# FAREY FRACTIONS WITH EQUAL NUMERATORS AND THE RANK OF UNIT FRACTIONS 

Rogelio Tomás García<br>CERN, Esplanade des Particules, Meyrin, Switzerland<br>rogelio.tomas@cern.ch

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#### Abstract

Analytical expressions are derived for the number of fractions with equal numerators in the Farey sequence of order $n, F_{n}$, and in the truncated Farey sequence $F_{n}^{1 / k}$ containing all Farey fractions below $1 / k$, with $1 \leq k \leq n$. These developments lead to an expression for the rank of $1 / k$ in $F_{n}$, or equivalently $\left|F_{n}^{1 / k}\right|$, and to remarkable relations between the ranks of different unit fractions. Furthermore, the results are extended to Farey fractions of the form $2 / k$.


## 1. Introduction

The Farey sequence $F_{n}$ of order $n \in \mathbb{N}$ is an ascending sequence of irreducible fractions between $0 / 1$ and $1 / 1$ whose denominators do not exceed $n$; see, for example, $[1,2,3,4,5]$. Throughout the paper we exclude the fraction $0 / 1$ from $F_{n}$. We also define the sequence $F_{n}^{1 / k}$ as

$$
F_{n}^{1 / k}=\left\{\alpha \in F_{n}: \alpha<1 / k\right\}, \quad k>0
$$

The number of Farey fractions with denominators equal to $d$ in $F_{n}$ is well known to be given by Euler's totient function, $\varphi(d)$, when $d \leq n$. It is also well known that the sum of all denominators in $F_{n}$ is twice the sum of all numerators [6]. However, expressions for the number of fractions with equal numerators in $F_{n}$ are not given in the literature. Here, we define $\mathcal{N}_{n}(h)$ as the number of fractions with numerators equal to $h$ in $F_{n}$. A closely related quantity is derived in Proposition 1.29 in [5], defined as the number of Farey fractions in $F_{n}$ with numerators below or equal to $m$ and given by

$$
\begin{equation*}
\sum_{h=1}^{m} \mathcal{N}_{n}(h)=\frac{1}{2}+\sum_{d \geq 1} \mu(d)\left\lfloor\frac{m}{d}\right\rfloor\left(\left\lfloor\frac{n}{d}\right\rfloor-\frac{1}{2}\left\lfloor\frac{m}{d}\right\rfloor\right) \tag{1}
\end{equation*}
$$

note that we have removed 1 from the expression in Proposition 1.29 in [5] as we exclude 0/1 from $F_{n}$.

We define $\mathcal{N}_{n}^{1 / k}(h)$ as the number of fractions with numerators equal to $h$ in $F_{n}^{1 / k}$. In Section 2, analytical expressions for $\mathcal{N}_{n}(h)$ and $\mathcal{N}_{n}^{1 / k}(h)$ are derived that allow us to reveal some remarkable properties of $\mathcal{N}_{n}(h)$ as, for example,

$$
\mathcal{N}_{n+p h}(h)=\mathcal{N}_{n}(h)+p \varphi(h), \text { for any integer } p \geq 0
$$

from Corollary 2.
We define $I_{n}(1 / k)=\left|F_{n}^{1 / k}\right|$ as the rank of $1 / k$ in $F_{n}$. In Section 2.1 new analytical expressions for $I_{n}(1 / k)$ are developed using the results in Section 2 for $\mathcal{N}_{n}^{1 / k}(h)$. These expressions could help in the development of efficient algorithms to compute the rank of Farey fractions and the related "order statistics" problem [5, 7]. Furthermore, $I_{n}(1 / k)$ appears when deriving estimates for the number of resonance lines $[3,8]$ and for estimates of partial Franel sums [9]. In Section 2.2 the previous results are easily extended to $\mathcal{N}_{n}^{2 / k}(h)$ and $I_{n}(2 / k)$.

## 2. Results

Lemma 1. For given positive integers $k>1, n$ and $h$, the number of Farey fractions between $1 / k$ and $1 /(k-1)$ in $F_{n}$ with numerators equal to $h$ is $\varphi(h)$ when $n \geq k h-1$. Furthermore, the number of Farey fractions between $1 / k$ and $2 /(2 k-1)$ in $F_{n}$ with numerators equal to $h$ is $\varphi(h) / 2$, for $h>2$.

Proof. Assuming that $n \geq h k-1$, we define $F_{n}^{\prime}$ as the subsequence of $F_{n} \cap\left[\frac{1}{k}, \frac{1}{k-1}\right]$ that includes only the Farey fractions with numerators below or equal to $h$, i.e.,

$$
F_{n}^{\prime}=\left\{\frac{u}{l} \in F_{n} \cap\left[\frac{1}{k}, \frac{1}{k-1}\right]: u \leq h\right\}
$$

We define the map $M$ between $F_{h}$ and $F_{n}^{\prime}$ and its inverse map $M^{-1}$ as

$$
\begin{aligned}
M & : \quad F_{h} \rightarrow F_{n}^{\prime}, \frac{t}{q} \mapsto \frac{q}{q k-t} \\
M^{-1} & : \quad F_{n}^{\prime} \rightarrow F_{h}, \frac{u}{l} \mapsto \frac{u k-l}{u}
\end{aligned}
$$

To complete the proof, we just need to demonstrate that $M$ is bijective.
To prove that $M$ is injective, with $\frac{t}{q} \in F_{h}$, we have to show that $\frac{q}{q k-t} \in F_{n}^{\prime}$, which is the case as $\operatorname{gcd}(q, q k-t)=1$ since $\operatorname{gcd}(q, t)=1, q k-t \leq n$ since it is assumed that $n \geq h k-1, \frac{q}{q k-t} \in\left[\frac{1}{k}, \frac{1}{k-1}\right]$ since $\frac{t}{q} \in F_{h}$, and $q \leq h$ since $\frac{t}{q} \in F_{h}$. To prove that $M^{-1}$ is injective, with $\frac{u}{l} \in F_{n}^{\prime}$, we have to show that $\frac{u k-l}{u} \in F_{h}$, which


Figure 1: Illustration of the bijective map $M$ between $F_{h}$ (bottom row) and $F_{n}^{\prime}$ (top row) as defined in the proof of Lemma 1. The curved arrows represent the mediant operation, where we recall that the mediant between the fractions $\frac{a}{b}$ and $\frac{c}{d}$ is defined as $\frac{a+c}{b+d}$.
is the case as $\operatorname{gcd}(u k-l, u)=1$ since $\operatorname{gcd}(u, l)=1, u \leq h$ by definition of $F_{n}^{\prime}$, and $0 \leq \frac{u k-l}{u} \leq 1$ since $\frac{u}{l} \in\left[\frac{1}{k}, \frac{1}{k-1}\right]$.

Since $2 /(2 k-1)$ is the image of $1 / 2$, there are $\varphi(h) / 2$ fractions with numerators equal to $h$ between $1 / k$ and $2 /(2 k-1)$, for $h>2$.

Actually, $M$ is similar to the map defined in Theorem 1 in [9] and the demonstration follows a similar logic. An illustration of the map $M$ is given in Figure 1.

Theorem 1. For given integers $h$ and $n$, such that $0<h<n, \mathcal{N}_{n}(h)$ is given by

$$
\mathcal{N}_{n}(h)=n \frac{\varphi(h)}{h}-\varphi(h)-\sum_{d \mid h} \mu(d)\left\{\frac{n}{d}\right\}+\delta_{1 h}
$$

where $\mu(x)$ is the Möbius function, $\{x\}$ represents the fractional part of $x$ and $\delta_{x y}$ is the Kronecker delta symbol.

Proof. For given $n$ and $h$ we are going to count all Farey fractions of the form $h / k$ such that $n \geq k>h>0$. Note that this excludes the fraction $1 / 1$ and it needs to
be added explicitly via the term $\delta_{1 h}$,

$$
\begin{aligned}
\mathcal{N}_{n}(h) & =\delta_{1 h}+\sum_{\substack{k=h+1 \\
\operatorname{gcd}(h, k)=1}}^{n} 1=\delta_{1 h}+\sum_{k=h+1}^{n} \sum_{d \mid \operatorname{gcd}(h, k)} \mu(d) \\
& =\delta_{1 h}+\sum_{k=h+1}^{n} \sum_{\substack{d|h \\
d| k}} \mu(d)=\delta_{1 h}+\sum_{d \mid h} \sum_{\substack{k=h+1 \\
d \mid k}}^{n} \mu(d) \\
& =\delta_{1 h}+\sum_{d \mid h}\left(\sum_{\substack{k=1 \\
d \mid k}}^{n} \mu(d)-\sum_{\substack{k=1 \\
d \mid k}}^{h} \mu(d)\right) \\
& =\delta_{1 h}+\sum_{d \mid h} \mu(d)\left(\left\lfloor\frac{n}{d}\right\rfloor-\left\lfloor\frac{h}{d}\right\rfloor\right)
\end{aligned}
$$

where we have used the fact that

$$
\sum_{d \mid n} \mu(d)=\left\{\begin{array}{l}
1 \text { if } n=1 \\
0 \text { if } n>1
\end{array} \text { and } \sum_{\substack{k=1 \\
d \mid k}}^{n} 1=\left\lfloor\frac{n}{d}\right\rfloor .\right.
$$

Then, using $\lfloor x\rfloor=x-\{x\}$ and $\sum_{d \mid h} \mu(d) / d=\varphi(h) / h$, we have

$$
\mathcal{N}_{n}(h)=n \frac{\varphi(h)}{h}-\varphi(h)-\sum_{d \mid h} \mu(d)\left\{\frac{n}{d}\right\}+\delta_{1 h} .
$$

Corollary 1. Let $m$ and $n$ be positive integers, and let $p$ be a prime such that $p^{m} \leq n$. Then we have

$$
\mathcal{N}_{n}\left(p^{m}\right)=\left\lceil\left(n-p^{m}\right)\left(1-\frac{1}{p}\right)\right\rceil .
$$

Proof. This is derived from Theorem 1 using the facts that

$$
\varphi\left(p^{m}\right)=p^{m}-p^{(m-1)} \text { and } \sum_{d \mid p^{m}} \mu(d)\left\{\frac{n}{d}\right\}=-\left\{\frac{n}{p}\right\}
$$

for $p$ a prime, yielding

$$
\mathcal{N}_{n}\left(p^{m}\right)=\left(n-p^{m}\right)\left(1-\frac{1}{p}\right)+\left\{\frac{n}{p}\right\}=n-p^{m}+p^{(m-1)}-\left\lfloor\frac{n}{p}\right\rfloor .
$$

Corollary 2. For given integers $h$, $n$ and $m$, such that $0<h<n$ and $m \geq 0$, we have

$$
\mathcal{N}_{n+m h}(h)=\mathcal{N}_{n}(h)+m \varphi(h)
$$

Proof. This is derived from Theorem 1 and realizing the fact that

$$
\sum_{d \mid h} \mu(d)\left\{\frac{n}{d}\right\}=\sum_{d \mid h} \mu(d)\left\{\frac{n+m h}{d}\right\}
$$

for any $m \geq 0$.
Corollary 3. For given integers $h$ and $n$, such that $0<h<n / 2$, we have

$$
\mathcal{N}_{n}(2 h)=\left\{\begin{array}{l}
\mathcal{N}_{n}(h)-\varphi(h) \text { if } h \text { is even } \\
\frac{1}{2}\left(\mathcal{N}_{n}(h)-\varphi(h)-\delta_{1 h}+\sum_{d \mid h} \mu(d) w(n / d)\right) \text { if } h \text { is odd },
\end{array}\right.
$$

where $w(x) \equiv\lfloor x\rfloor \bmod 2$.
Proof. For the case in which $h$ is even we assume it is expressed as $h=2^{m} p$, with $m>0$. By Theorem 1 we have

$$
\begin{aligned}
\mathcal{N}_{n}(2 h) & =n \frac{\varphi(2 h)}{2 h}-\varphi(2 h)-\sum_{d \mid 2 h} \mu(d)\left\{\frac{n}{d}\right\} \\
& =n \frac{\varphi(h)}{h}-2 \varphi(h)-\sum_{d \mid h} \mu(d)\left\{\frac{n}{d}\right\}-\sum_{d \mid p} \mu\left(2^{(m+1)} d\right)\left\{\frac{n}{2^{m+1} d}\right\} \\
& =\mathcal{N}_{n}(h)-\varphi(h)
\end{aligned}
$$

where we have used the fact that $\varphi(2 h)=2 \varphi(h)$ for even $h$ and $\mu\left(2^{(m+1)} d\right)=0$ for $m>0$.

For the case in which $h$ is odd, we have $\varphi(2 h)=\varphi(h)$ and for $d \mid h$ we have $\mu(2 d)=-\mu(d)$. Hence,

$$
\begin{aligned}
\mathcal{N}_{n}(2 h) & =n \frac{\varphi(h)}{2 h}-\varphi(h)-\sum_{d \mid 2 h} \mu(d)\left\{\frac{n}{d}\right\} \\
& =n \frac{\varphi(h)}{2 h}-\varphi(h)-\sum_{d \mid h} \mu(d)\left\{\frac{n}{d}\right\}-\sum_{d \mid h} \mu(2 d)\left\{\frac{n}{2 d}\right\} \\
& =\frac{1}{2}\left(\mathcal{N}_{n}(h)-\delta_{1 h}-\varphi(h)+\sum_{d \mid h} \mu(d)\left(2\left\{\frac{n}{2 d}\right\}-\left\{\frac{n}{d}\right\}\right)\right)
\end{aligned}
$$

To complete the proof, we define $w(x)=2\{x / 2\}-\{x\} \equiv\lfloor x\rfloor \bmod 2$.

As an illustration of the above results, let us inspect $F_{5}$ and $F_{8}$ :

$$
\begin{aligned}
F_{5} & =\left\{\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\right\}, \\
F_{8} & =\left\{\frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{1}{1}\right\} .
\end{aligned}
$$

The sequence $F_{5}$ has two fractions with numerators equal to 3 and indeed, from Theorem 1,

$$
\mathcal{N}_{5}(3)=5 \frac{\varphi(3)}{3}-\varphi(3)-\mu(3)\left\{\frac{5}{3}\right\}=2
$$

as $\varphi(3)=2$ and $\mu(3)=-1$. The sequence $F_{8}$ has four fractions with numerators equal to 3 and indeed, from Corollary 2,

$$
\mathcal{N}_{5+1 \cdot 3}(3)=\mathcal{N}_{5}(3)+1 \cdot \varphi(3)=4
$$

Using the numerical values $\mathcal{N}_{8}(2)=3, \varphi(2)=1$, and $\mathcal{N}_{8}(4)=2$, and Corollary 3, we obtain

$$
\mathcal{N}_{8}(2 \cdot 2)=\mathcal{N}_{8}(2)-\varphi(2)=2
$$

Using the numerical values $\mathcal{N}_{8}(6)=1, \mathcal{N}_{8}(3)=4, w(8)=w(8 / 3)=0$, and Corollary 3 , we obtain

$$
\mathcal{N}_{8}(2 \cdot 3)=\frac{1}{2}\left(\mathcal{N}_{8}(3)-\varphi(3)+\mu(1) w(8)+\mu(3) w(8 / 3)\right)=1
$$

Corollary 4. For given integers $m$ and $n$ such that $0<m<n$, the number of Farey fractions in $F_{n}$ with numerators below or equal to $m$ is given by

$$
\sum_{h=1}^{m} \mathcal{N}_{n}(h)=1+n \sum_{h=1}^{m} \frac{\varphi(h)}{h}-\Phi(m)-\sum_{h=1}^{m} \sum_{d \mid h} \mu(d)\left\{\frac{n}{d}\right\}
$$

where $\Phi(n)=\sum_{j=1}^{n} \varphi(j)$ is the totient summatory function.
Proof. This is directly derived from Theorem 1.
Corollary 4 is equivalent to Equation (1), from Proposition 1.29 in [5].
Corollary 5. For given integers $h$, $n$, and $k$ such that $0<h<n$ and $0<k<n / h$, we have

$$
\mathcal{N}_{n}^{1 / k}(h)=n \frac{\varphi(h)}{h}-k \varphi(h)-\sum_{d \mid h} \mu(d)\left\{\frac{n}{d}\right\}
$$

Proof. From Lemma 1 we know that the number of Farey fractions between $1 / k$ and $1 /(k-1)$ with numerators equal to $h$ is $\varphi(h)$ when $1<k<n / h$. Therefore, we can establish the following relation,

$$
\mathcal{N}_{n}^{1 / k}(h)=\mathcal{N}_{n}(h)-\delta_{1 h}-(k-1) \varphi(h) \text { for } k<n / h .
$$

The term $\delta_{1 h}$ needs to be subtracted, as $F_{n}^{1 / k}$ does not include the fraction $1 / k$ by definition.

The total number of Farey fractions in $F_{n}$ is given by $\sum_{i=1}^{n} \varphi(i)$ and the sum of the numerators of Farey fractions in $F_{n}$ is known to be $\left(1+\sum_{i=1}^{n} i \varphi(i)\right) / 2$. Hence, the following equalities can be established and validated with Theorem 1:

$$
\begin{aligned}
\sum_{h=1}^{n} \mathcal{N}_{n}(h) & =\sum_{i=1}^{n} \varphi(i) \\
\sum_{h=1}^{n} h \mathcal{N}_{n}(h) & =\frac{1}{2}\left(1+\sum_{i=1}^{n} i \varphi(i)\right) .
\end{aligned}
$$

### 2.1. Rank of Unit Fractions in $\boldsymbol{F}_{\boldsymbol{n}}$

Theorem 2. For given integers $k$ and $n$ such that $0<k \leq n$, we have

$$
I_{n}(1 / k)=n \sum_{j=1}^{\lfloor n / k\rfloor} \frac{\varphi(j)}{j}-k \Phi(\lfloor n / k\rfloor)-\sum_{j=1}^{\lfloor n / k\rfloor} \sum_{d \backslash j} \mu(d)\left\{\frac{n}{d}\right\}
$$

where $\Phi(n)=\sum_{j=1}^{n} \varphi(j)$ is the totient summatory function.
Proof. We obtain $I_{n}(1 / k)$ by adding the number of fractions with numerators equal to $j, \mathcal{N}_{n}^{1 / k}(j)$, from $j=1$ up to $j=\lfloor n / k\rfloor$,

$$
I_{n}(1 / k)=\sum_{j=1}^{\lfloor n / k\rfloor} \mathcal{N}_{n}^{1 / k}(j)
$$

To complete the proof we replace $\mathcal{N}_{n}^{1 / k}(j)$ with the expression given in Corollary 5.

Theorem 2 is a generalization of Theorem 3 in [9]. When $n$ is a multiple of all integers between 1 and $\lfloor n / k\rfloor$, the fractional part $\{n / d\}$ is equal to zero and Theorem 2 takes the form of Theorem 3 in [9]. Theorem 2 allows to establish remarkable equalities between the ranks of different unit fractions, as shown in Corollary 6.
Corollary 6. For any positive integers $c$, $k$, and $p$, and defining $k^{\prime}=k+$ $p \operatorname{lcm}(1,2, \ldots, c)$, where $\operatorname{lcm}$ is the least common multiple function, we have

$$
\begin{aligned}
I_{c k^{\prime}}\left(1 / k^{\prime}\right) & =I_{c k}(1 / k)+\left(k^{\prime}-k\right)\left(c \sum_{j=1}^{c} \frac{\varphi(j)}{j}-\Phi(c)\right) \\
I_{c k^{\prime}}\left(1 / k^{\prime}\right)-I_{c k}(1 / k) & =I_{c\left(k^{\prime}-k\right)}\left(1 /\left(k^{\prime}-k\right)\right) \\
\frac{I_{c\left(k^{\prime}-k\right)}\left(1 /\left(k^{\prime}-k\right)\right)}{k^{\prime}-k} & =c \sum_{j=1}^{c} \frac{\varphi(j)}{j}-\Phi(c) .
\end{aligned}
$$

Proof. The first identity is derived from Theorem 2 by realizing the fact that

$$
\sum_{j=1}^{c} \sum_{d \mid j} \mu(d)\left\{\frac{n}{d}\right\}=\sum_{j=1}^{c} \sum_{d \mid j} \mu(d)\left\{\frac{n+p \operatorname{lcm}(1,2, \ldots, c)}{d}\right\}
$$

for any integers $n>0$ and $p \geq 0$. The other two identities are easily derived.

### 2.2. Rank of Fractions of the Form $2 / \boldsymbol{k}$ in $\boldsymbol{F}_{\boldsymbol{n}}$

Corollary 7. For given positive integers $h, n$ and $k$, where $k$ is odd, such that $h<n$ and $k<2 n / h-1$, we have

$$
\mathcal{N}_{n}^{2 / k}(h)=n \frac{\varphi(h)}{h}-\frac{k}{2} \varphi(h)-\sum_{d \mid h} \mu(d)\left\{\frac{n}{d}\right\}
$$

Proof. By virtue of Lemma 1,

$$
\mathcal{N}_{n}^{2 /\left(2 k^{\prime}-1\right)}(h)=\mathcal{N}_{n}^{1 / k^{\prime}}(h)+\frac{1}{2} \varphi(h)
$$

for $0<k^{\prime}<n / h$. Replacing $\mathcal{N}_{n}^{1 / k^{\prime}}(h)$ by the expression given in Corollary 5 , we have

$$
\mathcal{N}_{n}^{2 /\left(2 k^{\prime}-1\right)}(h)=n \frac{\varphi(h)}{h}-\left(k^{\prime}-\frac{1}{2}\right) \varphi(h)-\sum_{d \mid h} \mu(d)\left\{\frac{n}{d}\right\}
$$

By defining $k=2 k^{\prime}-1$, we have

$$
\mathcal{N}_{n}^{2 / k}(h)=n \frac{\varphi(h)}{h}-\frac{k}{2} \varphi(h)-\sum_{d \mid h} \mu(d)\left\{\frac{n}{d}\right\}
$$

for $0<k<2 n / h-1$.
Corollary 8. For $k$ and $n$ integers such that $k$ is odd and $n \geq k>2$, the rank of $2 / k$ in $F_{n}$ is given by

$$
I_{n}(2 / k)=n \sum_{j=1}^{\lfloor 2 n / k\rfloor} \frac{\varphi(j)}{j}-\frac{k}{2} \Phi(\lfloor 2 n / k\rfloor)-\sum_{j=1}^{\lfloor 2 n / k\rfloor} \sum_{d \mid j} \mu(d)\left\{\frac{n}{d}\right\}
$$

Furthermore, for $c$ and $p$ positive integers such that $c$ is even and defining $n=c k / 2$, $n^{\prime}=c k^{\prime} / 2$, and $k^{\prime}=k+p \operatorname{lcm}(1,2, \ldots, c)$, we have

$$
I_{n^{\prime}}\left(2 / k^{\prime}\right)=I_{n}(2 / k)+\frac{k^{\prime}-k}{2}\left(c \sum_{j=1}^{c} \frac{\varphi(j)}{j}-\Phi(c)\right)
$$

Proof. This follows from Corollary 7 with the same steps as in the proofs of Theorem 2 and Corollary 6.

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