

FAREY FRACTIONS WITH EQUAL NUMERATORS AND THE RANK OF UNIT FRACTIONS

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Abstract

Analytical expressions are derived for the number of fractions with equal numerators in the Farey sequence of order n, F_n , and in the truncated Farey sequence $F_n^{1/k}$ containing all Farey fractions below 1/k, with $1 \le k \le n$. These developments lead to an expression for the rank of 1/k in F_n , or equivalently $\left|F_n^{1/k}\right|$, and to remarkable relations between the ranks of different unit fractions. Furthermore, the results are extended to Farey fractions of the form 2/k.

1. Introduction

The Farey sequence F_n of order $n \in \mathbb{N}$ is an ascending sequence of irreducible fractions between 0/1 and 1/1 whose denominators do not exceed n; see, for example, [1, 2, 3, 4, 5]. Throughout the paper we exclude the fraction 0/1 from F_n . We also define the sequence $F_n^{1/k}$ as

$$F_n^{1/k} = \{ \alpha \in F_n : \alpha < 1/k \}, \quad k > 0.$$

The number of Farey fractions with denominators equal to d in F_n is well known to be given by Euler's totient function, $\varphi(d)$, when $d \leq n$. It is also well known that the sum of all denominators in F_n is twice the sum of all numerators [6]. However, expressions for the number of fractions with equal numerators in F_n are not given in the literature. Here, we define $\mathcal{N}_n(h)$ as the number of fractions with numerators equal to h in F_n . A closely related quantity is derived in Proposition 1.29 in [5], defined as the number of Farey fractions in F_n with numerators below or equal to m and given by

$$\sum_{h=1}^{m} \mathcal{N}_n(h) = \frac{1}{2} + \sum_{d \ge 1} \mu(d) \left\lfloor \frac{m}{d} \right\rfloor \left(\left\lfloor \frac{n}{d} \right\rfloor - \frac{1}{2} \left\lfloor \frac{m}{d} \right\rfloor \right); \tag{1}$$

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note that we have removed 1 from the expression in Proposition 1.29 in [5] as we exclude 0/1 from F_n .

We define $\mathcal{N}_n^{1/k}(h)$ as the number of fractions with numerators equal to h in $F_n^{1/k}$. In Section 2, analytical expressions for $\mathcal{N}_n(h)$ and $\mathcal{N}_n^{1/k}(h)$ are derived that allow us to reveal some remarkable properties of $\mathcal{N}_n(h)$ as, for example,

$$\mathcal{N}_{n+ph}(h) = \mathcal{N}_n(h) + p\varphi(h)$$
, for any integer $p \ge 0$,

from Corollary 2.

We define $I_n(1/k) = |F_n^{1/k}|$ as the rank of 1/k in F_n . In Section 2.1 new analytical expressions for $I_n(1/k)$ are developed using the results in Section 2 for $\mathcal{N}_n^{1/k}(h)$. These expressions could help in the development of efficient algorithms to compute the rank of Farey fractions and the related "order statistics" problem [5, 7]. Furthermore, $I_n(1/k)$ appears when deriving estimates for the number of resonance lines [3, 8] and for estimates of partial Franel sums [9]. In Section 2.2 the previous results are easily extended to $\mathcal{N}_n^{2/k}(h)$ and $I_n(2/k)$.

2. Results

Lemma 1. For given positive integers k > 1, n and h, the number of Farey fractions between 1/k and 1/(k-1) in F_n with numerators equal to h is $\varphi(h)$ when $n \ge kh-1$. Furthermore, the number of Farey fractions between 1/k and 2/(2k-1) in F_n with numerators equal to h is $\varphi(h)/2$, for h > 2.

Proof. Assuming that $n \ge hk-1$, we define F'_n as the subsequence of $F_n \cap \left[\frac{1}{k}, \frac{1}{k-1}\right]$ that includes only the Farey fractions with numerators below or equal to h, i.e.,

$$F'_n = \left\{ \frac{u}{l} \in F_n \cap \left[\frac{1}{k}, \frac{1}{k-1} \right] : u \le h \right\}.$$

We define the map M between F_h and F'_n and its inverse map M^{-1} as

$$M : F_h \to F'_n , \frac{t}{q} \mapsto \frac{q}{qk-t},$$
$$M^{-1} : F'_n \to F_h , \frac{u}{l} \mapsto \frac{uk-l}{u}.$$

To complete the proof, we just need to demonstrate that M is bijective.

To prove that M is injective, with $\frac{t}{q} \in F_h$, we have to show that $\frac{q}{qk-t} \in F'_n$, which is the case as gcd(q, qk - t) = 1 since gcd(q, t) = 1, $qk - t \leq n$ since it is assumed that $n \geq hk-1$, $\frac{q}{qk-t} \in [\frac{1}{k}, \frac{1}{k-1}]$ since $\frac{t}{q} \in F_h$, and $q \leq h$ since $\frac{t}{q} \in F_h$. To prove that M^{-1} is injective, with $\frac{u}{l} \in F'_n$, we have to show that $\frac{uk-l}{u} \in F_h$, which



Figure 1: Illustration of the bijective map M between F_h (bottom row) and F'_n (top row) as defined in the proof of Lemma 1. The curved arrows represent the mediant operation, where we recall that the *mediant* between the fractions $\frac{a}{b}$ and $\frac{c}{d}$ is defined as $\frac{a+c}{b+d}$.

is the case as gcd(uk - l, u) = 1 since gcd(u, l) = 1, $u \le h$ by definition of F'_n , and $0 \leq \frac{uk-l}{u} \leq 1$ since $\frac{u}{l} \in [\frac{1}{k}, \frac{1}{k-1}]$. Since 2/(2k-1) is the image of 1/2, there are $\varphi(h)/2$ fractions with numerators

equal to h between 1/k and 2/(2k-1), for h > 2.

Actually, M is similar to the map defined in Theorem 1 in [9] and the demonstration follows a similar logic. An illustration of the map M is given in Figure 1.

Theorem 1. For given integers h and n, such that 0 < h < n, $\mathcal{N}_n(h)$ is given by

$$\mathcal{N}_n(h) = n \frac{\varphi(h)}{h} - \varphi(h) - \sum_{d|h} \mu(d) \left\{ \frac{n}{d} \right\} + \delta_{1h},$$

where $\mu(x)$ is the Möbius function, $\{x\}$ represents the fractional part of x and δ_{xy} is the Kronecker delta symbol.

Proof. For given n and h we are going to count all Farey fractions of the form h/ksuch that $n \ge k > h > 0$. Note that this excludes the fraction 1/1 and it needs to be added explicitly via the term δ_{1h} ,

$$\mathcal{N}_{n}(h) = \delta_{1h} + \sum_{\substack{k=h+1\\ \gcd(h,k)=1}}^{n} 1 = \delta_{1h} + \sum_{\substack{k=h+1\\ d|gcd(h,k)=1}}^{n} \sum_{\substack{k=h+1\\ d|k}} \mu(d) = \delta_{1h} + \sum_{\substack{k=h+1\\ d|k}}^{n} \sum_{\substack{k=h+1\\ d|k}}^{n} \mu(d) = \delta_{1h} + \sum_{\substack{k=h+1\\ d|k}}^{n} \sum_{\substack{k=h+1\\ d|k}}^{n} \mu(d) - \sum_{\substack{k=1\\ d|k}}^{h} \mu(d) = \delta_{1h} + \sum_{\substack{k=h+1\\ d|k}}^{n} \mu(d) - \sum_{\substack{k=1\\ d|k}}^{h} \mu(d) = \delta_{1h} + \sum_{\substack{k=h+1\\ d|k}}^{n} \mu(d) \left(\left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{h}{d} \right\rfloor \right),$$

where we have used the fact that

$$\sum_{d|n} \mu(d) = \begin{cases} 1 \text{ if } n = 1\\ 0 \text{ if } n > 1 \end{cases} \text{ and } \sum_{\substack{k=1\\d|k}}^n 1 = \left\lfloor \frac{n}{d} \right\rfloor.$$

Then, using $\lfloor x \rfloor = x - \{x\}$ and $\sum_{d|h} \mu(d)/d = \varphi(h)/h$, we have

$$\mathcal{N}_n(h) = n \frac{\varphi(h)}{h} - \varphi(h) - \sum_{d|h} \mu(d) \left\{ \frac{n}{d} \right\} + \delta_{1h}.$$

Corollary 1. Let m and n be positive integers, and let p be a prime such that $p^m \leq n$. Then we have

$$\mathcal{N}_n(p^m) = \left\lceil (n-p^m)\left(1-\frac{1}{p}\right) \right\rceil.$$

Proof. This is derived from Theorem 1 using the facts that

$$\varphi(p^m) = p^m - p^{(m-1)} \text{ and } \sum_{d|p^m} \mu(d) \left\{\frac{n}{d}\right\} = -\left\{\frac{n}{p}\right\}$$

for p a prime, yielding

$$\mathcal{N}_n(p^m) = (n - p^m) \left(1 - \frac{1}{p}\right) + \left\{\frac{n}{p}\right\} = n - p^m + p^{(m-1)} - \left\lfloor\frac{n}{p}\right\rfloor.$$

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Corollary 2. For given integers h, n and m, such that 0 < h < n and $m \ge 0$, we have

$$\mathcal{N}_{n+mh}(h) = \mathcal{N}_n(h) + m\varphi(h).$$

Proof. This is derived from Theorem 1 and realizing the fact that

$$\sum_{d|h} \mu(d) \left\{ \frac{n}{d} \right\} = \sum_{d|h} \mu(d) \left\{ \frac{n+mh}{d} \right\}$$

for any $m \ge 0$.

Corollary 3. For given integers h and n, such that 0 < h < n/2, we have

$$\mathcal{N}_n(2h) = \begin{cases} \mathcal{N}_n(h) - \varphi(h) \text{ if } h \text{ is even} \\ \frac{1}{2} \left(\mathcal{N}_n(h) - \varphi(h) - \delta_{1h} + \sum_{d|h} \mu(d) w(n/d) \right) \text{ if } h \text{ is odd,} \end{cases}$$

where $w(x) \equiv \lfloor x \rfloor \mod 2$.

Proof. For the case in which h is even we assume it is expressed as $h = 2^m p$, with m > 0. By Theorem 1 we have

$$\mathcal{N}_{n}(2h) = n\frac{\varphi(2h)}{2h} - \varphi(2h) - \sum_{d|2h} \mu(d) \left\{\frac{n}{d}\right\}$$
$$= n\frac{\varphi(h)}{h} - 2\varphi(h) - \sum_{d|h} \mu(d) \left\{\frac{n}{d}\right\} - \sum_{d|p} \mu(2^{(m+1)}d) \left\{\frac{n}{2^{m+1}d}\right\}$$
$$= \mathcal{N}_{n}(h) - \varphi(h),$$

where we have used the fact that $\varphi(2h) = 2\varphi(h)$ for even h and $\mu(2^{(m+1)}d) = 0$ for m > 0.

For the case in which h is odd, we have $\varphi(2h) = \varphi(h)$ and for d|h we have $\mu(2d) = -\mu(d)$. Hence,

$$\mathcal{N}_{n}(2h) = n\frac{\varphi(h)}{2h} - \varphi(h) - \sum_{d|2h} \mu(d) \left\{\frac{n}{d}\right\}$$
$$= n\frac{\varphi(h)}{2h} - \varphi(h) - \sum_{d|h} \mu(d) \left\{\frac{n}{d}\right\} - \sum_{d|h} \mu(2d) \left\{\frac{n}{2d}\right\}$$
$$= \frac{1}{2} \left(\mathcal{N}_{n}(h) - \delta_{1h} - \varphi(h) + \sum_{d|h} \mu(d) \left(2\left\{\frac{n}{2d}\right\} - \left\{\frac{n}{d}\right\}\right)\right).$$

To complete the proof, we define $w(x) = 2\{x/2\} - \{x\} \equiv \lfloor x \rfloor \mbox{ mod } 2.$

As an illustration of the above results, let us inspect F_5 and F_8 :

$$F_{5} = \left\{ \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \right\},\$$

$$F_{8} = \left\{ \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{1}{1} \right\}.$$

The sequence F_5 has two fractions with numerators equal to 3 and indeed, from Theorem 1,

$$\mathcal{N}_5(3) = 5\frac{\varphi(3)}{3} - \varphi(3) - \mu(3)\left\{\frac{5}{3}\right\} = 2,$$

as $\varphi(3) = 2$ and $\mu(3) = -1$. The sequence F_8 has four fractions with numerators equal to 3 and indeed, from Corollary 2,

$$\mathcal{N}_{5+1\cdot 3}(3) = \mathcal{N}_5(3) + 1 \cdot \varphi(3) = 4.$$

Using the numerical values $\mathcal{N}_8(2) = 3$, $\varphi(2) = 1$, and $\mathcal{N}_8(4) = 2$, and Corollary 3, we obtain

$$\mathcal{N}_8(2\cdot 2) = \mathcal{N}_8(2) - \varphi(2) = 2$$

Using the numerical values $\mathcal{N}_8(6) = 1$, $\mathcal{N}_8(3) = 4$, w(8) = w(8/3) = 0, and Corollary 3, we obtain

$$\mathcal{N}_8(2\cdot 3) = \frac{1}{2}(\mathcal{N}_8(3) - \varphi(3) + \mu(1)w(8) + \mu(3)w(8/3)) = 1.$$

Corollary 4. For given integers m and n such that 0 < m < n, the number of Farey fractions in F_n with numerators below or equal to m is given by

$$\sum_{h=1}^{m} \mathcal{N}_{n}(h) = 1 + n \sum_{h=1}^{m} \frac{\varphi(h)}{h} - \Phi(m) - \sum_{h=1}^{m} \sum_{d|h} \mu(d) \left\{ \frac{n}{d} \right\},$$

where $\Phi(n) = \sum_{j=1}^{n} \varphi(j)$ is the totient summatory function.

Proof. This is directly derived from Theorem 1.

Corollary 4 is equivalent to Equation (1), from Proposition 1.29 in [5].

Corollary 5. For given integers h, n, and k such that 0 < h < n and 0 < k < n/h, we have

$$\mathcal{N}_n^{1/k}(h) = n \frac{\varphi(h)}{h} - k\varphi(h) - \sum_{d|h} \mu(d) \left\{ \frac{n}{d} \right\}.$$

Proof. From Lemma 1 we know that the number of Farey fractions between 1/k and 1/(k-1) with numerators equal to h is $\varphi(h)$ when 1 < k < n/h. Therefore, we can establish the following relation,

$$\mathcal{N}_n^{1/k}(h) = \mathcal{N}_n(h) - \delta_{1h} - (k-1)\varphi(h) \text{ for } k < n/h.$$

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The term δ_{1h} needs to be subtracted, as $F_n^{1/k}$ does not include the fraction 1/k by definition.

The total number of Farey fractions in F_n is given by $\sum_{i=1}^{n} \varphi(i)$ and the sum of the numerators of Farey fractions in F_n is known to be $(1 + \sum_{i=1}^{n} i\varphi(i))/2$. Hence, the following equalities can be established and validated with Theorem 1:

$$\sum_{h=1}^{n} \mathcal{N}_{n}(h) = \sum_{i=1}^{n} \varphi(i) ,$$
$$\sum_{h=1}^{n} h \mathcal{N}_{n}(h) = \frac{1}{2} \left(1 + \sum_{i=1}^{n} i \varphi(i) \right) .$$

2.1. Rank of Unit Fractions in F_n

Theorem 2. For given integers k and n such that $0 < k \le n$, we have

$$I_n(1/k) = n \sum_{j=1}^{\lfloor n/k \rfloor} \frac{\varphi(j)}{j} - k\Phi(\lfloor n/k \rfloor) - \sum_{j=1}^{\lfloor n/k \rfloor} \sum_{d|j} \mu(d) \left\{ \frac{n}{d} \right\}$$

where $\Phi(n) = \sum_{j=1}^n \varphi(j)$ is the totient summatory function.

Proof. We obtain $I_n(1/k)$ by adding the number of fractions with numerators equal to j, $\mathcal{N}_n^{1/k}(j)$, from j = 1 up to $j = \lfloor n/k \rfloor$,

$$I_n(1/k) = \sum_{j=1}^{\lfloor n/k \rfloor} \mathcal{N}_n^{1/k}(j).$$

To complete the proof we replace $\mathcal{N}_n^{1/k}(j)$ with the expression given in Corollary 5.

Theorem 2 is a generalization of Theorem 3 in [9]. When n is a multiple of all integers between 1 and $\lfloor n/k \rfloor$, the fractional part $\{n/d\}$ is equal to zero and Theorem 2 takes the form of Theorem 3 in [9]. Theorem 2 allows to establish remarkable equalities between the ranks of different unit fractions, as shown in Corollary 6.

Corollary 6. For any positive integers c, k, and p, and defining $k' = k + p \operatorname{lcm}(1, 2, ..., c)$, where lcm is the least common multiple function, we have

$$I_{ck'}(1/k') = I_{ck}(1/k) + (k'-k) \left(c \sum_{j=1}^{c} \frac{\varphi(j)}{j} - \Phi(c) \right)$$
$$I_{ck'}(1/k') - I_{ck}(1/k) = I_{c(k'-k)}(1/(k'-k)),$$
$$\frac{I_{c(k'-k)}(1/(k'-k))}{k'-k} = c \sum_{j=1}^{c} \frac{\varphi(j)}{j} - \Phi(c).$$

Proof. The first identity is derived from Theorem 2 by realizing the fact that

$$\sum_{j=1}^{c} \sum_{d|j} \mu(d) \left\{ \frac{n}{d} \right\} = \sum_{j=1}^{c} \sum_{d|j} \mu(d) \left\{ \frac{n+p \, \mathrm{lcm}(1,2,...,c)}{d} \right\}$$

for any integers n > 0 and $p \ge 0$. The other two identities are easily derived. \Box

2.2. Rank of Fractions of the Form 2/k in F_n

Corollary 7. For given positive integers h, n and k, where k is odd, such that h < n and k < 2n/h - 1, we have

$$\mathcal{N}_n^{2/k}(h) = n \frac{\varphi(h)}{h} - \frac{k}{2} \varphi(h) - \sum_{d|h} \mu(d) \left\{ \frac{n}{d} \right\}.$$

Proof. By virtue of Lemma 1,

$$\mathcal{N}_{n}^{2/(2k'-1)}(h) = \mathcal{N}_{n}^{1/k'}(h) + \frac{1}{2}\varphi(h)$$

for 0 < k' < n/h. Replacing $\mathcal{N}_n^{1/k'}(h)$ by the expression given in Corollary 5, we have

$$\mathcal{N}_n^{2/(2k'-1)}(h) = n\frac{\varphi(h)}{h} - \left(k' - \frac{1}{2}\right)\varphi(h) - \sum_{d|h}\mu(d)\left\{\frac{n}{d}\right\}.$$

By defining k = 2k' - 1, we have

$$\mathcal{N}_n^{2/k}(h) = n\frac{\varphi(h)}{h} - \frac{k}{2}\varphi(h) - \sum_{d|h} \mu(d) \left\{\frac{n}{d}\right\}$$

for 0 < k < 2n/h - 1.

Corollary 8. For k and n integers such that k is odd and $n \ge k > 2$, the rank of 2/k in F_n is given by

$$I_n(2/k) = n \sum_{j=1}^{\lfloor 2n/k \rfloor} \frac{\varphi(j)}{j} - \frac{k}{2} \Phi(\lfloor 2n/k \rfloor) - \sum_{j=1}^{\lfloor 2n/k \rfloor} \sum_{d|j} \mu(d) \left\{ \frac{n}{d} \right\}.$$

Furthermore, for c and p positive integers such that c is even and defining n = ck/2, n' = ck'/2, and $k' = k + p \operatorname{lcm}(1, 2, ..., c)$, we have

$$I_{n'}(2/k') = I_n(2/k) + \frac{k'-k}{2} \left(c \sum_{j=1}^c \frac{\varphi(j)}{j} - \Phi(c) \right).$$

Proof. This follows from Corollary 7 with the same steps as in the proofs of Theorem 2 and Corollary 6. \Box

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