SUMS OF DISTINCT EISENSTEIN PRIMES

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Abstract

Let $\mathscr{E}$ be the set of all Eisenstein integers $a+b\omega$ with $a \geq 0$. $b \geq 0$, where $\omega = e^{\frac{2\pi i}{3}}$. We prove that almost all numbers from $\mathscr{E}$ are sums (empty sum allowed) of distinct primes belonging to $\mathscr{E}$.

1. Introduction

Richert [5], using elementary methods, proved that every integer greater than 6 is the sum of distinct primes (not necessarily odd). Riddell [6] proved a somewhat more precise result. Namely, if $n \geq 4$, then every integer in the closed interval $[7, 3+\sum_{k=4}^{n} p_k]$ can be partitioned into distinct primes not exceeding p_n. Dressler [2], using a stronger Bertrand's postulate, showed that every positive integer, except 1, 2, 4, 6 and 9, is the sum of distinct odd primes. Kløve [3] extended these results to sums of Gaussian primes. In this short note we prove a similar theorem on sums of distinct Eisenstein primes.

The Eisenstein integers, denoted $\mathbb{Z}[\omega]$, is a subring of $\mathbb{C}$ defined as follows

$$\mathbb{Z}[\omega] = \{a + b\omega|\ a, b \in \mathbb{Z} \text{ and } \omega = e^{\frac{2\pi i}{3}}\}.$$

Note that the minimal polynomial of ω is the quadratic $x^2 + x + 1$ or the third cyclotomic polynomial, and hence $\mathbb{Z}[\omega]$ like the Gaussian integers $\mathbb{Z}[i]$ is an imaginary quadratic integer ring. This ring is also a unique factorization domain.

Define $\mathscr{E} = \{a + b\omega|\ a, b \in \mathbb{Z}_{\geq 0}\}$ and let

$$A = \{1, 3, 4, 6, 8, 9, 10, 12, 14, 15, 20, 21, 26, 27, 32, 37, 38, 44, 50, 67, 79\},$$

$$A^* = \{a\omega|\ a \in A\}, \quad \text{and} \quad B = A \cup A^* \cup \{1 + \omega, 2 + 2\omega, 2 + 4\omega, 4 + 2\omega\}.$$

We prove the following theorem.

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Theorem 1. *Each element of $\mathscr{E}\backslash B$ may be expressed as a sum of distinct Eisenstein primes in $\mathscr{E}$.*

2. Preliminaries

In this section we shall collect some auxiliary results.

It is well-known that every prime in $\mathscr{E}$ falls into one of the three categories:

I $b = 0$ and $a = p$ prime with $p \equiv 2 \pmod 3$;

II $a = 0$ and $b = p$ prime with $p \equiv 2 \pmod 3$;

III $a + b\omega$ where the norm $N(a + b\omega) = a^2 - ab + b^2 = p$ is a prime such that $p = 3$ or $p \equiv 1 \pmod 3$.

We adopt the notation introduced by Kløve. The star operation $*$ is defined by

$$(a + b\omega)^* = b + a\omega \text{ for } a + b\omega \in \mathscr{E}.$$

Note that this is equivalent to multiplication of $a + b\omega$ by the unit ω^2 and taking a conjugate. Further, for any two subsets H_1 and H_2 of $\mathscr{E}$ we write $H_1 \S H_2$ if each element of H_1 may be represented as a sum of distinct elements of H_2.

With the above notations, we have the following analogue of Kløve's lemma.

Lemma 1. *Let H be a subset of $\mathscr{E}$ containing a sequence $X = \{x_1, x_2, \ldots\}$ of distinct positive integers such that $x_{n+1} \leq 2x_n$ for $n \geq 1$. If, for fixed b,*

$$\{m + b\omega : a < m \leq a + x_1\} \S H \setminus X,$$

then $\{m + b\omega : m > a\} \S H$.

Proof. Let $M_n = \{m + b\omega |\ a < m \leq a + x_n\}$. Proof is by induction that $M_n \S H \setminus \{x_n, x_{n+1}, \ldots\}$. □

Since $\mathscr{E}^* = \mathscr{E}$, we immediately get the following.

Corollary 1. *Let H be a subset of $\mathscr{E}$ containing a sequence $Y = \{y_1\omega, y_2\omega, \ldots\}$ where y_i's are distinct positive integers such that $y_{n+1} \leq 2y_n$ for $n \geq 1$. If, for fixed a, we have $\{a + m\omega : b < m \leq b + y_1\} \S H \setminus Y$, then $\{a + m\omega : m > b\} \S H$.*

We will also need the following lemmata due to Breusch [1] and Makowski [4], respectively.

Lemma 2. *If $x \geq 7$, then between x and $2x$ there is at least one prime of the form $6k - 1$.*

Lemma 3. *Every integer greater than* 161 *is the sum of distinct primes of the form* $6k-1$.

3. Proof

Proof of Theorem 1. Let P be the set of primes of type III and let $Q = \{q_1 = 2, q_2 = 5, \ldots\}$ be the increasing sequence of primes congruent to 2 (mod 3). By Lemma 2 we have that $q_{n+1} < 2q_n$ for $n \geq 3$. Now $2+\omega, 3+\omega, 4+\omega, 6+\omega, 7+\omega, 9+\omega \in P$ and

$$5+\omega = (3+\omega)+2,\ 8+\omega = (3+\omega)+5,\ 10+\omega = (3+\omega)+2+5,$$
$$11+\omega = (4+\omega)+2+5 = (6+\omega)+5 = (9+\omega)+2,\ 12+\omega = (7+\omega)+5.$$

Hence, $\{m+\omega : 1 < m \leq 12\} \S P \cup \{2,5\}$ and, by Lemma 1,

$$\{m+\omega : m > 1\} \S P \cup Q.$$

Similarly, we have $1+2\omega, 3+2\omega, 5+2\omega, 9+2\omega, 11+2\omega \in P$ and

$$6+2\omega = (1+2\omega)+5,\ 7+2\omega = (5+2\omega)+2,\ 8+2\omega = (1+2\omega)+2+5,$$
$$10+2\omega = (5+2\omega)+5,\ 12+2\omega = (5+2\omega)+2+5,\ 13+2\omega = (11+2\omega)+2,$$
$$14+2\omega = (9+2\omega)+5,\ 15+2\omega = (6+\omega)+(9+\omega).$$

Thus, $\{m+2\omega : 4 < m \leq 15\} \S P \cup \{2,5\}$ and, again by Lemma 1,

$$\{m+2\omega : m = 1, 3 \text{ or } m > 4\} \S P \cup Q.$$

In a similar manner we prove that

$$\{m+3\omega : m \geq 1\} \S P \cup Q \quad \text{and} \quad \{m+4\omega : m = 1 \text{ or } m > 2\} \S P \cup Q.$$

Adding 2ω or 5ω to each element in these sets we get

$$\{m+n\omega : 0 < n \leq 11\} \S P \cup Q \cup \{2\omega, 5\omega\} \text{ for } m > 4.$$

By the Corollary 1, we obtain $\{m+n\omega : n > 0, m > 4\} \S P \cup Q \cup Q^*$. Since $P^* = P$ we get

$$\{n+m\omega : n > 0, m > 4\} \S P \cup Q \cup Q^*.$$

Combining, we get

$$\{m+n\omega : n > 0, m > 0\} \setminus \{1+\omega, 2+2\omega, 2+4\omega, 4+2\omega\} \S P \cup Q \cup Q^*.$$

By Lemma 3 $\{n : n > 161\} \S Q$. Simple calculations show that $\{n : 1 \leq n \leq 161,\ n \notin A\} \S Q$. If $N = \{1, 2, \ldots\}$, we have

$$N \setminus A \S Q, \text{ hence } N^* \setminus A^* \S Q^*.$$

This completes the proof of the theorem. □

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