ON 2-NEAR PERFECT NUMBERS

Vedant Aryan
Hopkins School, New Haven, Connecticut
varyan24@students.hopkins.edu

Dev Madhavani
Harvard University, Cambridge, Massachusetts
devmadhavani@college.harvard.edu

Savan Parikh
Yale University, New Haven, Connecticut
savan.parikh@yale.edu

Ingrid Slattery
Yale University, New Haven, Connecticut
ingrid.slattery@yale.edu

Joshua Zelinsky
Department of Mathematics, Hopkins School, New Haven, Connecticut
jzelinsky@hopkins.edu

Received: 10/13/23, Revised: 11/28/23, Accepted: 6/4/24, Published: 6/19/24

Abstract

Let $\sigma(n)$ be the sum of the positive divisors of n. A number n is said to be 2-near perfect if $\sigma(n) = 2n + d_1 + d_2$, where d_1 and d_2 are distinct positive divisors of n. We give a complete description of those n which are 2-near perfect and of the form $n = 2^k p^i$ where p is prime and $i \in \{1, 2\}$. We also prove related results under the additional restriction where $d_1 d_2 = n$.

1. Introduction

A *perfect number* is a positive integer that is equal to the sum of its proper positive divisors. Equivalently, a perfect number is an integer n such that $\sigma(n) = 2n$, where $\sigma(n)$ is the sum of all positive divisors of n. Perfect numbers have been studied since antiquity. The idea of perfect numbers has been generalized in a variety of ways. A classic generalization is the notion of a *multiply perfect number*, defined

DOI: 10.5281/zenodo.12167628

as an integer n which satisfies $\sigma(n) = mn$ for some other integer m. Sierpiñski [6] introduced the term *pseudoperfect* number to mean a number $2n$ such that n is the sum of some subset of its divisors. Pollack and Shevelev [5] separated the pseudoperfect numbers into separate types by introducing the idea of an s-near perfect number. A number n is *s-near perfect* if $2n$ is the sum of all its positive divisors excepting s of them. For example, while 12 is not perfect, it is 1-near perfect, since $1 + 2 + 3 + 6 + 12 = 2(12)$. Here the divisor which has been removed from the set is 4. We will refer to divisors removed from the set as *omitted divisors*. Note that any pseudoperfect number is s-near perfect for some s, and one can think of perfect numbers as 0-near perfect numbers. In some sense, multiply perfect numbers are a multiplicative generalization of perfect numbers, while s-near perfect numbers are a more additive generalization.

A number n is said to be *abundant* if it satisfies $\sigma(n) > 2n$. If n is s-near perfect for some $s > 0$, then n must be abundant. However, it is possible for a number to be abundant while not being s-near perfect for any s. An example is 70, where $\sigma(70) = 144$. Numbers which are abundant but not s-near perfect for any s are said to be *weird*. A classic open problem is whether there are any odd weird numbers.

In addition to the more general notion of s-near perfect numbers, Pollack and Shevelev [5] also used the term near perfect number to mean 1-near perfect number. Another classic open problem is whether there is any n such that $\sigma(n) = 2n + 1$. Such numbers are called *quasiperfect numbers*. Note that any quasiperfect number is a 1-near perfect number with omitted divisor 1.

Pollack and Shevelev constructed the following three distinct families of 1-near perfect numbers:

1. $2^{t-1}(2^t - 2^k - 1)$ where $2^t - 2^k - 1$ is prime. Here 2^k is the omitted divisor.

2. $2^{2p-1}(2^p - 1)$ where $2^p - 1$ is prime. Here $2^p(2^p - 1)$ is the omitted divisor.

3. $2^{p-1}(2^p - 1)^2$ where $2^p - 1$ is prime. Here $2^p - 1$ is the omitted divisor.

Subsequent work by Ren and Chen [7] showed that all near perfect numbers with two distinct prime factors must be either 40, or one of the three families above.

The only known near perfect odd number is $173369889 = (3^4)(7^2)(11^2)(19^2)$. Tang, Ma, and Feng [8] showed that this is the only odd near perfect number with four or fewer distinct prime divisors. Cohen, Cordwell, Epstein, Kwan, Lott, and Miller proved general asymptotics for s-near perfect numbers for $s \geq 4$. Recent work by Hasanalizade [2] gave a partial classification of near perfect numbers which are also Fibonacci or Lucas numbers. Li and Liao [3] classified all even near perfect numbers of the form $2^a p_1 p_2$ where p_1 and p_2 are distinct primes.

The main results of this paper are twofold. First, we give a complete description of 2-near perfect numbers of the form $2^k p$ or $2^k p^2$ where p is prime. Second, we use these characterizations to introduce a closely related notion of strongly 2-near perfect numbers, and give a characterization of those of the form $2^k p$.

In particular, we have the following two main results.

Theorem 1. *Assume n is a 2-near perfect number with omitted divisors d_1 and d_2. Assume further that $n = 2^k p$ where p is prime and k is a positive integer. Then one must have, without loss of generality, one of four situations.*

1. $p = 2^k - 1$. *Here we have $d_1 = 1$ and $d_2 = p$.*
2. $p = 2^{k+1} - 2^a - 2^b - 1$ *for some $a, b \in \mathbb{N}$. Here $d_1 = 2^a$ and $d_2 = 2^b$.*
3. $p = \frac{2^{k+1}-2^a-1}{1+2^b}$ *for some $a, b \in \mathbb{N}$. Here $d_1 = 2^a$ and $d_2 = 2^b p$.*
4. $p = \frac{2^{k+1}-1}{1+2^a+2^b}$ *for some $a, b \in \mathbb{N}$. Here $d_1 = 2^a p$ and $d_2 = 2^b p$.*

Theorem 2. *Assume that n is a 2-near perfect number with omitted divisors d_1 and d_2. Assume further that $n = 2^k p^2$ where p is prime. Then $n \in \{18, 36, 200\}$.*

We recall the following basic facts about $\sigma(n)$:

Lemma 1. *The function $\sigma(n)$ has the following properties:*

1. *$\sigma(n)$ is multiplicative. That is, $\sigma(ab) = \sigma(a)\sigma(b)$ whenever a and b are relatively prime.*
2. *For a prime p, $\sigma(p^k) = p^k + p^{k-1} + \cdots + 1 = \frac{p^{k+1}-1}{p-1}$.*

2. Proof of Theorem 1

Let us now prove Theorem 1

Proof. Assume we have a 2-near perfect number of the form $n = 2^k p$ with two omitted divisors d_1, d_2, $d_1 \neq d_2$, and odd prime p. Because n is near perfect, we have that

$$\sigma(n) = 2n + d_1 + d_2. \tag{1}$$

Using Lemma 1, we then have:

$$\sigma(n) = \sigma(2^k p) = (2^{k+1} - 1)(p + 1) \tag{2}$$

So, setting Equation (1) equal to Equation (2), we have

$$(2^{k+1} - 1)(p + 1) = 2n + d_1 + d_2 = 2^{k+1} p + d_1 + d_2,$$

and hence

$$p = 2^{k+1} - 1 - d_1 - d_2. \tag{3}$$

Because p is odd, we have that $2^{k+1}-1-d_1-d_2$ is odd. Since $2^{k+1}-1$ is always odd, we have that $-(d_1+d_2)$ must be even. If $-(d_1+d_2)$ is even, d_1 and d_2 must be of the same parity. We thus need to consider two situations: where d_1, d_2 are both odd, and where they are both even. We will call the first situation Case 1, and we shall separate the second situation, Case 2, into three separate subcases without loss of generality.

`Case 1`: In this case, d_1, d_2 are both odd. The only odd divisors of n are 1 and p, so we can, without loss of generality, set $d_1 = 1$ and $d_2 = p$ to find $p = 2^{k+1}-1-1-p$, and hence $p = 2^k - 1$. Thus, our first family of 2-near perfect numbers correspond to the family of Mersenne primes and have the form $2^k(2^k-1)$ (twice an even perfect number).

We now consider the situation where d_1 and d_2 are both even. We shall break this down into three subcases, depending on the types of values for d_1 and d_2.

`Case 2.1`: In this case we have $d_1 = 2^a, d_2 = 2^b$, where $0 < a < b \leq k$. We can then use the definitions of d_1 and d_2 in Equation (3) to find that $p = 2^{k+1} - 2^a - 2^b - 1$. This is our second family of 2-near perfect numbers.

`Case 2.2`: In this case, d_1, d_2 are both even, and $d_1 = 2^a, d_2 = 2^b p$, and $a, b \in (0, k]$. We use a similar strategy, and plug our definitions of d_1, d_2 into Equation (3) to obtain $p = 2^{k+1} - 2^a - 2^b p - 1$, so that $p(1+2^b) = 2^{k+1} - 2^a - 1$, which becomes $p = \frac{2^{k+1}-2^a-1}{1+2^b}$. This is our third family of 2-near perfect numbers.

`Case 2.3`: In this case, d_1, d_2 are both even and we have $d_1 = 2^a p d_2 = 2^b p$, and $0 < a < b \leq k$. Using the same technique, Equation (3) yields $p = 2^{k+1} - 1 - 2^a p - 2^b p$, and hence, $p(1+2^a+2^b) = 2^{k+1}-1$, which implies that $p = \frac{2^{k+1}-1}{1+2^a+2^b}$. This is the fourth and final family of 2-near perfect numbers. $\square$

3. Proof of Theorem 2

One major technique we will use is what we call the *discriminant sandwich* method: we show that a given Diophantine equation has only a restricted set of possible solutions. We do so by showing that the equation is a quadratic equation in one variable, and thus in order to have integer valued solutions, the discriminant must be a perfect square. However, we will show that the discriminant must, except in a limited set of cases, be shown to be strictly between two consecutive perfect squares, and thus aside from those situations, we have no solution. Discriminant sandwiching will be used extensively in what follows.

Lemma 2. *Let a and k be positive integers such that $D = 2^{2k+2} + 2^{k+2} - 2^{a+2} - 7$.*

If $0 \le a \le k$ and D is a perfect square, then $k = a = 1$.

Proof. Let us assume that D is a perfect square. For all a, note that

$$2^{2k+2} + 2^{k+2} - 2^{a+2} - 7 < 2^{2k+2} + 2^{k+2} + 2 = (2^{k+1} + 1)^2.$$

Thus, if we have

$$D = 2^{2k+2} + 2^{k+2} - 2^{a+2} - 7 > 2^{2k+2} = (2^{k+1})^2 \tag{4}$$

then the quantity in question cannot be a perfect square because it is sandwiched between two consecutive perfect squares. So we must have that

$$2^{2k+2} + 2^{k+2} - 2^{a+2} - 7 \le 2^{2k+2},$$

and therefore

$$2^{k+2} - 2^{a+2} \le 7. \tag{5}$$

If $k > a \ge 1$, then from Equation (5) we have that $2^{k+2} - 2^{k+1} \le 2^{k+2} - 2^{a+2} \le 7$. Thus, $2^{k+2} - 2^{k+1} = 2^{k+1} \le 7$, which implies that $k \le 1$. However, given the conditions for this case, no solutions are possible.

Now, consider the case when $k = a \ge 1$. In this case, we have

$$D = 2^{2k+2} + 2^{k+2} - 2^{k+2} - 7 = 2^{2k+2} - 7.$$

Given this, note that $2^{2k+2} - 7 < 2^{2k+2} = (2^{k+1})^2$. Using the same logic as earlier, we see that if $D > (2^{k+1} - 1)^2$, then D will be sandwiched between two consecutive perfect squares, and thus will not be a square itself. Thus, we can assume that

$$2^{2k+2} - 7 \le (2^{k+1} - 1)^2 = 2^{2k+2} - 2^{k+2} + 1,$$

which implies that $k \le 1$. The bounds for this case require $k \ge 1$, so the only solution possible is $(a, k) = (1, 1)$. $\square$

Essentially, Lemma 2 is the sandwiching part of the discriminant sandwich we will use in the proof of Proposition 1 below.

Lemma 3. *Let b and k be non-negative integers and p be an odd number such that*

$$(2^{k+1} - 1)(p^2 + p + 1) = 2^{k+1}p^2 + 2^b p + 1. \tag{6}$$

Then $p \mid 2^k - 1$ and $p + 1 \mid 2^b - 2$.

Proof. Assume one has a solution to Equation (6). Then, if we take the equation modulo p, we get that $2^{k+1} - 1 \equiv 1 \mod p$, and hence $p \mid 2^{k+1} - 2 = 2(2^k - 1)$. Since p is odd, we have $p \mid 2^k - 1$. To prove the second half, observe that we can rewrite our initial equation as $2^{k+1}(p + 1) = 2^b p + p^2 + p + 2$, which implies that $p + 1 \mid 2^b p + p^2 + p + 2$. Hence $p + 1 \mid 2^b p + p^2 - p$, and so $p + 1 \mid p(2^b + p - 1)$. Since p and $p + 1$ are relatively prime, we have then that $p + 1 \mid 2^b + p - 1$. Finally, we take away another multiple of $p + 1$ to get $p + 1 \mid 2^b - 2$. $\square$

Lemma 4. *The equation*

$$(2^{k+1}-1)(p^2+p+1) = 2^{k+1}p^2 + 2^b p + 1 \tag{7}$$

has no solutions where p is odd and $2 \le b \le k-1$.

Proof. Assume we have a solution to the equation. From Lemma 3, we choose an integer x such that $x(p+1) = 2^b - 2$. Note that x must be odd since $p+1$ is even, and $2^b - 2$ is not divisible by 4. We then have $2^b = xp + x + 2$. When we substitute this back into Equation (7), and solve for 2^{k+1}, we get that $2^{k+1} = xp + p + 2$. By taking the difference of these two expressions, we obtain

$$2^{k+1} - 2^b = (xp + p + 2) - (xp + x + 2) = p - x.$$

Because $b \le k-1$, it follows that $2^{k+1} - 2^b > 2^k$, and hence we have $p - x > 2^k$, and hence $p > 2^k + 1$. But this contradicts Lemma 3, since we must have $p \mid 2^k - 1$. □

Lemma 5. *Let n be a 2-near perfect number of the form $n = 2^k p^2$ where p is an odd prime. Assume further that the omitted divisors of n are d_1 and d_2. Then, we have*

$$d_1 + d_2 = -p^2 + (2^{k+1}-1)p + (2^{k+1}-1), \tag{8}$$

and d_1 and d_2 are of opposite parity.

Proof. Assume the truth of the above hypotheses. Then we have $\sigma(n) - d_1 - d_2 = 2n$. This is equivalent to $\sigma(2^k p^2) - d_1 - d_2 = 2(2^k p^2) = 2^{k+1}p^2$, which can be rewritten as $(2^{k+1}-1)(p^2+p+1) - d_1 - d_2 = 2^{k+1}p^2$, and hence

$$d_1 + d_2 = -2^{k+1}p^2 + (2^{k+1}-1)(p^2+p+1).$$

This last line is equivalent to Equation (8). Since the right-hand side of Equation (8) is odd, d_1 and d_2 must be of opposite parity. □

Let us look at the possible divisors of n, assuming $n = 2^k p^2$. Every possible divisor can be of one of three types. Type I divisors are powers of 2, that is, $d = 2^a$ for some $0 \le a \le k$. Type II divisors are p or p^2, that is, $d = p^m$ where $m \in \{1, 2\}$. Type III divisors are of the form $d = 2^b p^j$ where $0 < b \le k$ and $j \in \{1, 2\}$.

We may then, without loss of generality, break down our situation into the following six cases as listed in Table 1 below depending on all the possible combinations of omitted divisor types, d_1 and d_2.

We will handle each of these six cases separately. But before we do, we will observe that Cases 4 and 6 are both trivial since they require that both d_1 and d_2 are of the same parity, which contradicts Lemma 5. We thus only consider Cases 1, 2, 3, and 5.

Case	d_1	d_2
1	I	I
2	I	II
3	I	III
4	II	II
5	II	III
6	III	III

Table 1: The six possible cases

Proposition 1. *If n is a 2-near perfect number of the form $n = 2^k p^2$, where p is an odd prime, with omitted divisors of Case 1 form, then $n = 18$, and the omitted divisors are* 1 *and* 2*.*

Proof. Assume we are in Case 1. In this case, both d_1 and d_2 must be distinct powers of 2. Since $d_1 + d_2$ is odd, one of the omitted divisors must be odd (and hence equal to 1). Without loss of generality, we will set $d_1 = 1$, and $d_2 = 2^a$ where $1 \le a \le k$. Putting this into Equation (8), we get that

$$p^2 - (2^{k+1} - 1)p - (2^{k+1} - 2^a - 2) = 0. \tag{9}$$

Equation (9) is a quadratic equation in p. Thus, in order to have a solution, its discriminant, defined as $D = 2^{2k+2} + 2^{k+2} - 2^{a+2} - 7$, must be a perfect square. From Lemma 2, D is only a perfect square if $k = a = 1$. In this case, Equation (9) becomes just $p^2 - 3p = 0$. Thus, one must have $p = 3$, and so $n = 18$ with $d_1 = 1, d_2 = 2$. One can verify this result by observing that $2(18) = \sigma(18) - (1 + 2)$. □

Proposition 2. *There are no 2-near perfect numbers of the form $2^k p^2$ with omitted divisors of the Case 2.*

Proof. We will apply the discriminant sandwich method to this situation. Again we are working with $n = 2^k p^2$, but we now have omitted divisors of the form $d_1 = 2^a$ and $d_2 = p^m$ where $a \in (0, k]$ and $m \in [1, 2]$. We will break Case 2 down into two subcases, depending on whether $m = 1$ or $m = 2$.

We first consider the situation where $m = 1$. Applying this to Equation (8), we obtain

$$0 = p^2 - (2^{k+1} - 2)p - (2^{k+1} - 2^a - 1). \tag{10}$$

Equation (10) has an even discriminant D, which means that if D is a perfect square, it must be divisible by 4. Thus, we can define

$$D' = \frac{D}{4} = 2^{2k} - 2^a,$$

and just as well assume that D' is a perfect square. Note that D' is still even, so we can skip over checks against odd squares. We have that $2^{2k} - 2^a < (2^k)^2$, and so $2^{2k} - 2^a \leq (2^k - 2)^2$. With a little algebra we then obtain that $2^{k+2} - 2^a \leq 4$. Without loss of generalization, we may write $k = a + m, m \in [0, k), a > 0$. We then have that $2^a(2^{m+2} - 1) = 2^{a+m+2} - 2^a \leq 4$. It is evident after plugging in the minimum values for a and m that no solution exists. We thus have shown that when $m = 1$, no solution exists.

We now consider the case when $m = 2$. We then obtain from Equation (8),

$$0 = 2p^2 - (2^{k+1} - 1)p - 2^{k+1} + 2^a + 1. \tag{11}$$

We then need that the discriminant D, defined as

$$D = 2^{2k+2} - 2^{k+2} + 2^{k+4} - 2^{a+3} - 7.$$

is a perfect square. We thus must have $D < (2^{k+1} + 3)^2$. Since D is odd, it cannot be equal to the next smallest square, which is even. So, $D \leq (2^{k+1} + 1)^2$. We thus have

$$2^{2k+2} - 2^{k+2} + 2^{k+4} - 2^{a+3} - 7 \leq 2^{2k+2} + 2^{k+2} + 1,$$

which implies that $2^k - 2^a \leq 1$. Thus, we can only have a solution when $k = a$ or we have $k = 1$ and $a = 0$. However, since d_1 and d_2 must be of opposite parity, we cannot have $a = 0$. Thus, we need consider only the case when $k = a$. Our expression for D simplifies so that we have $D = 2^{2k+2} + 2^{k+2} - 7$. But this quantity cannot be a perfect square since $(2^{k+1})^2 < 2^{2k+2} + 2^{k+2} - 7 < (2^{k+1} + 1)^2$, and so D is again sandwiched between two consecutive perfect squares. Thus, there are no solutions for Case 2 when $m = 2$. Since no solutions exist for all possible cases for m, Proposition 2 has been proven. □

Lemma 6. *If p is an odd number such that*

$$(2^{k+1} - 1)(p^2 + p + 1) = 2^{k+1}p^2 + 2^b p^2 + 1 \tag{12}$$

where k and b are positive integers, then $p \mid 2^k - 1$.

Proof. Assume one has a solution to Equation (12). Then, if we take the equation modulo p, we get that $2^{k+1} - 1 \equiv 1 \mod p$, and hence $p \mid 2^{k+1} - 2 = 2(2^k - 1)$. Since p is odd, we have $p \mid 2^k - 1$. □

Lemma 7. *If p is an odd number which is a solution to Equation (12), then $p+1 \mid 2^b + 2$.*

Proof. Assume one has a solution to Equation (12). We can rewrite this as

$$2^{k+1}(p + 1) = (2^b + 1)p^2 + p + 2$$

which implies that $p+1 \mid (2^b+1)p^2+p+2$, and hence Since p and $p+1$ are relatively prime, $p+1 \mid (2^b+1)p-1$, and so by similar logic, $p+1 \mid 2^b+2$, which is the needed relation. □

Note that Lemma 7 is distinct from Lemma 3, since the equations needed are different, and one has a positive 2 on the right-hand side, and the other a negative 2 on the right-hand side.

Proposition 3. *Let $n = 2^k p^2$ be a 2-near perfect number with omitted divisors of the Case 3 form with omitted divisors* 1 *and* $2^b p^2$. *Then $n = 36$, and our omitted divisors are* 1 *and* 18.

Proof. Assuming the hypotheses, from Equation (8), we have some b such $b \leq k$, and p prime such that

$$(2^{k+1}-1)(p^2+p+1) = 2^{k+1}p^2 + 2^b p^2 + 1. \tag{13}$$

By Lemmas 6 and 7, we have $p \mid 2^k - 1$ and $p+1 \mid 2^b+2$. Thus, there is a positive integer z such that $z(p+1) = 2^b + 2$, and $z(p+1) - 2 = 2^b$. If we take Equation (13) modulo 2^b we also get that $2^b \mid p^2+p+2$.

We also have

$$\begin{aligned} z(p+1)-2 &\mid p^2+p+z(p+1) \\ z(p+1)-2 &\mid p(p+1)+z(p+1) \\ z(p+1)-2 &\mid (p+1)(p+z). \end{aligned} \tag{14}$$

Let Q be some integer such that $Q \mid z(p+1)-2$ and $Q \mid p+1$. Then Q will divide any linear combination of those terms. Thus we have

$$Q \mid z(p+1)-2-z(p+1) = -2.$$

Thus, the only possible common factors of $z(p+1)-2$ and $(p+1)$ are 1 and 2. Hence Equation (14) may be strengthened to

$$z(p+1)-2 \mid 2(p+z). \tag{15}$$

Therefore, we know that $z(p+1)-2 \leq 2(p+z)$, which implies that $z \leq \frac{2p+2}{p-1} = 2 + \frac{4}{p-1}$.

Since $p \geq 3$, we have $z \leq 3$, and hence have only three cases, $z = 1$, $z = 2$, or $z = 3$. One can easily check that if $p = 3$ then the only case which leads to integer values is when $z = 1$ and $b = 1$. Here $n = 36$, and our omitted divisors are $d_1 = 1, d_2 = 18$. Thus, we may assume that $p > 3$, which implies $z = 1$ or $z = 2$. However, if $b = 1$, we get a contradiction if $p > 3$. Thus, we may assume that $b > 1$ which forces z to be odd, and so $z = 1$.

From Equation (15), we have $p-1 \mid 2(p+1)$. So there is some m such that $m(p-1) = 2(p+1)$. If $m = 1$ then we get a negative value for p, and if $m = 2$, we immediately get a contradiction. So we may assume that $m \geq 3$. If $m = 3$, then we have $3(p-1) = 2(p+1)$, which yields $p = 5$ which quickly leads to a contradiction. We thus must have $m \geq 4$. However, if $4(p-1) \leq 2(p+1)$, then one must have $p = 3$, but we are in the situation where $p > 3$.

Thus, our only possibility is when $n = 36$. □

Proposition 4. *If n is a 2-near perfect number of the form $n = 2^k p^2$, where p is an odd prime, with omitted divisors of Case V form, then $n = 200$.*

Proof. Assume we have such an n, with omitted divisors d_1 and d_2. Then, without loss of generality, we may assume that $d_1 = p^j$ for $j \in [1,2]$ and $d_2 = 2^b p^g$ for some $g \in [1,2]$, and $1 \leq b \leq k$. We may break this down into four cases as outlined in the table below.

Case	d_1	d_2
1	p	$2^b p$
2	p	$2^b p^2$
3	p^2	$2^b p$
4	p^2	$2^b p^2$

`Case 1`: This case can be handled by the discriminant sandwich technique. Equation (8) becomes

$$p + 2^b p = -p^2 + (2^{k+1} - 1)p + (2^{k+1} - 1). \tag{16}$$

The relevant discriminant from Equation (16) is

$$D = 2^{2k+2} - 2^{k+b+2} + 2^{2b} + 2^{b+2}.$$

We have then $(2^{k+1} - 2^b)^2 < D < (2^{k+1} - 2^b + 1)^2$, so D cannot be a perfect square. Thus the equation has no solutions.

`Case 2`: In this case Equation (8) becomes

$$p + 2^b p^2 = -p^2 + (2^{k+1} - 1)p + (2^{k+1} - 1). \tag{17}$$

The relevant discriminant from Equation (17) is

$$D = 2^{2k+2} + 2^{k+b+3} - 2^{b+2}. \tag{18}$$

We wish to show that such a D is not a perfect square. We do so by splitting into two subcases, when b is even and when b is odd. In both cases we will get a contradiction from assuming that D is a perfect square.

First, assume that b is even. Then 2^{b+2} is a perfect square, and so if D is a perfect square, we may factor out 2^{b+2} from Equation (18). In that case, we have then that $D_0 = 2^{2k-b} + 2^{k+1} - 1$ must be a perfect square. However, if $b < k$, then $D_0 \equiv 3 \pmod 4$, and thus D_0 cannot be a perfect square. Thus, we must have $b = k$. In that situation we have $D_0 = 2^k + 2^{k+1} - 1$. However, since b is even and greater than 1, we must then have $k \geq 2$. Thus, we still have $D_0 \equiv 3 \pmod 4$.

Now, for our second subcase, assume that b is odd. Then 2^{b+1} is a perfect square, and so we may factor that quantity out of D and still have a perfect square. We thus have that $D_1 = 2^{2k-b+1} + 2^{k+2} - 2$ must be a perfect square. However, we have then that $D_1 \equiv 2 \pmod 4$, and so we have again reached a contradiction.

Since both cases lead to a contradiction, we conclude that the relevant discriminant is never a perfect square, and thus the equation has no solutions.

`Case 3`: where $d_1 = p^2$ and $d_2 = 2^b p^2$. Equation (8) then becomes

$$p^2 + 2^b p = -p^2 + (2^{k+1} - 1)p + 2^{k+1} - 1.$$

Hence,

$$-2p^2 + (2^{k+1} - 2^b - 1)p + 2^{k+1} - 1 = 0. \tag{19}$$

Equation (19) has a corresponding discriminant value given by

$$D = x^2 - 2x(2^b - 3) + 2^{2b} + 2^{b+1} - 7, \tag{20}$$

where $x = 2^{k+1}$. We note that if $b = 1$, then we get that either $p = -1$ or $p = \frac{2^{k+1}-1}{2}$, neither of which is a prime. Thus, we may assume that $b > 1$. Since $b > 1$, we have $2^{b+3} > 16$, which implies that$2^{b+1} - 7 > -6 \cdot 2^b + 9$. We then obtain that

$$x^2 - 2x(2^b - 3) + 2^{2b} + 2^{b+1} - 7 > x^2 - 2x(2^b - 3) + 2^{2b} - 6 \cdot 2^b + 9,$$

which implies that $D > x^2 - 2x(2^b - 3) + (2^b - 3)^2 = (x - (2^b - 3))^2$.

If we have a solution to our original equation, D must be a perfect square. Equation (20) also shows that D must be odd. Thus, we cannot have $D = (x - (2^b - 2))^2$, and thus we have

$$D \geq (x - (2^b - 1))^2. \tag{21}$$

Equation (21) then implies that

$$3 \cdot 2^b \geq 2^{k+1} + 8, \tag{22}$$

and hence,

$$4 \cdot 2^b > 2^{k+1}. \tag{23}$$

Inequality (23) implies that $b > k - 1$. Since we have that $b \leq k$, and b is a natural number, we conclude that $b = k$. We thus may replace k with b in Equation (19) to obtain $p^2 + 2^b p = -p^2 + (2^{b+1} - 1)p + 2^{b+1} - 1$, which is equivalent to

$$2^b(p + 2) = 2p^2 + p + 1. \tag{24}$$

Thus, we have $p + 2 \mid 2p^2 + p + 1$. We then have

$$p + 2 \mid (2p^2 + p + 1) + (3 - 2p)(p + 2) = 7.$$

Since $p + 2 \mid 7$, we must have $p = 5$. We then can solve to get that $b = k = 3$. This yields $n = (2^3)(5^2) = 200$, which is in fact a 2-near perfect number of the desired form. Here our omitted divisors are 25 and 40.

`Case 4`: We have $d_1 = p^2$ and $d_2 = 2^b p^2$. In this situation Equation (8) becomes

$$(2^{k+1} - 1)(p^2 + p + 1) - 2^{k+1}p^2 = p^2 + 2^b p^2,$$

which can be rewritten as

$$2^{k+1}(p + 1) = 2p^2 + 2^b p^2 + p + 1. \tag{25}$$

Therefore, we have the following:

$$(p + 1) \mid (2p^2 + 2^b p^2 + p + 1),$$

$$(p + 1) \mid (2p^2 + 2^b p^2 + p + 1) - (p + 1) = 2p^2 + 2^b p^2,$$

and thus $(p + 1) \mid p^2(2 + 2^b)$. Since $p + 1$ and p^2 are relatively prime this implies $(p + 1) \mid (2 + 2^b)$.

Thus, there exists a positive integer z such that $z(p + 1) = 2 + 2^b$, and hence

$$2^b = z(p + 1) - 2. \tag{26}$$

We note that $p + 1$ is even and the only way which $2 + 2^b$ can be divisible by 4 is if $b = 1$ (which does not lead to a solution). Thus, z is odd.

If we take Equation (25) modulo 2^b, we obtain that

$$2^b \mid 2p^2 + p + 1. \tag{27}$$

We note that Equation (27) allows one to obtain a finite set of possible values b for any given fixed choice of p, and then use each to solve for k. Thus, with only a small amount of effort, we may verify that we must have $p > 23$. We may combine Equation (27) with Equation (26) to obtain $z(p+1) - 2 \mid 2p^2 + p + 1$. We then have

$$zp + z - 2 \mid 2p^2 + p + 1$$

and

$$zp + z - 2 \mid 2z(2p^2 + p + 1) - (4p - 2)(zp + z - 2) = 4z + 8p - 4.$$

Hence,

$$zp + z - 2 \mid 4(z + 2p - 1). \tag{28}$$

Consider now the situation where we have equality in the relationship in Equation (28). Then we have $zp+z-2 = 4(z+2p-1)$, which is equivalent to $z(p+1) = 8p+2$. Thus, $p + 1 \mid 8p + 2$, and hence $p + 1 \mid 6p$. Since $p + 1$ and p are relatively prime, this forces us to have $p + 1 \mid 6$, and hence we must have $p = 5$, which we can verify does not work. Thus, we must have some integer $m \geq 2$ such that $m(zp + z - 2) = 4(z + 2p - 1)$, and thus we have $zp + z - 2 \leq 2(z + 2p - 1)$, which is equivalent to

$$z \leq \frac{4p}{p-1}. \tag{29}$$

We have that $p \geq 7$, and thus, Inequality (29) implies that $z < 6$. Since z is odd, we must then have $z = 1$, $z = 3$, or $z = 5$. If we have $z = 1$, then Equation (28) implies that $p - 1 \mid 8p$. But since $p - 1$ is relatively prime to p, we must have $p - 1 \mid 8$, which is impossible since $p > 23$. Using similar logic, for $z = 3$, we obtain from Equation (28) that

$$3p + 1 \mid 4(2p + 2) = 8(p + 1)$$

$$3p + 1 \mid 8p + 8 - 8(3p + 1) = -16p.$$

Thus, $3p + 1$ is relatively prime to p, so we must have that $3p + 1 \mid 16$. But, once again, we have $p > 23$, so there cannot be any solution. Finally, for $z = 5$, we obtain, from Equation (28), that $5p + 3 \mid 4(2p + 4) = 8p + 16$, and thus

$$5p + 3 \mid 3(8p + 16) = 24p + 48$$

and

$$5p + 3 \mid 24p + 48 - 16(5p + 3) = -56p.$$

Since $p \neq 3$, $5p + 3$ and p are relatively prime, so we must have $5p + 3 \mid 56$. But, once again this contradicts $p > 23$. □

Proof of Theorem 2. Theorem 2 now follows since Proposition 1, Proposition 2, Proposition 3, and Proposition 4 exhaust all possible options for the omitted divisors. □

4. Strongly 2-Near Perfect Numbers

A slightly different way of defining a number to be pseudoperfect is to say that a number n is pseudoperfect if there is a set S which is a subset of the positive

n	$\sigma(n)$	d_1	d_2
156	392	2	78
352	756	8	44
6832	15376	4	1708
60976	122512	148	412
91648	184140	128	716
152812	306432	302	506
260865	539136	15	17391

Table 2: Strongly 2-near perfect numbers under one million

divisors of n such that the sum of the elements in S sums to $2n$. The last author and McCormack [4] studied what they called strongly pseudoperfect numbers. A number n is said to be *strongly pseudoperfect* if there is a subset S of divisors of n where the sum of the elements sums to $2n$, and where we also have the property that $d \in S$ if and only if $\frac{n}{d} \in S$. It is natural to combine the notion of 2-near perfect and strongly pseudoperfect as follows: we say that a number n is *strongly 2-near perfect* if n is strongly pseudoperfect and also 2-near perfect. Note that this is equivalent to n having a divisor d such that

$$\sigma(n) - d - \frac{n}{d} = 2n.$$

Table 2 gives all seven strongly 2-near perfect numbers less than one million.

In this section, we will give a description of all strongly 2-near perfect numbers n of the form $n = 2^k p$ for a prime p.

Lemma 8. *If n is a strong 2-near perfect number of the form $2^k p$ for some odd prime p and natural number k, then $p = \frac{2^{k+1}-2^a-1}{1+2^{k-a}}$.*

Proof. Assume we have a strong 2-near perfect number. By looking at our four families of numbers which arise from Theorem 1, we can see that only numbers in the third family might possibly be strongly 2-near perfect. In the first family, the product of omitted divisors $d_1 d_2$ is odd, so one cannot have $d_1 d_2 = n$ since n is even. In the second family, we have $d_1 d_2$ is a power of 2, and thus is not n. In our fourth family, we have $p^2 \mid d_1 d_2$ so $d_1 d_2 \neq n$.

Thus, we may assume that we have a number arising from the third family. In that situation, from $d_1 d_2 = n$ we get that $a + b = k$, from which the result follows. □

Lemma 9. *Assume that n is a strong 2-near perfect number of the form $n = 2^k p$ with $p = \frac{2^{k+1}-2^a-1}{1+2^{k-a}}$. Then $k < 2a$.*

Proof. Assume that n is a strong 2-near perfect number of the form $n = 2^k p$ with $p = \frac{2^{k+1}-2^a-1}{1+2^{k-a}}$, and that $k \geq 2a$. Thus, we have $1 + 2^{k-a} \mid 2^{k+1} - 2^a - 1$, which implies that

$$1+2^{k-a} \mid 2^{k+1} - 2^a - 1 + (1+2^{k-a}) = 2^{k+1} - 2^a + 2^{k-a} = 2^a(2^{k+1-a} + 2^{k-2a} - 1).$$

Since $k \geq 2a$, $2^{k+1-a} + 2^{k-2a} - 1$ is a positive integer. We also have that $(1 + 2^{k-a}, 2^a) = 1$, so we have then that

$$1 + 2^{k-a} \mid 2^{k+1-a} + 2^{k-2a} - 1. \tag{30}$$

Consider the situation where $k = 2a$. Then Equation (30) becomes $1+2^a \mid 2^{a+1}$, which has no solutions. So, we may assume that $k > 2a$.

We have from Equation (30) that there is some m such that

$$m(1 + 2^{k-a}) = 2^{k+1-a} + 2^{k-2a} - 1. \tag{31}$$

Note that the right-hand side of the equation is odd, so m must be odd. If $m = 1$ then we have $(1 + 2^{k-a}) = 2^{k+1-a} + 2^{k-2a} - 1$, which implies that

$$1 + 2^{k-a-1} = 2^{k-a} - 2^{k-2a-1}. \tag{32}$$

The left-hand side of Equation (32) is odd, and the only way for the right-hand side to be odd is if $k - 2a - 1 = 0$. The only solution of this system of equations is when $k = 3$ and $a = 1$, which forces $p = \frac{13}{5}$ which is not an integer. Thus, we have $m \neq 1$, and so $m \geq 3$.

We thus have $3(1 + 2^{k-a-1}) \leq 2^{k-a} - 2^{k-2a-1}$, which is impossible. □

Proposition 5. *Assume that n is a strong 2-near perfect number of the form $n = 2^k p$ with $p = \frac{2^{k+1}-2^a-1}{1+2^{k-a}}$. Then $k = a + 2$, and the omitted divisors are $d_1 = 2^a$ and $d_2 = 4p$, with $p = \frac{2^{a+3}-2^a-1}{5}$, and $a \equiv 3 \pmod 4$.*

First, we need to prove the following lemma.

Lemma 10. *If $2^b + 1 \mid 3(2^a) + 1$, then $b = 2$.*

Proof. Assume that $2^b + 1 \mid 3(2^a) + 1$. We thus have for some positive integer m,

$$m(2^b + 1) = 3(2^a) + 1. \tag{33}$$

Notice that $3(2^a)+1$ is never divisible by 3, and thus b must be even, and m cannot be divisible by 3. If $m = 1$ then we have $2^b + 1 = 3(2^a) + 1$ which would imply we would have $3 \mid 2^b$, which cannot happen. Thus, we may assume that $m \geq 5$. This implies that $a > b$. Set $a = bq + r$ where $0 \leq r < b$. We have

$$(3)(2^{qb+r} + 1) = (3)((2^b)^q 2^r + 1) \equiv 3((-1)^q 2^r + 1) \pmod{2^b + 1} \equiv 0. \tag{34}$$

Now, we separate into two cases, depending on whether q is even or odd. If q is even, then Equation (34) yields that

$$3(2^r) + 1 \equiv 0 \pmod{2^b + 1}, \tag{35}$$

and so

$$2^b + 1 \mid 3(2^r) + 1. \tag{36}$$

Equation (36) implies that $r \geq b - 1$. We thus have $r = b - 1$. Thus,

$$2^b + 1 \mid 3(2^{b-1}) + 1 = 2^b + 1 + 2^{b-1}$$

which is impossible.

We now consider the case where q is odd. Then Equation (34) implies that

$$-3(2^r) + 1 \equiv 0 \pmod{2^b + 1}. \tag{37}$$

Thus, $2^b + 1 \mid 3(2^r) - 1$, which similarly leads to a contradiction, unless $b = 2$ and $r = 1$. □

One might wonder if Lemma 10 can be strengthened to conclude that if p is a prime where $p \mid 2^b + 1$ for some even b and $p \mid 3(2^a) + 1$ for some a, then one must have $p = 5$. However, this is in fact not true. In particular, note that $29 \mid 2^{14} + 1$, but it is also true that $29 \mid 3(2^9) + 1$. We now prove Proposition 5.

Proof of Proposition 5. Assume that n is a strongly 2-near perfect number of the form $n = 2^k p$ with $p = \frac{2^{k+1} - 2^a - 1}{1 + 2^{k-a}}$. Our proof is complete if we can show that we must have $k = a + 2$. If we have $k = a + b$, then this is the same as $2^b + 1 \mid 2^{a+b+1} - 2^a - 1$, which implies that $2^b + 1 \mid 2^{a+b+1} - 2^a + 2^b$. We have

$$2^b + 1 \mid 2^{a+b+1} - 2^a - 1 - 2^{a+1}(1 + 2^b) = -3(2^a) - 1$$

and so $2^b + 1 \mid 3(2^a) + 1$, which allows us to apply Lemma 10, to conclude that $b = 2$, and the rest follows from noting that all 2-near perfect of this form are in Case 3. □

We list in Table 3 the first few values of a where $\frac{2^{a+3} - 2^a - 1}{5}$ is prime, and its corresponding prime p, each of which corresponds to a strong 2-near perfect number of the form $2^{a+2}p$. We do not include the last two primes as they are too big to fit on one line. Standard heuristic arguments suggest that there should be infinitely many primes of the form $\frac{2^{a+3} - 2^a - 1}{5}$.

a	p
3	11
7	179
19	734003
27	187904819
31	3006477107
39	769658139443
151	3996293539576687666963200714458586381871690547
199	–
451	–

Table 3: Primes of the form $\frac{2^{a+3}-2^a-1}{5}$

5. Open Problems

One obvious direction is to try to extend the classification of 2-near perfect numbers to classify all of the form $2^k p^m$ where $m \geq 3$.

Conjecture 1. There are only finitely many 2-near perfect numbers of the form $2^k p^m$ where $m \geq 2$.

A slightly weaker conjecture is the following.

Conjecture 2. For any fixed $m \geq 2$, there are only finitely many 2-near perfect numbers of the form $2^k p^m$.

Another potential for further research is to change the signs in the relationship $\sigma(n) = 2n + d_1 + d_2$. The two other options are $\sigma(n) = 2n - d_1 - d_2$ and $\sigma(n) = 2n + d_1 - d_2$. It seems likely that the main method used in this paper, including the discriminant sandwich, would be successful for the first of these two situations, but the situation with mixed signs on the divisors may be more difficult.

Acknowledgement. This paper was written as part of the Hopkins School Mathematics Seminar 2022-2023. Steven J. Miller suggested the problem of changing signs as discussed in the final section. The referee helpfully identified a flaw in an earlier version of the proof of Proposition 4.

References

[1] P. Cohen, K. Cordwell, A. Epstein, C. Kwan, A. Lott, and S. J. Miller, On near perfect numbers, *Acta Arith.* **194** (2020), 341-366.

[2] Hasanalizade, E. On near-perfect numbers of special forms, *Bull. Aust. Math. Soc.* **108** (3) (2023), 366-372.

[3] Y. Li and Q. Liao, A class of new near perfect numbers, *J. Korean Math. Soc.*, **52** (4) (2015), 751-763.

[4] T. McCormack and J. Zelinsky, Weighted versions of the arithmetic-geometric mean and Zaremba's function, `arXiv:2312.11661`.

[5] P. Pollack and V. Shevelev, On perfect and near perfect numbers, *J. Number Theory* **132** (12) (2012), 3037-3046.

[6] W. Sierpiñski, Sur les nombres pseudoparfaits, *Matematiĉki Vesnik* **2** (33) (1965), 212-213.

[7] X. Ren and Y. Chen, On near perfect numbers with two distinct prime factors, *Bull. Aust. Math. Soc.* **88** (3) (2013), 520-524 .

[8] M. Tang, X. Ma, and M. Feng, On near perfect numbers, *Colloq. Math.* **144** (2) (2016), 157-188.