_1!\cdot c_2!\cdot c_3!\cdots c_n! 1^{c_1}\cdot 2^{c_2}\cdot 3^{c_3}\cdots n^{c_n}} \\ &= \sum_{c_1,c_2,\dots\geq 0} \frac{z^{c_1+2c_2+3c_3\cdots} u^{c_1+c_2+c_3+\cdots} x_1^{c_1} x_2^{c_2}\cdots}{c_1!\cdot c_2!\cdot c_3!\cdots c_n! 1^{c_1}\cdot 2^{c_2}\cdot 3^{c_3}\cdots n^{c_n}} \\ &= \left(\sum_{c_1\geq 0}\frac{(ux_1 z)^{c_1}}{c_1!}\right)\left(\sum_{c_2\geq 0}\frac{(ux_2\frac{z^2}{2})^{c_2}}{c_2!}\right)\left(\sum_{c_3\geq 0}\frac{(ux_3\frac{z^3}{3})^{c_3}}{c_3!}\right)\cdots \\ &= \exp(ux_1 z).\exp\left(ux_2\frac{z^2}{2}\right)\exp\left(ux_3\frac{z^3}{3}\right)\cdots \\ &= \exp u\left(x_1 z + x_2\frac{z^2}{2}+\cdots\right).\end{aligned}$$

□

Definition 2. A permutation without any fixed point is called a *derangement*.

Let $d(n)$ be the total number of derangements of n objects. Using the inclusion–exclusion principle, one can prove that

$$d(n) = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} - \cdots + (-1)^n \frac{1}{n!}\right).$$

The number of derangements with k cycles is denoted by $d(n,k)$; this is also the number of permutations with k cycles with a length of at least 2. We have

$$d(n+1,k) = n(d(n,k) + d(n-1,k-1)),\ n \geq 1,\ \text{and } d(0,0) = 1. \tag{1}$$

The proofs of the previous relations may be found in [9].

2.2. Unimodal Log-Concave Sequences

Let us recall the following definitions and facts about unimodal sequences.

Definition 3. A real positive sequence $(a_j)_{j=0}^{n}$ is said to be *unimodal* if there exist integers k_0 and k_1, with $k_0 \leq k_1$, such that

$$a_0 \leq a_1 \leq \ldots < a_{k_0} = a_{k_0+1} = \ldots = a_{k_1} > a_{k_1+1} \geq \ldots \geq a_n.$$

The integers $j,$ where $k_0 \leq j \leq k_1$, are the *modes* of the sequence.

Another property stronger than unimodality is described in the following definition.

Definition 4. A positive sequence, $(a_j)_{j=0}^{n}$, is said to be *log-concave* if

$$a_j^2 \geq a_{j-1}a_{j+1} \text{ for } 1 \leq j \leq n-1.$$

A real sequence, $(a_j)_{j=0}^{n}$, is said to have *no internal zeros* (NIZ) if $i < j$ and $a_i,\ a_j$ are non-zero; then, $a_l \neq 0$ for every $l, i \leq l \leq j$. A NIZ log-concave sequence is obviously unimodal; however, the converse is not necessarily true. In fact, the sequence 1, 1, 3, 6, 7, 2, 1 is unimodal but not log-concave. The importance of the NIZ property is illustrated by the following example: the sequence 1, 3, 2, 0, 0, 1 is log-concave but not unimodal.

If inequalities in the log-concavity definition are strict, the sequence is said to be *strongly log-concave* (SLC), and, in this case, it has at most two consecutive modes.

One important consequence of the real-rootedness of a polynomial is given by the following classical result of Newton.

Theorem 1. *If the polynomial* $\sum_{j=0}^{n} a_j x^j$*, associated with the sequence* $(a_j)_{j=0}^{n}$, $n \geq 2$*, has only real zeros, then*

$$a_j^2 \geq \frac{j+1}{j}.\frac{n-j+1}{n-j}a_{j-1}a_{j+1},\ 1 \leq j \leq n-1. \tag{2}$$

Proof. The result is proved by induction on n. For $n = 2$, the polynomial

$$a_0 + a_1 x + a_2 x^2$$

has real zeros if and only if $\Delta = a_1^2 - 4a_0a_2 \geq 0$. This is Relation (2) for $n = 2$. Suppose now that the statement holds for $(n-1)$. Let $L(x) = \sum_{j=0}^{n} a_j x^j$ be a polynomial with only real zeros. By Rolle's theorem, its derivative $L'(x) = \sum_{j=0}^{n-1} b_j x^j$ also has only real zeros ($b_j = (j+1)a_{j+1},\ 0 \leq j \leq n-1$). Using the induction hypothesis,

$$b_j^2 \geq \frac{j+1}{j} . \frac{n-j}{n-j-1} b_{j-1} b_{j+1},\ 1 \leq j \leq n-2,$$

or

$$a_{j+1}^2 \geq \frac{j+2}{j+1} . \frac{n-j}{n-j-1} a_j a_{j+2},\ 1 \leq j \leq n-2.$$

The remaining relation, $a_1^2 \geq \frac{2n}{n-1} a_2 a_0$, is obtained by applying the induction hypothesis to $(L^r(x))'$, the derivative of $L^r(x) = \sum_{j=0}^{n} a_{n-j} x^j$, which is the reciprocal of $L(x)$. □

If the positive sequence $(a_j)_{j=0}^n,\ n \geq 2$, satisfies the hypothesis of the previous theorem, more information about it is supplied by the following corollary.

Corollary 1. *If the sequence $(a_j)_{j=0}^n$ is positive and satisfies the conditions of the previous theorem, then it is SLC, and, in this case, it has a single maximum or a plateau of two elements.*

Proof. We may suppose $a_n = 1$. So,

$$L(x) = \sum_{j=0}^{n} a_j x^j = \prod_{j=1}^{n} (x - \alpha_i).$$

Since the coefficients (a_j) are the elementary symmetric functions of α_i, then, necessarily, all α_i are negative. If $a_j = 0$ for one coefficient, then $\alpha_i = 0$ for all $1 \leq i \leq n$, because a_j is the symmetric function of order $(n-j)$ of α_i. Now, we may suppose that $a_i > 0$ for all $1 \leq i \leq n$. Newton's inequalities yield

$$a_j^2 \geq \frac{j+1}{j} . \frac{n-j+1}{n-j} a_{j-1} a_{j+1} > a_{j-1} a_{j+1},\ 1 \leq j \leq n-1.$$

The previous inequalities may be written as

$$\frac{a_1}{a_0} > \frac{a_2}{a_1} > \frac{a_3}{a_2} > \cdots > \frac{a_n}{a_{n-1}}.$$

Thus, the sequence $(a_j)_{j=0}^n$ is either decreasing (if $1 > \frac{a_1}{a_0}$) or increasing (if $\frac{a_n}{a_{n-1}} > 1$), or there exists an integer l, $(1 \leq l \leq n-1)$, such that

$$\frac{a_1}{a_0} > \frac{a_2}{a_1} > \cdots > \frac{a_l}{a_{l-1}} > 1 \geq \frac{a_{l+1}}{a_l} \cdots > \frac{a_n}{a_{n-1}}.$$

This means that the sequence is unimodal with mode l. Note that we have at most one integer l such that $\frac{a_{l+1}}{a_l} = 1$. This is the case where we have a plateau of two elements. □

3. The Sequence $a(n, k, j)$

In the following proposition, the value of $a(n, k, j)$ is explicitly given.

Proposition 3. *Let $n \geq k \geq 1$ be positive integers. The number $a(n, k, j)$ of permutations with k cycles that have j fixed points satisfies the following:*

(i) $a(n, k, j) = \binom{n}{j} d(n-j, k-j), \ 0 \leq j \leq k$;

(ii) $\displaystyle\sum_{n,k,j \geq 0} a(n, k, j) v^j u^k \frac{z^n}{n!} = \frac{e^{zu(v-1)}}{(1-z)^u}$;

(iii) $ja(n, k, j) = na(n-1, k-1, j-1), \ 1 \leq j \leq k$;

(iv) $\displaystyle\sum_{j=0}^{k} a(n, k, j) = c(n, k)$.

Proof. For (i), the number $a(n, k, j)$ is computed as follows: we choose j fixed points among n elements in $\binom{n}{j}$ ways; there remain $(n-j)$ elements, which will be placed into $(k-j)$ cycles that have a length of at least 2. This is performed in $d(n-j, k-j)$ ways. Therefore, the total number is $\binom{n}{j} d(n-j, k-j)$. The generating function of the sequence $a(n, k, j)$ is a consequence of Proposition 2,

$$\begin{aligned}
\sum_{n,k,j \geq 0} a(n, k, j) v^j u^k \frac{{}^{j}z^n}{n!} &= \Phi(z, u, v, 1, \ 1, 1, \ \ldots, \ ) \\
&= \exp\left\{ u \left(vz + \frac{z^2}{2} + \cdots \right) \right\} \\
&= \exp\left(uvz + u \left(\ln \frac{1}{1-z} - z \right) \right).
\end{aligned}$$

For (iii), note that for $j \geq 1$ we have

$$\begin{aligned}a(n,k,j) &= \binom{n}{j} d(n-j,k-j) = \frac{n}{j}\binom{n-1}{j-1} d(n-j,k-j)\\ &= \frac{n}{j}\binom{n-1}{j-1} d(n-1-(j-1),k-1-(j-1))\\ &= \frac{n}{j} a(n-1,k-1,j-1).\end{aligned}$$

Relation (iv) is obvious. This concludes the proof. □

The aim of the following section is to prove that the polynomial $P(x)$ has only real zeros. The proof is based on two results. The next theorem is due to Schur (a proof of it may be found in [20]). The second one concerns the reality of zeros of the generating polynomial associated with the number of derangements.

Theorem 2 (Schur)**.** *Let* $\sum_{k=0}^{n} a_k x^k$ *and* $\sum_{k=0}^{m} c_k x^k$ *be two real polynomials having only real zeros. Suppose that all the zeros of one of them are on the same side of the real axis; then, the polynomial* $\sum_{k=0}^{d} k! a_k c_k x^k$ *has only real zeros, where* $d = \min(n,m)$.

The second result we need is as follows.

Theorem 3. *For every integer* $n \geq 2$*, the polynomial* $D_n(x) = \sum_{k=1}^{n} d(n+k,k)x^{k-1}$ *has only real zeros.*

Proof. We proceed by induction on n. For $n = 2$, the polynomial reduces to

$$D_2(x) = d(3,1) + d(4,2)x = 2 + 3x,$$

and the result holds trivially. Suppose the result holds for $n \geq 2$, and consider

$$D_{n+1}(x) = \sum_{k=1}^{n+1} d(n+k+1,k)x^{k-1}.$$

Using Equation (1), we obtain

$$\begin{aligned}D_{n+1}(x) &= \sum_{k=1}^{n+1} (n+k)(d(n+k,k) + d(n+k-1,k-1))x^{k-1}\\ &= ((n+2)x + n + 1))D_n(x) + x(x+1)D_n^{'}(x).\end{aligned}$$

Let

$$H_n(x) = (x+1)x^{n+1}D_n(x).$$

By the induction hypothesis, the polynomial H_n has $2n+1$ real zeros. By Rolle's theorem, $H_n^{'}$ has $2n$ real zeros; however, $H_n^{'}(x) = x^n D_{n+1}(x)$, and the degree of the polynomial D_{n+1} is n. This means that all the zeros of D_{n+1} are real. □

The following theorem constitutes the principal result of this section.

Theorem 4. *Let $n \geq k \geq 1$ be two positive integers. Then, the polynomial $P(x)$ has only real zeros.*

Proof. First, suppose $n = k$. Then, $a(n,n,j) = \binom{n}{j} d(n-j, n-j) = 0$ except for $j = n$. Thus,

$$P(x) = \sum_{j=0}^{k} a(n,k,j)x^j = x^n,$$

and its zeros are real. We know that $d(n,k) = 0$ if $n < 2k$. It follows then that $a(n,k,j) = 0$ for $n - j < 2(k-j)$ or $j < 2k - n$. For this, we consider two cases.

`Case 1:` $n - k \geq k$. In this case, $a(n,k,j) \neq 0$ for all j, and $0 \leq j \leq k-1$. So, using Theorem 3, the polynomial

$$D_l(x) = \sum_{j=1}^{l} d(l+j, j)x^{j-1} = \sum_{j=0}^{l-1} d(l+j+1, j+1)x^j$$

has only real zeros for every $l \geq 2$. Theorem 2 can be applied to $D_{n-k}(x)$ and $(x+1)^{k-1}$ to obtain the polynomial

$$\phi(x) = \sum_{j=0}^{k-1} \frac{d(n-k+j+1, j+1)}{(k-j-1)!} x^j,$$

which has only real zeros. Its reciprocal polynomial, $\phi_r(x) = \sum_{j=0}^{k-1} \frac{d(n-j,k-j)}{j!} x^j$, has this property too. Once again, Theorem 2 can be applied to $\phi_r(x)$ and $(x+1)^n$. The resulting polynomial is $P(x)$, which has only real zeros.

`Case 2:` $n - k < k$. In this case, the coefficients $a(n,k,j)$ equal 0 for $j < 2k - n$. The polynomial $P(x) = \sum_{j=0}^{k} a(n,k,j)x^j$ becomes

$$P(x) = \sum_{j=2k-n}^{k-1} a(n,k,j)x^j.$$

Theorem 2 applied to $D_{n-k}(x)$ and $(x+1)^{k-1}$ gives the polynomial

$$h(x) = \sum_{j=0}^{n-k-1} \frac{d(n-k+j+1, j+1)}{(k-j-1)!} x^j,$$

which has only real zeros. The same property holds for its reciprocal polynomial

$$h_r(x) = \sum_{j=0}^{n-k-1} \frac{d(2(n-k)-j, n-k-j)}{(2k-n+j)!} x^j.$$

Apply Theorem 2 to $h_r(x)$ and $(x+1)^{2n-2k}$. We obtain the polynomial

$$g(x) = \sum_{j=0}^{n-k-1} \frac{d(2(n-k)-j, n-k-j)}{(2n-2k-j)!(2k-n+j)!} x^j,$$

which has only real zeros. This completes the proof since $P(x) = n! x^{2k-n} g(x)$. □

The following corollary arises as a direct consequence of the previous theorem.

Corollary 2. *The sequence $(a(n,k,j))_{j=0}^{k}$ is SLC in j, and it is unimodal with a peak or a plateau with two elements.*

4. A Central Limit Theorem for $a(n,k,j)$

In what follows, we study the distribution of the fixed points in the set of k-permutations. For this, consider the family of random variables $(X_{n,k})_{1\le k\le n}$ on the set $\hat{s}(n,k)$ of k-permutations defined by

$$\Pr(X_{n,k} = j) = \frac{a(n,k,j)}{c(n,k)},\ 0 \le j \le k.$$

We use a variant of Lindeberg's theorem to establish the asymptotic normality of the sequence $(a(n,k,j))_j$. Proposition 3 is needed to compute the mean and the variance of the random variable $X_{n,k}$. For a certain range of $k = k(n)$, the variance becomes infinitely large, ensuring the applicability of Lindeberg's theorem. It is pertinent to recall the relevant definitions.

Definition 5. A positive real sequence $(b(n,k))_{k=0}^{n}$, with $B_n = \sum_{k=0}^{n} b(n,k) \neq 0$, is said to *satisfy a central limit theorem* (or is *asymptotically normal*) with *mean μ_n and variance σ_n^2*, if

$$\lim_{n\longrightarrow+\infty} \sup_{x\in\mathbb{R}} \left| \sum_{0\le k\le \mu_n + x\sigma_n} \frac{b(n,k)}{B_n} - (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt \right| = 0.$$

The sequence *satisfies a local limit theorem on $I \subseteq \mathbb{R}$, with mean μ_n and variance σ_n^2*, if

$$\lim_{n\longrightarrow+\infty} \sup_{x\in I} \left| \frac{\sigma_n b(n, \mu_n + x\sigma_n)}{B_n} - (2\pi)^{-1/2} e^{-\frac{x^2}{2}} \right| = 0.$$

The following theorem is a consequence of the Lindeberg central limit theorem; for details, see [8].

Theorem 5. *Let $(Q_n)_{n\geq 1}$ be a sequence of real polynomials with only real negative zeros. The sequence of the coefficients of the $(Q_n)_{n\geq 1}$ satisfies a central limit theorem with $\mu_n = \frac{Q'_n(1)}{Q_n(1)}$ and $\sigma_n^2 = \left(\frac{Q''_n(1)}{Q_n(1)} + \frac{Q'_n(1)}{Q_n(1)} - \left(\frac{Q'_n(1)}{Q_n(1)}\right)^2\right)$ provided that $\lim_{n\longrightarrow+\infty} \sigma_n^2 = +\infty$. If, in addition, the sequence of the coefficients of each Q_n has no internal zeros, then the sequence of the coefficients satisfies a local limit theorem on $\mathbb{R}$.*

Let us evaluate the mean and the variance of $X_{n,k}$. We have the following.

Proposition 4. *The mean $\mu_{n,k}$ and the variance $\sigma_{n;k}^2$ of the random variable $X_{n,k}$ are given by,*

$$\begin{aligned}
\mu_{n,k} &= n\frac{c(n-1,k-1)}{c(n,k)}\\
\sigma_{n,k}^2 &= n\frac{c(n-1,k-1)}{c(n,k)}\left(1+(n-1)\frac{c(n-2,k-2)}{c(n-1,k-1)} - n\frac{c(n-1,k-1)}{c(n,k)}\right)\\
&= \mu_{n,k}\left(1+\mu_{n-1,k-1}-\mu_{n;k}\right).
\end{aligned}$$

Proof. Consider Assertion (3) of Proposition 3: $ja(n,k,j) = na(n-1,k-1,j-1)$. Summing over j, we obtain

$$\sum_{j=1}^{k} ja(n,k,j) = \sum_{j=1}^{k} na(n-1,k-1,j-1) = nc(n-1,k-1);$$

then, we obtain

$$\mu_{n,k} = \frac{\sum_{j=1}^{k} ja(n,k,j)}{\sum_{j=0}^{k} a(n,k,j)} = n\frac{c(n-1,k-1)}{c(n,k)}.$$

Recall that

$$\begin{aligned}
\sigma_{n;k}^2 &= \sum_{j\geq 0}(\mu_{n,k}-j)^2 \Pr(X_{n,k}=j)\\
&= -\mu_{n,k}^2 + \sum_{j\geq 0} j^2 \Pr(X_{n,k}=j).
\end{aligned}$$

To evaluate $\sum_{j\geq 1} j^2 a(n,k,j)$, differentiate the generating function in Proposition 3 with respect to v. We obtain

$$\sum_{n,k\geq 0}\left(\sum_{j\geq 1} j a(n,k,j) v^{j-1}\right) u^k \frac{z^n}{n!} = \frac{zue^{zu(v-1)}}{(1-z)^u}.$$

Multiplying by v and differentiating again with respect to v yields

$$\sum_{n,k\geq 0}\left(\sum_{j\geq 1} j^2 a(n,k,j) v^{j-1}\right) u^k \frac{z^n}{n!} = \frac{zue^{zu(v-1)} + z^2u^2ve^{zu(v-1)}}{(1-z)^u}.$$

Let $v=1$ in the previous relation; one has

$$\sum_{n,k\geq 0}\left(\sum_{j\geq 1} j^2 a(n,k,j)\right) u^k \frac{z^n}{n!} = \frac{zu + z^2u^2}{(1-z)^u}.$$

Equating the coefficients of $u^k \frac{z^n}{n!}$ on both sides gives

$$\sum_{j\geq 1} j^2 a(n,k,j) = nc(n-1,k-1) + n(n-1)c(n-2,k-2).$$

Finally,

$$\begin{aligned}
\sigma^2_{n;k} &= -\mu^2_{n,k} + \sum_{j\geq 0} j^2 \Pr(X_{n,k}=j) \\
&= -\left(n\frac{c(n-1,k-1)}{c(n,k)}\right)^2 + \frac{nc(n-1,k-1)}{c(n,k)} + \frac{n(n-1)c(n-2,k-2)}{c(n,k)} \\
&= \frac{nc(n-1,k-1)}{c(n,k)}\left(1 - \frac{nc(n-1,k-1)}{c(n,k)} + \frac{(n-1)c(n-2,k-2)}{c(n-1,k-1)}\right) \\
&= \mu_{n,k}\left(1 - \mu_{n,k} + \mu_{n-1,k-1}\right).
\end{aligned}$$

The proof is concluded. □

In order to apply the preceding theorem, we need explicit equivalents of $\mu_{n,k}$ and $\sigma^2_{n,k}$. To this end, we use an asymptotic expansion of $c(n,k)$ due to Moser and Wyman. When k is small or very large we obtain a degenerate law. For an intermediate value of k we obtain a normal law. We recall the definition of a degenerate law.

Definition 6. A random variable X is *degenerate* if $P(X=a)=1$ for some real constant a.

In the following theorem, we show that the sequence of random variables $(X_{n,k})$ is degenerate for the extreme values of $k = k(n)$ (small and large values of k, with $n \longrightarrow +\infty$). It is noteworthy that convergence in probability is stronger than convergence in distribution. However, if the the limit is constant (degenerate) then convergence in distribution implies convergence in probability.

Theorem 6. *The sequence* $(X_{n,k})$ *of random variables is degenerate in the two following cases.*
1) *For* $k = o(\ln n)$ *or* $n - o(n^\alpha) \le k \le n,\ 0 < \alpha < 1/2$, $(X_{n,k})$ *converges in probability to a degenerate law at* 0.
2) *If* k *is large enough and* $\lim_{n\longrightarrow+\infty} \frac{(n-k)^2}{k} = 0$ *then* $(X_{n,k})$ *converges in probability to a degenerate law at* n.

Proof. For $k = o(\ln n)$, $c(n,k) \sim \frac{(n-1)!(\ln n+\gamma)^{k-1}}{(k-1)!}$ (see [19]). We deduce that

$$\mu_{n,k} \sim \frac{k}{\ln n} \longrightarrow 0.$$

Consequently, $P(X_{n,k} = 0) = (1 + o(1)) \sim 1$. This is expected. Indeed, if k is small and n large enough, there is no place for fixed points. If $n - o(n^\alpha) \le k,\ 0 < \alpha < 1/2$, the asymptotic expansion of $c(n,k)$ in this range is given by (see [19])

$$c(n,k) \sim \binom{n}{k} \left(\frac{k}{2}\right)^{n-k}.$$

Since $n - o(n^\alpha) \le k,\ 0 < \alpha < 1/2$, it follows that $\lim_{n\longrightarrow+\infty} \frac{(n-k)^2}{k} = 0$; in this case, we have

$$\mu_{n,k} \sim k\left(1 - \frac{1}{k}\right)^{n-k} \longrightarrow k.$$

Thus, $P(X_{n,k} = k) = (1 + o(1)) \sim 1$. In this situation, the result is expected; if k is large, almost all cycles have a length of one, that is, they are fixed points. □

For k such that $\frac{k}{\ln n} \longrightarrow +\infty$ as $n \longrightarrow +\infty$ and $k \le n - O(n^\alpha),\ 0 < \alpha < 1$ by the work of Moser and Wyman (see [19, Equation 5.7]) provided the first two terms of a formula are convenient for calculations. More precisely, they gave

$$c(n,k) \simeq \frac{n!u^k}{k!(1-e^{-u})^n\sqrt{2\pi k K_1}}\left(1 + \frac{1}{k}\left(\frac{K_3}{8K_1^2} - \frac{5K_2^2}{24K_1^3}\right)\right),$$

with

$$\begin{aligned}
\frac{e^u - 1}{u} &= \frac{n}{k} = \lambda,\\
K_1 &= \lambda(e^u - \lambda)),\\
K_2 &= \lambda\left(2\lambda^2 - (3\lambda+1)e^u + 2e^{2u}\right),\\
K_3 &= \lambda\left(-6\lambda^3 - (12\lambda^2 + 4\lambda + 1)e^u - (11\lambda + 6)e^{2u} + 6e^{3u}\right).
\end{aligned}$$

In the next theorem, employing the same method (and the same notation) as in [19], we give an asymptotic formula for $c(n,k)$ of order three. This result is important on its own, since, in the proof of Theorem 7, we give a complete asymptotic expansion of the Stirling numbers of the first kind (which can be compared with [16, 18, 19, 21, 22]).

Theorem 7 ([19])**.** *For n and k such that $\frac{k}{\ln n} \longrightarrow +\infty$, $n \longrightarrow +\infty$ and $k \leq n - O(n^{\alpha})$, $0 < \alpha < 1$, we have*

$$c(n,k) = \frac{n!u^k}{k!(1-e^{-u})^n\sqrt{2\pi k K_1}}\left(1 + \frac{b_1}{k} + \frac{b_2}{k^2} + \frac{b_3}{k^3} + o\left(\frac{1}{k^3}\right)\right),$$

where u is the unique positive real root of $\dfrac{e^u-1}{u} = \dfrac{n}{k} = \lambda$, and

$$\begin{aligned}
b_1 &= \frac{K_3}{8K_1^2} - \frac{5K_2^2}{24K_1^3},\\
b_2 &= \frac{35K_3^2}{384K_1^4} + \frac{7K_2K_4}{48K_1^4} - \frac{K_5}{48K_1^3} - \frac{35K_2^2K_3}{96K_1^5} + \frac{385K_2^4}{1152K_1^6},\\
b_3 &= \frac{K_7}{384K_1^4} - \frac{20K_2K_6 + 35K_3K_5 + 21K_4^2}{640K_1^5} + \frac{77K_2^2K_5}{384K_1^6} + \frac{77K_2K_3K_4}{128K_1^6} + \frac{385K_3^3}{3072K_1^6}\\
&\quad - \left(\frac{5005K_2^2K_3^2}{3072K_1^7} + \frac{1001K_2^3K_4}{1152K_1^7}\right) + \frac{25025K_2^4K_3}{9216K_1^8} - \frac{85085K_2^6}{82944K_1^9}.
\end{aligned}$$

The constants K_i, $1 \leq i \leq 7$, are given by

$$\begin{aligned}
K_1 &= \lambda(e^u - \lambda);\\
K_2 &= \lambda(2\lambda^2 - (3\lambda+1)e^u + 2e^{2u});\\
K_3 &= \lambda(-6\lambda^3 + (12\lambda^2 + 4\lambda + 1)e^u - (11\lambda + 6)e^{2u} + 6e^{3u});\\
K_4 &= \lambda\{24\lambda^4 - (60\lambda^3 + 20\lambda^2 + 5\lambda + 1)e^u + (70\lambda^2 + 40\lambda + 14)e^{2u}\\
&\qquad -(50\lambda + 36)e^{3u} + 24e^{4u}\};\\
K_5 &= \lambda\{-120\lambda^5 + (360\lambda^4 + 110\lambda^3 + 30\lambda^2 + 6\lambda + 1)e^u\\
&\qquad +(510\lambda^3 + 300\lambda^2 + 109\lambda + 30)e^{2u} + (450\lambda^2 + 345\lambda + 150)e^{3u}\\
&\quad +(274\lambda + 240)e^{4u} + 120e^{5u}\};\\
K_6 &= \lambda\{720\lambda^6 - (\lambda^5 + 840\lambda^4 + 210\lambda^3 + 42\lambda^2 + 7\lambda + 1)e^u\\
&\quad +(4200\lambda^4 + 2520\lambda^3 + 938\lambda^2 + 266\lambda + 62)e^{2u}\\
&\quad -(4410\lambda^3 + 3542\lambda^2 + 1624\lambda + 540)e^{3u} + (3248\lambda^2 + 3066\lambda + 1560)e^{4u}\\
&\quad -(1764\lambda + 1800)e^{5u} + 760e^{6u}\};
\end{aligned}$$

$$\begin{aligned}K_7 = \lambda\{&-5040\lambda^7 + (20160\lambda^6 + 6720\lambda^5 + 1680\lambda^4 + 336\lambda^3 + 56\lambda^2 + 8\lambda + 1)e^u \\ &- (38640\lambda^5 + 23520\lambda^4 + 8904\lambda^3 + 2576\lambda^2 + 615\lambda + 126)e^{2u} \\ &+ (47040\lambda^4 + 38976\lambda^3 + 18564\lambda^2 + 6476\lambda + 1806)e^{3u} \\ &- (40614\lambda^3 + 40376\lambda^2 + 21944\lambda + 8400)e^{4u} + (26264\lambda^2 + 29016\lambda + 16800)e^{5u} \\ &- (13068\lambda + 15120)e^{6u} + 5040e^{7u}\}.\end{aligned}$$

Proof. Using the generating function

$$\sum_{n\geq k}(-1)^n s(n,k)\frac{z^n}{n!} = \frac{\ln^k(1-z)}{k!},$$

and the Cauchy formula, we obtain

$$s(n,k) = \frac{(-1)^n n!}{2\pi i k!}\int_\Gamma \frac{\ln^k(1-z)}{z^{n+1}}dz,$$

where Γ is a circle around the origin, and its radius will be determined later. Let $z = re^{i\theta}$. Then,

$$\begin{aligned}s(n,k) &= \frac{(-1)^n n!}{2\pi k! r^n}\int_{-\pi}^{\pi} \frac{\ln^k(1-re^{i\theta})}{e^{in\theta}}d\theta \\ &= \frac{(-1)^n n!}{2\pi k! r^n}\int_{-\pi}^{\pi} \exp(k\ln(\ln(1-re^{i\theta})) - in\theta)d\theta \\ &= \frac{(-1)^n n!}{2\pi k! r^n}\int_{-\pi}^{\pi} \exp(F(\theta)d\theta,\end{aligned}$$

where $F(\theta) = k\ln(\ln(1-re^{i\theta})) - in\theta$.

In order to find an asymptotic equivalent of $c(n,k)$, we use the saddle point method: the value of the integral is independent of the path of integration. We choose one that passes through (or near) a saddle point z_0 ($F'(z_0) = 0$, $F(z_0) \neq 0$) and along a neighbourhood of z_0, the imaginary part of F, denoted $\operatorname{Im} F(z)$, is constant. By this choice, the saddle point corresponds to a local maximum in this neighborhood. Thus, the major contribution to the integral essentially comes from the small part of the path containing z_0. For a detailed discussion of this method, see ([11], Chapter VIII). The calculations are very long as shown by the first few derivatives of F:

$$\begin{aligned}
F'(\theta) &= -\frac{ikre^{i\theta}}{(1-re^{i\theta})\ln(1-re^{i\theta})} - in;\\
F''(\theta) &= k\frac{re^{i\theta}(1-re^{i\theta})\ln(1-re^{i\theta}) + r^2e^{2i\theta}\ln(1-re^{i\theta}) + r^2e^{2i\theta}}{(1-re^{i\theta})^2\ln^2(1-re^{i\theta})};\\
F^{(3)}(\theta) &= \frac{ikre^{i\theta}}{(1-re^{i\theta})\ln(1-re^{i\theta})} + \frac{3ikr^2e^{2i\theta}}{(1-re^{i\theta})^2\ln(1-re^{i\theta})}\\
&\quad + \frac{3ikr^2e^{2i\theta}}{(1-re^{i\theta})^2\ln^2(1-re^{i\theta})} + \frac{2ikr^3e^{3i\theta}}{(1-re^{i\theta})^3\ln(1-re^{i\theta})}\\
&\quad + \frac{3ikr^3e^{3i\theta}}{(1-re^{i\theta})^3\ln^3(1-re^{i\theta})} + \frac{2ikr^2e^{3i\theta}}{(1-re^{i\theta})^3\ln^3(1-re^{i\theta})}.
\end{aligned}$$

The fourth derivative of F, with respect to θ, is

$$\begin{aligned}
F^{(4)}(\theta) &= -\frac{kre^{i\theta}}{(1-re^{i\theta})\ln(1-re^{i\theta})} - \frac{7kr^2e^{2i\theta}}{(1-re^{i\theta})^2\ln(1-re^{i\theta})}\\
&\quad - \frac{7kr^2e^{2i\theta}}{(1-re^{i\theta})^2\ln^2(1-re^{i\theta})} - \frac{12kr^3e^{3i\theta}}{(1-re^{i\theta})^3\ln(1-re^{i\theta})}\\
&\quad - \frac{18kr^3e^{3i\theta}}{(1-re^{i\theta})^3\ln^2(1-re^{i\theta})} - \frac{12kr^3e^{3i\theta}}{(1-re^{i\theta})^3\ln^3(1-re^{i\theta})}\\
&\quad - \frac{6kr^4e^{4i\theta}}{(1-re^{i\theta})^4\ln(1-re^{i\theta})} - \frac{11kr^4e^{4i\theta}}{(1-re^{i\theta})^4\ln^2(1-re^{i\theta})}\\
&\quad - \frac{12kr^4e^{4i\theta}}{(1-re^{i\theta})^4\ln^3(1-re^{i\theta})} - \frac{6kr^4e^{4i\theta}}{(1-re^{i\theta})^4\ln^4(1-re^{i\theta})}.
\end{aligned}$$

The radius r is chosen such that $F^{'}(0) = 0$, or, explicitly,

$$-\frac{kr}{(1-r)\ln(1-r)} = n.$$

The equation $F^{'}(0) = F^{'}(r) = 0$ is equivalent to

$$\frac{e^u - 1}{u} = \frac{n}{k} = \lambda, \text{ with } r = 1 - e^{-u}.$$

The value of $F''(0)$ is given by

$$\begin{aligned}
F''(0) &= k\frac{r(1-r)\ln(1-r) + r^2\ln(1-r) + r^2}{(1-r)^2\ln^2(1-r)}\\
&= \frac{kr}{(1-r)\ln(1-r)} + \frac{kr^2}{(1-r)^2\ln(1-r)} + \frac{kr^2}{(1-r)^2\ln^2(1-r)}\\
&= -n + \frac{n^2}{k}\ln(1-r) + \frac{n^2}{k} = -n - \frac{n^2}{k}u + \frac{n^2}{k} = -kK_1.
\end{aligned}$$

Using the notation of the theorem, we obtain

$$\begin{aligned}
F^{(3)}(0) &= -\lambda(2\lambda^2 - (3\lambda+1)e^u + 2e^{2u})ik = -ikK_2;\\
F^{(4)}(0) &= \left(\lambda(-6\lambda^2 + (12\lambda^2+4\lambda+1)e^u - (11\lambda+6)e^{2u} + 6e^{3u})\right)k = kK_3;\\
F^{(5)}(0) &= -ikK_4;\\
&\ \ \vdots
\end{aligned}$$

We write

$$\begin{aligned}
\int_{-\pi}^{\pi} \exp(F(\theta))d\theta &= \int_{-\pi}^{-\epsilon} \exp(F(\theta))d\theta + \int_{-\epsilon}^{\epsilon} \exp(F(\theta))d\theta + \int_{\epsilon}^{\pi} \exp(F(\theta))d\theta\\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where $\epsilon = \ln k/\sqrt{k}$. We will prove that I_1 and I_3 are negligible, and then, the major contribution to the integral comes from I_2. The function $|\exp(F(\theta))|$ attains its (unique) maximum at $\theta = 0$. In addition, $\exp(F(\theta))$ is strictly decreasing in the interval $[0, \epsilon]$ since, around $\theta = 0$, F is real and well approximated by $F(0) + \frac{F''(0)}{2}\theta^2$ (recall that $F''(0) < 0$). We have

$$\begin{aligned}
\operatorname{Re}\left(e^{F(\theta)}\right) &= \exp\left(\frac{k}{2}\left(\ln\left(\frac{1}{4}\ln^2\left(1 - 2r\cos(\theta) + r^2\right) + \arctan^2\left(\frac{-r\sin\theta}{2 - r\cos\theta}\right)\right)\right)\right)\\
&\quad \times \cos\left(k\arctan\left(\frac{2\arctan\left(\frac{-r\sin\theta}{2-r\cos\theta}\right)}{\ln^2\left(1 - 2r\cos(\theta) + r^2\right)}\right) - n\theta\right).
\end{aligned}$$

Let $\operatorname{Re}(\exp(F(\theta))) = g(\theta)\exp(k\phi(\theta))$. Then,

$$|I_3| \le \int_{\epsilon}^{\pi} |\exp(F(\theta))|\, d\theta = \int_{\epsilon}^{\pi} g(\theta)\exp(k\phi(\theta))d\theta,$$

and since F has a *unique* critical point in $[0,\ \pi]$, the function $g(\theta)\exp(k\phi(\theta))$ is monotonically decreasing in $[\epsilon,\ \pi]$. An integration by parts yields

$$\begin{aligned}
\int_{\epsilon}^{\pi} \exp(\operatorname{Re} F(\theta))d\theta &= \int_{\epsilon}^{\pi} g(\theta)\exp(k\phi(\theta))d\theta\\
&= \frac{g(\theta)}{k\phi'(\theta)}\exp(k\phi(\theta))\Big|_{\epsilon}^{\pi} - \frac{1}{k}\int_{\epsilon}^{\pi} \frac{d}{d\theta}\left(\frac{g(\theta)}{\phi'(\theta)}\right)\exp(k\phi(\theta))d\theta\\
&= \frac{g(\pi)}{k\phi'(\pi)}\exp(k\phi(\pi)) - \frac{g(\epsilon)}{k\phi'(\epsilon)}\exp(k\phi(\epsilon))\\
&\quad - \frac{1}{k}\int_{\epsilon}^{\pi} \frac{d}{d\theta}\left(\frac{g(\theta)}{\phi'(\theta)}\right)\exp(k\phi(\theta))d\theta\\
&= O\left(\frac{1}{k}\right),
\end{aligned}$$

because all the terms are bounded. It follows that $\int_{\epsilon}^{\pi} \exp(\operatorname{Re} F(\theta))d\theta$ is negligible (as well as $\int_{-\pi}^{-\epsilon} \exp(\operatorname{Re} F(\theta))d\theta$). Thus,

$$s(n,k) \simeq \frac{(-1)^n n!}{2\pi k! r^n} \int_{-\epsilon}^{\epsilon} \exp(F(\theta))d\theta. \tag{3}$$

The next step is to evaluate the expression on the right-hand side of Relation (3). For this, expand the function $F(\theta)$ about $\theta = 0$ at any order $l \geq 2$.

We use the notation $a_i = \dfrac{F^{(i)}(0)}{i!}$. Relation (3) is now

$$\begin{aligned}
s(n,k) &\simeq \frac{(-1)^n n!}{2\pi k! r^n} \int_{-\epsilon}^{\epsilon} \exp(F(\theta))d\theta \\
&= \frac{(-1)^n n!}{2\pi k! r^n} \int_{-\epsilon}^{\epsilon} \exp\left(F(\theta) - F(0) + F(0)\right) d\theta \\
&= \frac{(-1)^n n! \exp\left(F(0)\right)}{2\pi k! r^n} \int_{-\epsilon}^{\epsilon} \exp\left(F(\theta) - F(0)\right) d\theta.
\end{aligned}$$

Since $c(n,k) = (-1)^{n+k} s(n,k)$ and $\exp\left(F(0)\right) = (-u)^k$, we have

$$\begin{aligned}
c(n,k) &\simeq \frac{n! u^k}{2\pi k! r^n} \int_{-\epsilon}^{\epsilon} \exp\left(F(\theta) - F(0)\right) d\theta \\
&= \frac{n! u^k}{2\pi k!(1-e^{-u})^n} \int_{-\epsilon}^{\epsilon} \exp\left(\sum_{j=2}^{l} a_j \theta^j + O(\theta^{l+1})\right) d\theta \\
&= \frac{n! u^k}{2\pi k!(1-e^{-u})^n} \int_{-\epsilon}^{\epsilon} \exp(a_2\theta^2) \exp\left(\sum_{j=3}^{l} a_j \theta^j + O(\theta^{l+1})\right) d\theta \\
&= \frac{n! u^k}{2\pi k!(1-e^{-u})^n} \left(\int_{-\epsilon}^{\epsilon} \exp(a_2\theta^2) \left(1 + \sum_{i=1}^{l} \frac{\left(\sum_{j=3}^{l} a_j \theta^j\right)^i}{i!}\right) d\theta + O(\theta^{l+1}) \right).
\end{aligned}$$

If we rearrange the sum in the last integral and drop the terms of order greater than $l+1$, we obtain

$$c(n,k) = \frac{n! u^k}{2\pi k!(1-e^{-u})^n} \left(\int_{-\epsilon}^{\epsilon} \exp(a_2\theta^2) \left(1 + \sum_{j=3}^{l} c_j \theta^j\right) d\theta + O(\theta^{l+1}) \right),$$

where

$$a_2 = -\frac{kK_1}{2},\ c_3 = a_3 = -\frac{ikK_2}{6},\ c_4 = a_4 = \frac{kK_3}{24},\ c_5 = a_5 = \frac{ikK_4}{120},$$

$$c_6 = a_6 + a_3^2/2 = -\frac{kK_5}{720} - \frac{k^2K_2^2}{72}, \ \cdots .$$

Let $y = \sqrt{kK_1}\theta$. The last integral becomes

$$c(n;k) = A\left(\int_{-\sqrt{K_1}\ln k}^{\sqrt{K_1}\ln k} e^{-y^2/2}\left(1 - i\frac{K_3}{6\sqrt{k}K_1^{\frac{3}{2}}}y^3 + \frac{K_3}{24kK_1^2}y^4 + \dots\right)dy + O(y^{l+1})\right),$$

where

$$A = \frac{n!u^k}{\sqrt{kK_1}2\pi k!(1-e^{-u})^n}.$$

Note that $\int_{-a}^{a} x^{2i+1}e^{-x^2/2}dx = 0$, and $\int_{\sqrt{K_1}\ln k}^{+\infty} y^{2j}\exp(-y^2/2)$ is small and may be dropped. Hence, we can complete the bounds of the integral. With $c_{n,k} = c(n,k)$, we obtain

$$c_{n,k} = A\left(\int_{-\infty}^{\infty} e^{-y^2/2}\left(1 + \frac{K_3y^4}{24kK_1^2} - \left(\frac{K_5y^6}{720k^2K_1^3} + \frac{K_2^2y^6}{72kK_1^3}\right) + \dots\right)dy + O(y^{l+1})\right).$$

Using the well-known values of the J_i,

$$J_i = \int_{-\infty}^{+\infty} x^i e^{-x^2/2}dx, \ J_0 = \sqrt{2\pi}, \ J_2 = \sqrt{2\pi}, \ J_4 = 3\sqrt{2\pi}, \ \dots,$$

we obtain a complete asymptotic formula for $c(n,k)$:

$$c(n,k) = \frac{n!u^k}{k!(1-e^{-u})^n\sqrt{2\pi kK_1}}\left(\sum_{j=0}^{l}\frac{b_i}{k^i} + O\left(\frac{1}{k^{l+1}}\right)\right),$$

where

$$b_0 = 1, \ b_1 = \frac{K_3}{8K_1^2} - \frac{5K_2^2}{24K_1^3}, \ b_2 = \frac{35K_3^2}{384K_1^4} + \frac{7K_2K_4}{48K_1^4} - \frac{K_5}{48K_1^3} - \frac{35K_2^2K_3}{96K_1^5} + \frac{385K_2^4}{1152K_1^6}, \ \cdots .$$

This completes the proof. □

Based on the previous sections, we can now establish the main result of this section.

Theorem 8. *For n and k such that*

$$\frac{k}{\ln n} \longrightarrow +\infty, n \longrightarrow +\infty, \ k \le n - O(n^\alpha), \ 0 < \alpha < 1,$$

the sequence $a(n,k,j)_{j\ge 0}$ is asymptotically normal with mean

$$\mu_{n,k} \sim ne^{-u},$$

and variance

$$\sigma_{n,k}^2 \sim \frac{k}{\ln n},$$

where u is the unique positive real root of $\dfrac{e^u-1}{u} = \dfrac{n}{k} = \lambda.$

Proof. The polynomial $P(x)$ has only real zeros. So, by Theorem 5 the sequence $a(n,\ k,\ j)$ is asymptotically normal, provided that $\lim\limits_{n\longrightarrow+\infty} \sigma_{n;k} = +\infty$. We have

$$\mu_{n,k} = n\frac{c(n-1,k-1)}{c(n,k)}.$$

Recall that u and v are, respectively, the positive real roots of the equations

$$f(u) = \frac{e^u-1}{u} = \frac{n}{k} = \lambda \quad \text{and} \quad f(v) = \frac{e^v-1}{v} = \frac{n-1}{k-1} = \lambda^{'}.$$

Using Theorem 7, we obtain

$$\mu_{n,k} = k\frac{v^{k-1}(1-e^{-u})^n(kK_1)^{1/2}\left(1+\frac{b_1^{'}}{k-1}+\frac{b_2^{'}}{(k-1)^2}+\frac{b_3^{'}}{(k-1)^3}+o\left(\frac{1}{k^3}\right)\right)}{u^k(1-e^{-v})^{n-1}((k-1)K_1^{'})^{1/2}\left(1+\frac{b_1}{k}+\frac{b_2}{k^2}+\frac{b_3}{k^3}+o\left(\frac{1}{k^3}\right)\right)}.$$

To avoid long and tedious calculations, we use only the first terms of the asymptotic formula proved in Theorem 7. This is enough to obtain a central limit theorem. Let us evaluate $(u-v)$. For this, let

$$f(u) = \frac{e^u-1}{u} = \frac{n}{k} = \lambda \quad \text{and} \quad f(v) = \frac{e^v-1}{v} = \frac{n-1}{k-1} = \lambda^{'};$$
$$u = g(\lambda) = f^{-1}(\lambda) \quad \text{and} \quad v = g(\lambda^{'}) = f^{-1}(\lambda^{'}).$$

The successive derivatives of g are

$$g^{'}(\lambda) = \frac{1}{f'(u)},\ g^{''}(\lambda) = -\frac{f^{''}(u)}{f'^3(u)},\ \ g^{^{(3)}}(\lambda) = -\frac{f^{^{(3)}}(u)f^{'}(u)-3f^{''}(u)}{f'^5(u)},\ \cdots .$$

Then,

$$v = g(\lambda^{'}) = g(\lambda)+(\lambda^{'}-\lambda)g^{'}(\lambda)+\frac{(\lambda^{'}-\lambda)^2}{2}g^{''}(\lambda)+O\left(\frac{1}{k^3}\right).$$

We also have

$$\lambda^{'}-\lambda = \frac{n}{k}\ -\frac{n-1}{k-1} \sim \frac{\lambda-1}{k},$$

and

$$f^{'}(u) = \frac{ue^u-e^u+1}{u^2},\ f^{''}(u) = \frac{u^2e^u-2ue^u+2e^u-2}{u^3}.$$

Then,

$$v = g(\lambda^{'}) = g(\lambda) + \frac{\lambda - 1}{kf'(u)} - \frac{1}{2}\left(\frac{\lambda - 1}{k}\right)^2 \frac{f^{''}(u)}{f'^3(u)} + O\left(\frac{1}{k^3}\right).$$

With $k = \frac{n}{\lambda}$ and $g(\lambda) = u$, we obtain

$$v = u + \frac{\lambda - 1}{kf'(u)} - \frac{1}{2}\left(\frac{\lambda - 1}{k}\right)^2 \frac{f^{''}(u)}{f'^3(u)} + O\left(\frac{1}{k^3}\right).$$

Replace $f^{^{(i)}}(u),\ i = 1, 2$, with their values to obtain

$$v = u + \frac{(\lambda - 1)u}{\lambda k(\lambda u - \lambda + 1)} - \frac{1}{2}\left(\frac{(\lambda - 1)}{k}\right)^2 \frac{u^2(1 + u\lambda) - 2u(1 + \lambda u) + 2\lambda u}{(\lambda u - \lambda + 1)^3} + O\left(\frac{1}{k^3}\right).$$

For the sake of simplicity let

$$v - u = \frac{B_k u}{k} + O\left(\frac{1}{k^3}\right),$$

where

$$B_k = \frac{(\lambda - 1)}{\lambda(\lambda u - \lambda + 1)} - \left(\frac{(\lambda - 1)^2}{2k}\right) \frac{u(1 + u\lambda) - 2(1 + \lambda u) + 2\lambda}{(\lambda u - \lambda + 1)^3}.$$

Next, we evaluate $\frac{v^k}{u^k}$:

$$\begin{aligned}
\frac{v^k}{u^k} &= \left(1 + \frac{v - u}{u}\right)^k = \exp\left(k \ln\left(1 + \frac{v - u}{u}\right)\right) \\
&= \exp k \left(\frac{B_k}{k} - \frac{B_k^2}{2k^2} + O\left(\frac{1}{k^3}\right)\right) \\
&= \exp\left(B_k - \frac{B_k^2}{2k} + O\left(\frac{1}{k^2}\right)\right).
\end{aligned}$$

The next quantity to compute is

$$\left(\frac{1 - e^{-u}}{1 - e^{-v}}\right)^n = \exp\left\{-n \ln\left(1 - \frac{e^{-v} - e^{-u}}{1 - e^{-u}}\right)\right\}.$$

We have $e^{-v} - e^{-u} = -(v - u)e^{-u} + \frac{(v-u)^2}{2}e^{-u} + O\left(\frac{1}{k^3}\right)$. This yields

$$\begin{aligned}
\left(\frac{1 - e^{-u}}{1 - e^{-v}}\right)^n &= \exp\left\{-n \ln\left(1 + \frac{(v - u)e^{-u}}{1 - e^{-u}} - \frac{(v - u)^2 e^{-u}}{2(1 - e^{-u})} + O\left(\frac{1}{k^3}\right)\right)\right\} \\
&= \exp\left\{-n \ln\left(1 + \frac{B_k u e^{-u}}{k(1 - e^{-u})} - \frac{B_k^2 u^2 e^{-u}}{2k^2(1 - e^{-u})} + O\left(\frac{1}{k^3}\right)\right)\right\}.
\end{aligned}$$

Since $1 - e^{-u} = \lambda u e^{-u}$ and $n = \lambda k$, we obtain

$$\left(\frac{1-e^{-u}}{1-e^{-v}}\right)^n = \exp\left\{-\lambda k \ln\left(1 + \frac{B_k}{\lambda k} - \frac{\lambda(\lambda-1)^2 u}{2k^2(\lambda u - \lambda + 1)^2} + O\left(\frac{1}{k^3}\right)\right)\right\}$$
$$= \exp\left(-B_k + \frac{\lambda^2(\lambda-1)^2 u}{2k(\lambda u - \lambda + 1)^2}\right)\left(1 + O\left(\frac{1}{k^2}\right)\right).$$

We note in passing that $O\left(\frac{1}{n^l}\right) = O\left(\frac{1}{k^l}\right)$ for $l \geq 1$. The ratio $\left(\frac{1+\frac{b_1'}{k-1}}{1+\frac{b_1}{k}}\right)$ is asymptotically equal to $1 + O\left(\frac{1}{k^2}\right)$. To get an asymptotic equivalent of $\left(\frac{kK_1}{(k-1)K_1'}\right)^{1/2}$, substitute K_1 and K_2 with their values, to obtain:

$$K_\lambda = \left(\frac{kK_1}{(k-1)K_1'}\right)^{1/2} = \exp\left(\frac{1}{2n} + O\left(\frac{1}{k^2}\right)\right)\exp\left(-\frac{1}{2}\ln\left(\frac{e^u - \lambda}{e^v - \lambda'}\right)\right).$$

Let

$$E_\lambda = \left(\frac{e^u - \lambda}{e^v - \lambda'}\right)^{1/2}.$$

From

$$e^u = \lambda u + 1,\ e^v = \lambda^{'} v + 1,\ \lambda^{'} - \lambda\ = \frac{\lambda - 1}{k} + o(1),\ v - u = \frac{B_k u}{k} + O\left(\frac{1}{k^3}\right),$$

we obtain

$$E_\lambda = \exp\left(-\frac{1}{2}\ln\left(\frac{\lambda^{'} v - \lambda^{'} + 1}{\lambda u - \lambda + 1}\right)\right)$$
$$= \exp\left(-\frac{1}{2}\ln\left(1 + \frac{B_k u}{k(\lambda u - \lambda + 1)} + \frac{(\lambda-1)u}{k(\lambda u - \lambda + 1)} - \frac{(\lambda-1)}{k(\lambda u - \lambda + 1)}\right)\right)$$
$$+ O\left(\frac{1}{k^3}\right).$$

Substituting B_k with its value yields

$$E_\lambda = \exp\left(-\frac{1}{2}\ln\left(1 + \frac{\lambda(\lambda-1)u}{k(\lambda u - \lambda + 1)^2} + \frac{(\lambda-1)u}{k(\lambda u - \lambda + 1)} - \frac{(\lambda-1)}{(\lambda u - \lambda + 1)}\right)\right)$$
$$+ \emptyset\left(\frac{1}{k^2}\right).$$

After expanding $\ln\left(1 + \frac{\lambda(\lambda-1)u}{k(\lambda u - \lambda + 1)^2} + \dots\right)$, we obtain

$$K_\lambda = \exp\left(-\frac{1}{2}\left(\frac{\lambda(\lambda-1)u}{k(\lambda u - \lambda + 1)^2} + \frac{(\lambda-1)u}{k(\lambda u - \lambda + 1)} - \frac{(\lambda-1)}{k(\lambda u - \lambda + 1)}\right) + \frac{1}{2n}\right)$$
$$+ O\left(\frac{1}{k^2}\right).$$

Recall that $\mu_{n,k}$ is given by

$$\mu_{n,k} = (1-e^{-v})\frac{k}{v}\left(\frac{v}{u}\right)^k \left(\frac{1-e^{-u}}{1-e^{-v}}\right)^n \left(\frac{kK_1}{(k-1)K_1'}\right)^{1/2} \frac{\left(1+\frac{b_1'}{k-1}\right)}{\left(1+\frac{b_1}{k}\right)}(1+o(1)).$$

Since $u \sim v$ and $1-e^{-u} \sim 1-e^{-v}$, we have

$$\mu_{n,k} \sim (1-e^{-u})\frac{k}{u}\left(\frac{v}{u}\right)^k \left(\frac{1-e^{-u}}{1-e^{-v}}\right)^n \left(\frac{kK_1}{(k-1)K_1'}\right)^{\frac{1}{2}} \frac{\left(1+\frac{b_1'}{k-1}\right)}{\left(1+\frac{b_1}{k}\right)}. \tag{4}$$

Remembering that $(1-e^{-u}) = \lambda u e^{-u}$, and replacing each term in Relation (4) with its asymptotic equivalent, we obtain

$$\mu_{n,k} \sim n \exp\left(-u - \frac{\lambda(\lambda-1)}{2k(\lambda u-\lambda+1)^2} + \frac{(\lambda-1)u}{2k(\lambda u-\lambda+1)^2} - \frac{(\lambda-1)}{2k(\lambda u-\lambda+1)}\right).$$

Keeping just the first term in the previous relation yields $\mu_{n,k} \sim ne^{-u}$ for large n and k in the indicated range. An equivalent value of $\sigma^2_{n,k}$ is deduced from $\mu_{n,k}$ as follows:

$$\sigma^2_{n,k} = \mu_{n,k}\left(1+\mu_{n-1,k-1}-\mu_{n;k}\right) \sim ne^{-u}(1+(n-1)e^{-v}-ne^{-u}).$$

Using the facts that $u \sim v$ and $e^{-v}-e^{-u} \sim (-v+u)e^{-u}$ leads to

$$\begin{aligned}
\sigma^2_{n,k} &\sim ne^{-u}(1+(n-1)e^{-v}-ne^{-u}) \\
&= ne^{-u}(1-e^{-v}+n(e^{-v}-e^{-u})) \\
&= ne^{-u}(\lambda' v e^{-v}+n(e^{-v}-e^{-u})) \\
&\sim ne^{-u}(\lambda u e^{-u}-n(v-u)e^{-u}).
\end{aligned}$$

From $v-u = \frac{B_k}{k}u + O\left(\frac{1}{k^3}\right)$ and $n = \lambda k$, we deduce

$$\begin{aligned}
\sigma^2_{n,k} &\sim ne^{-u}(\lambda u e^{-u} - \lambda B_k u e^{-u}) \\
&= n\lambda u e^{-2u}(1-B_k) \\
&\sim n\lambda u e^{-2u}\left(1-\frac{(\lambda-1)}{\lambda(\lambda u-\lambda+1)}\right) \\
&= kue^{-2u}\left(\frac{\lambda^2 u-\lambda^2+1}{\lambda^2 u-\lambda^2+\lambda}\right).
\end{aligned}$$

For large enough n and k, as in Theorem 8, $\dfrac{\lambda^2 u-\lambda^2+1}{\lambda^2 u-\lambda^2+\lambda} \sim 1$; hence,

$$\sigma^2_{n,k} \sim nu\lambda e^{-2u} = k\lambda^2 e^{-2u}.$$

The root u may be obtained by bootstrapping:

$$\lambda = \frac{e^u - 1}{u} \quad \text{implies} \quad u \sim \ln \lambda + \ln(\ln \lambda).$$

Substituting u with $\ln \lambda + \ln(\ln \lambda)$ yields

$$\sigma_{n,k}^2 \sim \frac{n}{\lambda \ln \lambda} = \frac{k}{\ln \lambda} \longrightarrow +\infty.$$

The proof is concluded. □

5. Conclusions and Further Questions

The original motivation for this work stemmed from a finding in [17]: when factoring a random n-digit number, the distribution of the number of digits in its prime factors is *almost* the same as the distribution of the cycle lengths in a permutation of n objects. In [6], we further explored the distribution of the exponent of the prime number 2 in the factorization of the integer n, comparing it with the number of fixed points in a k permutation of n objects. Additional instances of such similarities, and intriguing parallels in other combinatorial models, can be found in [1, 12, 13].

There is another subject where a similitude may be observed. Let X be an n-element set. Denote by $T(n, j)$, $0 \leq j \leq n$, the number of topologies one can define on X, and having j open sets, which are singletons. For $n = 2$, we have $T(2, 0) = 1, T(2, 1) = 2, T(2, 2) = 1$. More calculation gives $T(3, 0) = 4$, $T(3, 1) = 15$, $T(3, 2) = 9$, $T(3, 3) = 1$. Despite the challenging determination of this sequence, it seems to be interesting and may have many nice properties. We conjecture that the generating polynomial associated with $(T(n, j))_{j=0}^n$, $n \geq 2$, has only real zeros. A weaker conjecture is the log-concavity of the sequence $(T(n, j))_{j=0}^n$.

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