ON THE DISTRIBUTION OF $\phi(\sigma(N))$

Anup B. Dixit
Institute of Mathematical Sciences (HBNI), Chennai, Tamil Nadu, India.
anupdixit@imsc.res.in

Saunak Bhattacharjee
Indian Institute of Science Education and Research Tirupati, Andhra Pradesh, India.
saunakbhattacharjee@students.iisertirupati.ac.in

Received: 3/22/24, Accepted: 6/19/24, Published: 7/8/24

Abstract

Let $\phi(n)$ be the Euler totient function and $\sigma(n)$ denote the sum of divisors of n. In this note, we obtain explicit upper bounds on the number of positive integers $n \leq x$ such that $\phi(\sigma(n)) > cn$ for any $c > 0$. This is a refinement of a result of Alaoglu and Erdős.

1. Introduction

For any positive integer n, let $\phi(n)$ be the Euler-totient function given by

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

where p runs over distinct primes dividing n. Let $\sigma(n)$ be the sum of divisors of n, which is given by

$$\sigma(n) = \sum_{d|n} d = n \prod_{p^k \| n} \left(\frac{1 - p^{k+1}}{1 - p}\right).$$

Here the notation $p^k \| n$ means that p^k is the largest power of p dividing n. In 1944, L. Alaoglu and P. Erdős introduced the study of compositions of such arithmetic functions. In particular, they showed that for any real number $c > 0$,

$$\#\{n \leq x : \phi\left(\sigma\left(n\right)\right) \geq cn\} = o(x) \qquad \text{and} \qquad \#\{n \leq x : \sigma\left(\phi\left(n\right)\right) \leq cn\} = o(x).$$

DOI: 10.5281/zenodo.12685741

In [3], F. Luca and C. Pomerance obtained finer results on the distribution of $\sigma(\phi(n))$. The objective of this paper is to study the distribution of $\phi(\sigma(n))$.

Denote by $\log_k$ the k-fold iterated logarithm $\log\log\cdots\log$ (k-times). We show the following result.

Theorem 1. *For every $c > 0$,*

$$\#\{n \leq x : \phi(\sigma(n)) \geq cn\} \leq \frac{\pi^2 x}{6c\log_4 x} + O\left(\frac{x\log_3 x}{(\log x)^{\frac{1}{\log_3 x}}\log_4 x}\right),$$

where the implied constant only depends on c.

This implies that except for $O\left(\frac{x}{\log\log\log\log x}\right)$ integers less than x, $\phi(\sigma(n)) < cn$ for any $c > 0$. It is possible to replace the constant c above by a slowly decaying function. For a non-decreasing real function f, define

$$P_f(x) := \left\{n \leq x : \phi(\sigma(n)) \geq \frac{n}{f(n)}\right\}.$$

Then, we prove the following.

Theorem 2. *Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing function satisfying*

$$f(x) = o(\log_4 x).$$

Then,

$$|P_f(x)| = O\left(\frac{xf(x)}{\log_4 x} + \frac{x\log_3 x}{(\log x)^{\frac{1}{\log_3 x}}\log_4 x}\right) = o(x)$$

as $x \to \infty$. In other words, for almost all positive integers n, $\phi(\sigma(n)) < \frac{n}{f(n)}$.

Choosing $f(x) = \log_5 x$ in Theorem 2, we obtain the following corollary, which is an improvement of the result of Alaoglu and Erdős [2].

Corollary 1. *Except for $O\left(\frac{x\log_5 x}{\log_4 x}\right)$ positive integers $n \leq x$,*

$$\phi(\sigma(n)) \leq \frac{n}{\log_5 n}.$$

2. Preliminaries

A necessary component of our proof is to estimate the number of positive integers not greater than x which do not have certain prime factors. Such an estimate requires an application of Brun's sieve. For our purpose, we invoke the following result by P. Pollack and C. Pomerance [4, Lemma 3].

Lemma 1. *Let P be a set of primes and for $x > 1$, let*

$$A(x) = \sum_{\substack{p \le x \\ p \in P}} \frac{1}{p}.$$

Then uniformly for all choices of P, the proportion of $n \le x$ free of prime factors from P is $O\left(e^{-A(x)}\right)$.

We also recall the famous Siegel-Walfisz theorem (see [5, Corollary 11.21]).

Lemma 2 (Siegel-Walfisz)**.** *For $(a, q) = 1$, let $\pi(x; q, a)$ denote the number of primes $p \le x$ such that $p \equiv a (\bmod q)$. Let $A > 0$ be given. If $q \le (\log x)^A$, then*

$$\pi(x; q, a) = \frac{li(x)}{\phi(q)} + O\left(x \exp(-c\sqrt{\log x})\right),$$

where the implied constant only depends on A and $li(x) := \int_2^x \frac{1}{\log t}\, dt$.

For any prime p, define

$$S_p(x) := \#\{n \le x : p \nmid \sigma(n)\}.$$

The main ingredient in the proof of Theorem 1, which is also interesting in its own right, is an upper bound for $S_p(x)$.

Lemma 3. *For any prime p and $x \ge e^p$*

$$S_p(x) = O\left(x \left(\frac{\log\log x}{\log x}\right)^{\frac{1}{p-1}}\right),$$

where the implied constant is absolute.

Proof. Note that for any prime $q \equiv -1 \bmod p$, all n such that $q \| n$ satisfy $p \mid \sigma(n)$. Thus, to obtain an upper bound for $S_p(x)$, it suffices to estimate the number of $n \le x$ such that either $q \nmid n$ or $q^2 \mid n$ for a subset of primes $q \equiv -1 (\bmod p)$. By Lemma 2, for $x > e^p$, we have

$$\pi(x; p, -1) = \frac{x}{(p-1)\log x} + O\left(\frac{x}{(\log x)^2}\right),$$

where the implied constant is absolute. Now suppose x is sufficiently large such that $\log x > e^p$. Applying partial summation, we obtain

$$\begin{aligned}
\sum_{\substack{\log x < q < x \\ q \equiv -1 (\bmod p)}} \frac{1}{q} &= \frac{\pi(x; p, -1)}{x} - \frac{\pi(\log x; p, -1)}{\log x} + \int_{\log x}^{x} \frac{\pi(t; p, -1)}{t^2}\, dt \\
&= \frac{1}{p-1} \int_{\log x}^{x} \frac{1}{t \log t}\, dt + O\left(\frac{1}{\log_2 x}\right) \\
&= \frac{1}{p-1} (\log_2 x - \log_3 x) + O\left(\frac{1}{\log_2 x}\right).
\end{aligned}$$

Now, applying Lemma 1 with P being the set of primes $q \equiv -1 \bmod p$ and $\log x < q < x$, we obtain that the number of $n \le x$ free of prime factors from P is

$$O\left(x\left(\frac{\log_2 x}{\log x}\right)^{\frac{1}{p-1}}\right).$$

Since

$$\#\{n \le x : q^2 \mid n \text{ for prime } q \equiv -1 \bmod p \text{ and } \log x < q < x\} \ll x \sum_{\log x < q < x} \frac{1}{q^2} \ll \frac{x}{\log x},$$

we have the lemma. □

3. Proof of Theorems 1 and 2

Proof of Theorem 1. Note that

$$\phi\left(\sigma\left(n\right)\right) = \sigma\left(n\right) \prod_{p \mid \sigma(n)} \left(1 - \frac{1}{p}\right)$$

Denote by $P(y) := \prod_{p \le y} p$, the product of all primes $\le y$. If $P(y) | \sigma(n)$, then

$$\begin{aligned}\phi(\sigma(n)) &= \sigma(n) \prod_{p \mid \sigma(n)} \left(1 - \frac{1}{p}\right)\\ &\le \sigma(n) \prod_{p \le y} \left(1 - \frac{1}{p}\right) < \frac{\sigma(n)}{\log y},\end{aligned}$$

where the last inequality follows from Merten's theorem (see [5, Theorem 2.7 (e)]), namely

$$\prod_{p \le y} \left(1 - \frac{1}{p}\right) < \frac{1}{\log y}.$$

Thus, for any $c > 0$, the inequality $\phi(\sigma(n)) < cn$ holds if $P(y) \mid \sigma(n)$, $\sigma(n) < \delta n$, and $(\log y)^{-1} \le c/\delta$. We know that (see [1, Theorem 3.4])

$$\sum_{n \le x} \sigma(n) = \frac{\pi^2}{12} x^2 + O\left(x \log x\right).$$

Using partial summation, we get

$$\sum_{n \le x} \frac{\sigma(n)}{n} = \frac{\pi^2}{6} x + O\left(\log^2 x\right).$$

Hence,

$$\#\{n \le x : \sigma(n) \ge \delta n\} = \sum_{\substack{n \le x \\ \sigma(n) \ge \delta n}} 1 \le \frac{1}{\delta} \sum_{n \le x} \frac{\sigma(n)}{n}$$
$$= \frac{\pi^2}{6\delta} x + O\left(\frac{\log^2 x}{\delta}\right).$$

Therefore,

$$\#\{n \le x : \sigma(n) < \delta n\} \ge x\left(1 - \frac{\pi^2}{6\delta}\right) + O\left(\frac{\log^2 x}{\delta}\right). \quad (1)$$

From Lemma 3, we also have

$$\#\{n \le x : P(y) \nmid \sigma(n)\} \le \sum_{p \le y} |S_p(x)|$$
$$= O\left(x \left(\frac{\log_2 x}{\log x}\right)^{\frac{1}{y}} \frac{y}{\log y}\right).$$

Hence,

$$\#\{n \le x : P(y) \mid \sigma(n)\} \ge x\left(1 - O\left(\left(\frac{\log_2 x}{\log x}\right)^{\frac{1}{y}} \frac{y}{\log y}\right)\right). \quad (2)$$

Choosing

$$y = \log_3 x \quad \text{and} \quad \delta = c \log_4 x$$

in (1) and (2), we obtain

$$\#\{n \le x : \phi(\sigma(n)) < cn\} \ge x - \frac{\pi^2 x}{6c \log_4 x} + O\left(\frac{x \log_3 x}{(\log x)^{\frac{1}{\log_3 x}} \log_4 x}\right).$$

Hence,

$$\#\{n \le x : \phi(\sigma(n)) \ge cn\} \le \frac{\pi^2 x}{6c \log_4 x} + O\left(\frac{x \log_3 x}{(\log x)^{\frac{1}{\log_3 x}} \log_4 x}\right),$$

which proves Theorem 1. □

Proof of Theorem 2. We use the exact same method as in the proof of Theorem 1, with the choices

$$y = \log_3 x \quad \text{and} \quad \delta = \frac{\log_4 x}{f(x)}$$

in (1) and (2). This gives

$$\#\left\{n \le x : \phi(\sigma(n)) < \frac{n}{f(n)}\right\} \ge x - O\left(\frac{x f(x)}{\log_4 x} + \frac{x \log_3 x}{(\log x)^{\frac{1}{\log_3 x}} \log_4 x}\right).$$

This proves Theorem 2. □

4. Concluding Remarks

The study of composition of multiplicative arithmetic functions seems to be a difficult theme in general. This has also received scant attention, except for a very few instances such as [2] and [4]. For example, it is not clear if $\phi(\sigma(n))$ has a normal order. It would be desirable to develop a unified theory for such functions and perhaps construct families of multiplicative functions whose compositions have a finer distribution.

Acknowledgements. We thank Professor Jean-Marc Deshouillers for several fruitful discussions. We also thank the referee for comments on an earlier version of this paper.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York-Heidelberg, 1976.

[2] L. Alaoglu and P. Erdos, A conjecture in elementary number theory, *Bull. Amer. Math. Soc.* **50**, (1944), 881–882.

[3] F. Luca and C. Pomerance, On some problems of Makowski-Schinzel and Erdős concerning the arithmetical functions ϕ and σ, *Colloq. Math.* **92**, no. 1, (2002), 111–130.

[4] P. Pollack and C. Pomerance, Phi, primorials, and Poisson, *Illinois J. Math.* **64**, no. 3, (2020), 319-330.

[5] H. L. Montgomery, R. C. Vaughan, *Multiplicative Number Theory I. Classical Theory*, Cambridge University Press, Cambridge, 2007.