EXPLICIT IMPROVEMENTS TO THE BURGESS BOUND VIA PÓLYA–VINOGRADOV

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Abstract

We make explicit a theorem of Fromm and Goldmakher, which states that one can improve Burgess' bound for short character sums simply by improving the leading constant in the Pólya–Vinogradov inequality. Towards achieving this, we establish explicit versions of several estimates related to the mean values of real multiplicative functions and the Dickman function.

1. Introduction

Given a non-principal Dirichlet character $\chi \pmod{q}$, it is often the case that we need to consider the size of the corresponding *character sum*,

$$S_\chi(t) = \sum_{n \le t} \chi(n). \tag{1}$$

Owing to the orthogonality relation on residues modulo q, one only ever needs to consider the case that the character sum is *short*, i.e., $t \le q$. In this case, we have the trivial estimate,

$$|S_\chi(t)| \le t.$$

There are two standard non-trivial estimates for the size of Equation (1). First, the *Pólya–Vinogradov inequality*, $S_\chi(t) \ll \sqrt{q}\log q$ (henceforth referred to as the "P–V inequality"). Second, *Burgess' bound*, $S_\chi(t) \ll t^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}+\epsilon}$, for $\epsilon > 0$ and any integer $r > 2$ if q is square-free and $r = 2, 3$ for a general q. If the modulus q is a prime, then both of these estimates can be used to show that

$$S_\chi(t) = o\left(t\right), \tag{2}$$

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for $t \gg q^{\alpha}$ for certain small explicit $\alpha > 0$. One might consider the Burgess' bound to be a better result, however, unless the character sum is particularly long. Specifically, P–V inequality implies that Equation (2) holds for $t > q^{\frac{1}{2}+\epsilon}$, while Burgess' bound implies that Equation (2) holds for $t > q^{\frac{1}{4}+o(1)}$. The proof of the Burgess' bound also relies on advanced results due to Weil [23], while the standard proof of the P–V inequality is substantially easier. Finally, one of the best-known P–V inequalities is proved using the effective range of Burgess' bound, see [14].

Conversely, when working with explicit versions of these estimates, any improvement to the leading constant in the P–V inequality will immediately yield improvements in the leading constant for Burgess' bound (see, for example, [21] and [7]). Fromm and Goldmakher [9] have recently established that, in fact, improvements to the P–V inequality can be used to extract improvements to the effective range (with respect to t) in Burgess' bound. Precisely, they establish the following relationship.

Theorem 1. ([9, Theorem A]) *Suppose the P–V inequality can be improved to*

$$S_{\chi}(t) = o\left(\sqrt{q}\log q\right)$$

for all even primitive quadratic $\chi \pmod q$*. Then*

$$S_{\xi}(t) = o\left(t\right)$$

for all $t \gg_{\epsilon} p^{\epsilon}$ *for all odd primitive quadratic* $\xi \pmod p$*.*

Based on a suggestion Fromm and Goldmakher made in their paper, we will prove the following explicit version of Theorem 1. The interested reader may also consider the work of Mangerel [15], for a different approach to the relationship between P–V inequality and Burgess' bound.

Theorem 2. *Suppose the P–V inequality can be improved to*

$$S_{\chi}(t) \leq (c_1 + o\left(1\right))\sqrt{q}\log q,$$

for all even primitive quadratic $\chi \pmod q$*. Then for all odd primitive quadratic characters* $\xi \pmod p$ *we have* $S_{\xi}(t) < ct$ *for* $t > p^{\epsilon(c_1,c)}$*, with* $\epsilon(c_1, c) = 4\pi\frac{c_1}{\delta(c)^{3/2}} + o_t(1)$ *and* $\delta(c)$ *as in Lemma 1, such that* $\delta(c) \leq 2/7$*.*

The above result is particularly interesting, as it the first to show that a Burgess' bound-like result depends in a meaningful way on the leading constant in the P–V inequality. In Table 1, we compare $\epsilon(c_1, c)$ for various c using the best known constant in the P–V inequality and several powers of 10. From Table 1, one sees that $\epsilon(c_1, c)$ roughly decays in magnitude as c_1 does. However, even to obtain an improvement over the trivial bound would require significant improvements over the best available choices of c_1. One should expect this behaviour, since one also expects to be able to take c_1 tending to 0. Additionally, since the best c_1 in the P–V

	$c = 0.99$	0.5	0.25	0.025
c_1	$\delta(c) = 1.56 \cdot 10^{-10}$	$\delta(c) = 5.51 \cdot 10^{-11}$	$1.92 \cdot 10^{-11}$	$5.78 \cdot 10^{-13}$
1	$9.15 \cdot 10^{15}$	$4.35 \cdot 10^{16}$	$2.12 \cdot 10^{17}$	$4.05 \cdot 10^{19}$
$(2\pi^2)^{-1}$	$4.64 \cdot 10^{14}$	$2.21 \cdot 10^{15}$	$1.08 \cdot 10^{16}$	$2.05 \cdot 10^{18}$
10^{-5}	$9.15 \cdot 10^{10}$	$4.35 \cdot 10^{11}$	$2.12 \cdot 10^{12}$	$4.05 \cdot 10^{14}$
10^{-10}	$9.15 \cdot 10^{5}$	$4.35 \cdot 10^{6}$	$2.12 \cdot 10^{7}$	$4.05 \cdot 10^{9}$
10^{-15}	9.15	43.5	212	$4.05 \cdot 10^{4}$
10^{-20}	$8.45 \cdot 10^{-14}$	$4.35 \cdot 10^{-4}$	$2.12 \cdot 10^{-3}$	0.405

Table 1: Sample values for $\epsilon(c_1, c)$.

inequality is obtained via Burgess' bound, one does not expect to have $\epsilon(c_1, c) < 0.25$ for all c while c_1 is fixed. While there is room for improvement in $\epsilon(c_1, c)$, we believe that our result has significance as the first of its kind. This is also part of the reason, together with the heavy analytic machinery employed, why $\epsilon(c_1, c)$ is not yet optimal. We hope this result will increase the interest in the explicit correlation between P–V and Burgess' bound.

As an aside, note that in Theorem 2, we have still included some $o(1)$ terms. This is because many of the best known P–V results appear in this form. This choice also makes the exposition more concise. Further attempts in line with this article, in particular those using completely explicit P–V results like [8] or [2], should be able to make the result completely explicit.

In order to obtain Theorem 2, we must establish some notation. Let

$$\mathbf{M}_f(x) := \frac{1}{x} \sum_{n \le x} f(n) \quad \text{and} \quad \mathbf{L}_f(x) := \frac{1}{\log x} \sum_{n \le x} \frac{f(n)}{n}.$$

The result that allowed Fromm and Goldmakher to obtain Lemma A 1 is a correlation between the two functions defined above. This correlation assures us that if $\mathbf{M}_f(x)$ is bounded away from zero, then $\mathbf{L}_f(x)$ will be as well (for certain f). The proof of Theorem 2 relies on establishing an explicit version of Lemma B in [9].

Lemma 1. *Given $c > 0$ and $x_0 = x_0(c) \ge 1$ such that*

$$|\mathbf{M}_f(x)| \ge c \Rightarrow \mathbf{L}_f(x) \ge \delta(c),$$

with

$$\delta(c) := 0.2 \exp\left(-\frac{1}{K} \log\left(\frac{9.75 \cdot 10^5}{c}\right)\left(1.42\left(\frac{9.75 \cdot 10^5}{c}\right)^{\frac{1}{2K}} + 1/2\right)\right) + o_x(1),$$

for all completely multiplicative functions $f : \mathbb{Z} \to [-1, 1]$, $x > x_0$, $K \approx 0.3286$.

This result allows us to prove Theorem 2.

Proof of Theorem 2. Here we follow the proof of Theorem A in [9]. Using Lemma 2.1 [9] and assuming $|\mathbf{M}_{\xi}(x)| \geq c$, we obtain infinitely many characters χ such that

$$|S_{\chi}(N)| \geq \left(\frac{\sqrt{l}\delta(c)\epsilon}{2\pi\varphi(l)} + o(1)\right)\sqrt{q}\log q,$$

with l the least prime larger than $\frac{2}{\delta(c)}$ which satisfies $l \equiv 3 \pmod 4$. We therefore have a contradiction if $\frac{\sqrt{l}\delta(c)\epsilon}{2\pi\varphi(l)} > c_1$, i.e., when $\epsilon > 2\pi c_1 \frac{\varphi(l)}{\sqrt{l}\delta(c)}$. We can further simplify this by observing that we trivially have $\varphi(l) \leq l$, that is an optimal result for large l. Using the version of Bertrand's postulate for primes in arithmetic progressions in [3], with the assumption $\frac{2}{\delta(c)} \geq 7$, we have that $l \leq \frac{4}{\delta(c)}$. Note that assuming a smaller upper bound for $\delta(c)$ and using Corollary 6 in [1], it is possible to reduce the constant 4 to $2 + o(1)$. As this would lengthen the exposition, we decided not to do so to keep the result as concise as possible. Thus, we obtain

$$\epsilon > 4\pi\frac{c_1}{\delta(c)^{\frac{3}{2}}}.$$

□

The proof of Lemma 1 will require two results, which will make up the bulk of this article. The easier of these is the following explicit version of Theorem 2 in [13] applied to $(1 * f)(n)$ (another non-explicit version of this result can be found in [10]). First, for a given multiplicative function f, let us define

$$u := \sum_{p\leq x}\frac{1-f(p)}{p}.$$

We now introduce a completely multiplicative function f, as defined in (1.1) of [13], such that for $x > 2$, with some positive constants K, K_1 and $k_2 < 2$, holds

$$\begin{cases} 1 \leq f(p) \leq K & (p \leq x), \\ 1 \leq f(p^m) \leq K_1 K_2^m & (p \leq x,\ m \geq 2). \end{cases} \tag{3}$$

Theorem 3. *Let f be a completely multiplicative function as in (3). Then, we have*

$$\frac{1}{x}\sum_{n\leq x}(1 * f)(n) \geq (0.2 + o(1))\log x\ e^{-u\left(1.42e^{\frac{u}{2}}+\frac{1}{2}\right)} + o(1).$$

The second result, which is the harder to prove, is an explicit version of Theorem III.4.14 in Tenenbaum's book [20]. In our current application, we focus on functions $g(n)$ which are quadratic Dirichlet characters, but there are variants of this theorem which cover a much larger class of functions (for example, see the main theorem of [12]).

Theorem 4. *Let K be the unique solution to*

$$\frac{1}{2\pi}\int_0^{2\pi} |\cos(t) - K|\, dt = 1 - K.$$

Note that $K \approx 0.3286$. If f is a real, completely multiplicative function, we have, uniformly for $x \geq 1$,

$$|\mathbf{M}_f(x)| \leq (9.75 \cdot 10^5 + o\,(1)) \exp\left\{ -K \sum_{p \leq x} \frac{1 - f(p)}{p} \right\} + o\,(1)\,.$$

We can now easily prove Lemma 1.

Proof of Lemma 1. Here, we follow the proof of Lemma B in [9]. Theorem 4 gives

$$\left(\frac{|\mathbf{M}_f(x)| + o\,(1)}{9.75 \cdot 10^5 + o\,(1)}\right)^{-\frac{1}{2K}} \geq e^{\frac{u}{2}} \quad \text{and} \quad \frac{1}{K}\log\left(\frac{|\mathbf{M}_f(x)| + o\,(1)}{9.75 \cdot 10^5 + o\,(1)}\right)^{-1} \geq u. \tag{4}$$

It is easy to see that

$$\mathbf{L}_f(x) = \frac{1}{x \log x} \sum_{n \geq x} (1 * f)(n) + o\,(1)\,,$$

and, by Theorem 3, we obtain

$$\mathbf{L}_f(x) \geq (0.2 + o\,(1))\, e^{-u(1.42e^{u/2} + 1/2)} \log x + o\,(1)\,. \tag{5}$$

The result follows by substituting Equation (4) into Equation (5) and remembering that $|\mathbf{M}_f(x)| \geq c$. □

In Section 2 we will prove Theorem 3. In Section 3 we will prove a partially explicit version of an upper bound for the mean value of multiplicative functions, that works as an intermediate result for Theorem 4. In Section 4, we introduce some explicit bounds related to prime numbers; applying these results, to those obtained in the previous sections, we conclude with a proof of Theorem 4. To ease the understanding of the relationships between the results we introduce a scheme; see Figure 1.

2. Lower Bound for the Mean Value Theorem for a Non-Negative Multiplicative Function

The aim of this section is to prove Theorem 3. We start by giving an explicit lower bound for the Dickman function, $\rho(x)$, defined by

$$x\rho'(x) + \rho(x-1) = 0,$$

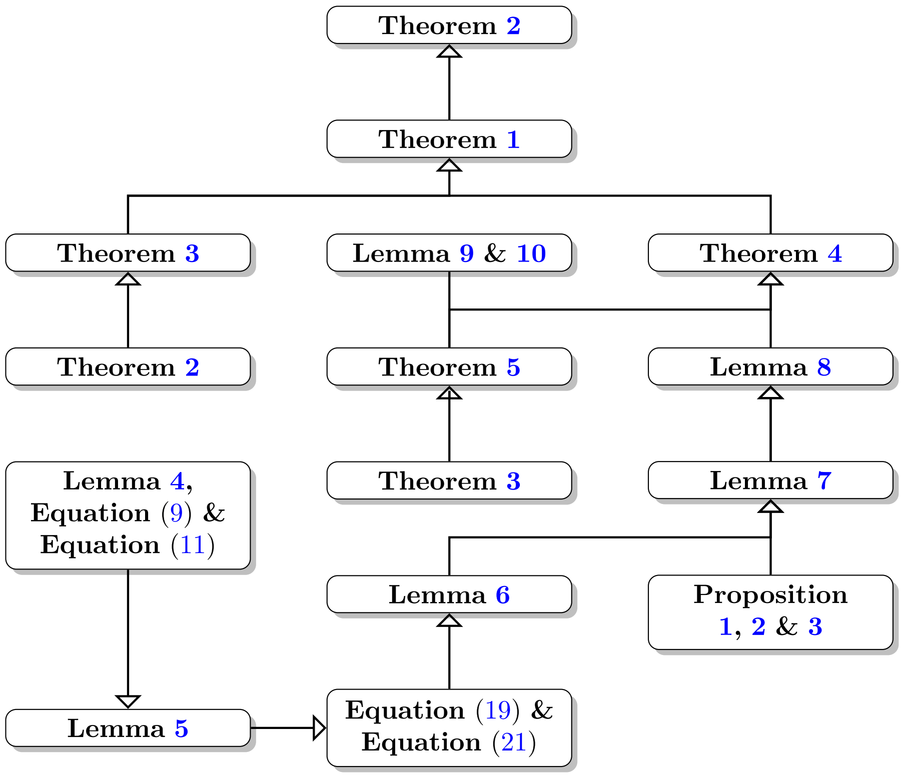

Figure 1: Relationship between the results.

with initial conditions $\rho(x) = 1$ for $0 \leq x \leq 1$. Note that we will follow Buchstab's approach from [4] for large x, alongside computations for small x.

Lemma 2. *Assuming $x \geq 1$, we have*

$$\rho(x) \geq x^{-1.42x}. \tag{6}$$

Proof. Using the the built-in Dickman function in Sage, we determine that for $1 \leq x \leq 130$ we can take as an exponent 1.15. Note that we are limited to this interval due to the computational complexity. We can thus use the following result due to Buchstab [4], that tells us that for $x \geq 6$ and $\delta = \frac{1}{\log x + 1 + \frac{\log x}{x}} < \frac{1}{3}$, we have

$$\rho(x) \geq \exp\left(-x\left(1 + \frac{1}{\log x}\right)\left(\log(x+\delta) + \log\frac{1}{\delta} - 1\right) - 2\log x\right), \tag{7}$$

and the result follows taking $x \geq 130$. □

It is worth noting that, by [4], the right size for the constant in the exponent of Equation (6) is $1 + o(1)$. Since we want a uniform result, a lower bound for the

$1 + o(1)$ term appears, by computation, to be 1.15. Obtaining this result appears difficult as Equation (7) does not give a good estimate for small values of x. One might get around this by obtaining an explicit version of other estimates for $\rho(x)$, such as the one in [5], but we have not pursued this here. We can now prove Theorem in 3.

Proof of Theorem 3. This is Theorem 2 in [13] with $K = 2$, $K_2 = 1.1$ and $z = 2$, used together with $\max(0, 1 - (1 * f)(p)) \leq \frac{1-f(p)}{2}$ and Lemma 2. We also need to note that

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(1 + \frac{(1 * f)(p)}{p} + \frac{(1 * f)(p^2)}{p^2} \cdots \right) = \prod_{p \leq x} \frac{1 - \frac{1}{p}}{1 - \frac{(1*f)(p)}{p}}$$

$$\geq e^{-u} \exp\left(\sum_{p \leq x} \frac{1}{p}\right) \exp\left(\sum_{p \leq x} \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right)\right).$$

We can conclude using Theorem 5 and Corollary 1 in [19], which gives

$$\exp\left(\sum_{p \leq x} \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right)\right) = \exp(M - \gamma) \geq \exp(-0.32),$$

with M the Meissel–Mertens constant and γ the Euler–Mascheroni constant. □

3. A Partially Explicit Upper Bound for the Mean Value of Multiplicative Functions

In this section, we aim to prove an explicit version of a theorem of Montgomery in [16], regarding the mean value of multiplicative functions. He restricted his interest, as will we, to completely multiplicative functions. The more general case involves technical changes (see [20]) which make the leading constant increase significantly.

We start by introducing for clarity a well-known, but useful, result.

Lemma 3. *Assuming $s = 1 + \alpha + i\tau$, with $\alpha \searrow 0$ and $|\tau| \leq 1/2$, we have*

$$\left|\frac{\zeta'}{\zeta}(s)\right| \leq \frac{1}{|s-1|} + \mathcal{O}(1).$$

Proof. By the Euler–Maclaurin formula, we have

$$\sum_{n \leq N} \frac{1}{n^s} = \int_1^N \frac{1}{x^s} dx + \frac{1}{2}\left(\frac{1}{N^s} + 1\right) - s\int_1^N \frac{1}{x^{s+1}}\left(\{x\} - \frac{1}{2}\right) dx.$$

Thus, taking $N \to \infty$,

$$\left| \sum_{n}^{\infty} \frac{1}{n^s} - \int_1^{\infty} \frac{1}{x^s} dx \right| \leq \frac{1}{2}(1+|s|).$$

Now, it follows from

$$\int_1^{\infty} \frac{(\log x)^{\ell}}{x^s} dx = \frac{\ell!}{(s-1)^{\ell+1}}$$

that

$$\left| \zeta(s) - \frac{1}{(s-1)} \right| \leq \frac{1}{2}(1+|s|).$$

Proceeding in the same way, we obtain

$$\left| \zeta'(s) + \frac{1}{(s-1)^2} \right| \leq \frac{1}{2}(1+|s|).$$

The result easily follows remembering that $\alpha \searrow 0$. □

Everything is in place to prove an explicit version of the inequality in [16]. Note that our result appears slightly different when compared with the cited one, as we have tailored the optimization of the constant for the current application.

Theorem 5. *Let g be a completely multiplicative function such that $|g(n)| \leq 1$. Set*

$$G(x) := \sum_{n \leq x} g(n), \qquad F(s) := \sum_{n=1}^{\infty} g(n) n^{-s}.$$

We define

$$H(\alpha)^2 := \sum_{k \in \mathbb{Z}} \frac{1}{(k-1/2)^2+1} \max_{\substack{\sigma=1+\alpha \\ |\tau-k| \leq \frac{1}{2}}} |F(s)|^2.$$

Then, for $x \geq x_0$ large enough,

$$G(x) \leq (3.14 + o(1)) \frac{x}{\log x} \int_{1/\log x}^{1} H(\alpha) \frac{d\alpha}{\alpha} + O_{x_0}\left(\frac{x}{\sqrt{\log x}} \right).$$

Proof. We now establish, for $x \geq x_0$, the following result:

$$\int_{\sqrt{x}}^{x} \frac{|G(t)|}{t^2} dt \leq \left(\sqrt{\frac{9.45}{2}} + o(1) \right) H\left(\frac{2}{\log x} \right) + O\left(\sqrt{\log x} \right). \tag{8}$$

By the Cauchy–Schwarz inequality, with $\alpha = 2/\log x$,

$$\int_{\sqrt{x}}^{x} \frac{|G(t)|}{t^2} dt \leq \left(\int_1^x \frac{(|G(t)| \log t)^2}{t^{3+2\alpha}} dt \int_{\sqrt{x}}^{x} \frac{1}{\log^2 t \; t^{1-2\alpha}} dt \right)^{1/2}.$$

We can observe that, with $n \in \mathbb{N}$,

$$\int_{\sqrt{x}}^{x} \frac{1}{\log^2 t\ t^{1-2\alpha}} dt \le \sum_{j=0}^{n-1} \int_{x^{\frac{1}{2}\left(\frac{j}{n}+1\right)}}^{x^{\frac{1}{2}\left(\frac{j+1}{n}+1\right)}} \frac{1}{\log^2 t\ t^{1-2\alpha}} dt$$

$$\le \frac{1}{\log^2 x} \sum_{j=0}^{n-1} \frac{4}{\left(\frac{j}{n}+1\right)^2} \int_{x^{\frac{1}{2}\left(\frac{j}{n}+1\right)}}^{x^{\frac{1}{2}\left(\frac{j+1}{n}+1\right)}} \frac{1}{t^{1-2\alpha}} dt$$

$$\le \frac{1}{\log^2 x\ \alpha} (e^{\frac{2}{n}} - 1) 2e^2 \sum_{j=0}^{n-1} \frac{e^{2j/n}}{(\frac{j}{n}+1)^2}$$

$$\le \frac{1}{\log^2 x\ \alpha} (e^{\frac{2}{n}} - 1) 2n \int_1^2 \frac{e^{2y}}{y^2} dy \le \frac{4 \cdot 9.45}{\log^2 x\ \alpha}.$$

Defining $K(t) := \sum_{n \le t} g(n) \log n$, then

$$G(t) \log t - K(t) \ll t.$$

Thus, taking $\alpha = 2/\log x$, the proof of Equation (8) reduces to that of

$$\int_1^x \frac{|K(t)|^2}{t^{3+2\alpha}} dt \le \left(\frac{1}{2} + o(1)\right) \frac{H(\alpha)^2}{\alpha}.$$

The equation

$$\int_0^\infty K(e^u) e^{-u\sigma} e^{-iur} du = \frac{-F'(s)}{s} \quad (\sigma > 1)$$

allows us to write Plancherel's formula as

$$\int_1^x \frac{|K(t)|^2}{t^{3+2\alpha}} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{F'(1+\alpha+i\tau)}{1+\alpha+i\tau} \right|^2 d\tau.$$

We assume T arbitrary large. For $|\tau| > T$ we have, by (4.46) in [20],

$$\int_{|\tau|>T} \left| \frac{F'(1+\alpha+i\tau)}{1+\alpha+i\tau} \right|^2 d\tau \ll \frac{1}{T} + \frac{1}{\alpha^3 T^2}.$$

We now estimate the contribution in the complementary range $|\tau| \le T$. We write

$$\int_{|\tau|\le T} \left| \frac{F'(1+\alpha+i\tau)}{1+\alpha+i\tau} \right|^2 d\tau \le \sum_{|k|\le T} \frac{1}{1+(k-1/2)^2} \int_{k-1/2}^{k+1/2} \left|F'(1+\alpha+i\tau)\right|^2 d\tau.$$

The right-hand side integral does not exceed

$$\max_{\substack{\sigma=1+\alpha \\ |\tau-k|\le 1/2}} |F(s)|^2 \int_{k-1/2}^{k+1/2} \left| \frac{F'}{F}(1+\alpha+i\tau) \right|^2 d\tau.$$

We can observe that

$$\left|\frac{F'}{F}(s)\right|^2 \leq \left|\frac{\zeta'}{\zeta}(s)\right|^2,$$

and, choosing $x \geq x_0$ to have $\alpha = 2/\log x$ small enough, by Lemma 3 we obtain

$$\int_{k-1/2}^{k+1/2} \left|\frac{F'}{F}(1+\alpha+i\tau)\right|^2 d\tau \leq \int_{-1/2}^{1/2} \left|\frac{\zeta'}{\zeta}(1+\alpha+i\tau)\right|^2 d\tau$$
$$\leq \int_{-1/2}^{1/2} \frac{1}{\alpha^2+\tau^2} d\tau + \mathcal{O}(1) = \frac{\pi}{\alpha} + \mathcal{O}(1).$$

Thus, Equation (8) is obtained taking $T \to \infty$. We now introduce (4.39) from [20]:

$$|G(x)| \leq \frac{x}{\log x} \int_1^x \frac{|G(t)|}{t^2} dt + O\left(\frac{x}{\log x}\right).$$

With the above result and using Equation (8) we can now finish the proof as follows:

$$\int_{e^2}^{x} \frac{|G(t)|}{t^2} dt \leq \frac{1}{\log 2} \int_{e^2}^{x} \frac{|G(t)|}{t^2} \int_t^{t^2} \frac{dy}{y\log y} dt$$
$$\leq \frac{1}{\log 2} \int_{e^2}^{x^2} \frac{dy}{y\log y} \int_{\sqrt{y}}^{y} \frac{|G(t)|}{t^2} dt$$
$$\leq \frac{1}{\log 2} \left(\sqrt{\frac{9.45}{2}} + o(1)\right) \int_{1/\log x}^{1} \frac{H(\alpha)}{\alpha} d\alpha + O_{x_0}\left(\sqrt{\log x}\right).$$

□

4. Explicit Mean Value Estimates for Real Multiplicative Functions

In this section we aim to prove Theorem 4. We will first, in Subsections 4.1 and 4.2, introduce some useful explicit results and then tackle Theorem 4 in Subsection 4.3.

4.1. Prime Counting Estimates

Take $\pi(x)$ to be the prime counting function. We provide two versions of the Prime Number Theorem (PNT), the first good for small x and the second for large x. Assuming $x \geq 59$, by [19] we have

$$\frac{x}{\log x}\left(1 + \frac{1}{2\log x}\right) \leq \pi(x) \leq \frac{x}{\log x}\left(1 + \frac{3}{2\log x}\right). \tag{9}$$

Defining

$$\mathrm{li}(x) = \int_0^x \frac{1}{\ln y} dy \tag{10}$$

and taking $x \geq 229$, by Corollary 2 [22], we have

$$|\pi(x) - \mathrm{li}(x)| \leq x \frac{0.2795}{(\log x)^{3/4}} \exp\left(-\sqrt{\frac{\log x}{6.455}}\right). \tag{11}$$

Note that there is a better version of the PNT due to Platt and Trudgian [18]. However, we will turn the above result into a uniform one and the improvement obtained using Platt and Trudgian's result is not clear and would make the following exposition longer and more complicated. We also note that another way to improve the result could be using the improved zero-free region for the Riemann zeta function given in [17]. We now provide some useful bounds on $\mathrm{li}(x)$.

Lemma 4. *For $x \geq 2$ we have*

$$\mathrm{li}(x) \geq \frac{x}{\log x}\left(1 + \frac{1}{\log x}\right).$$

Proof. By repeatedly integrating Equation (10) by parts, we have

$$\mathrm{li}(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + \int_0^x \frac{2}{\ln^3 y} dy,$$

and the result follows by observing that the last integral is positive for $x \geq 2$. □

From [1, Lemma 5.9] we have, for $x \geq 1865$,

$$\mathrm{li}(x) \leq \frac{x}{\log x}\left(1 + \frac{3}{2\log x}\right) + \mathrm{li}(2). \tag{12}$$

We can now prove the main lemma.

Lemma 5. *For all $x \geq 2$ we have*

$$|\pi(x) - \mathrm{li}(x)| \leq 0.4897 \frac{x}{\log x}, \tag{13}$$

$$|\pi(x) - \mathrm{li}(x)| \leq 1.3597 \frac{x}{\log^2 x}, \tag{14}$$

and

$$|\pi(x) - \mathrm{li}(x)| \leq 0.1522 x \exp\left(-\sqrt{\frac{\log x}{6.455}}\right). \tag{15}$$

Proof. We first prove Equation (13) and Equation (14). For $2 \le x \le 10^5$, we obtain these inequalities by computation. Assuming $x \ge 10^5$, we obtain these inequalities using Equation (9), Lemma 4, and Equation (12).

Now, we prove Equation (15). For $2 \le x \le 10^3$, we obtain the inequality by computation. Assuming $x \ge 10^3$ we obtain the inequality using Equation (11). □

Let f be a 2π-periodic function of bounded variation on $[0, 2\pi]$. Writing $S(f) := \sup_t |f(t)|$, $V(f) := \int_0^{2\pi} |d\{f(t)\}|$, we can now prove the following results. Assuming $w > 1$, by Equation (13) we obtain

$$\left| \frac{|\pi(x) - \mathrm{li}(x)| f(\tau \log t)}{t} \right|_w^z \le \frac{0.9794}{\log w} S(f), \tag{16}$$

and, by Equation (14),

$$\left| \int_w^z \frac{|\pi(x) - \mathrm{li}(x)| f(\tau \log t)}{t^2} dt \right| \le \frac{1.3597}{\log w} S(f). \tag{17}$$

By Equation (15) we obtain

$$\begin{aligned} \left| \int_w^z \frac{R(t)}{t} d\{f(\tau \log t)\} \right| &\le 0.1522 \int_{\tau \log w}^{\tau \log z} \exp\left(-\sqrt{\frac{v}{6.455\tau}} \right) |df(v)| \\ &\le 0.1522 \int_{\tau \log w}^{\tau \log w + 2\pi} \sum_{k=0}^{\infty} \exp\left(-\sqrt{\frac{v + 2\pi k}{6.455\tau}} \right) |df(v)| \\ &\le 0.1522 V(f) \sum_{k=0}^{\infty} \exp\left(-\sqrt{\frac{\tau \log w + 2\pi k}{6.455\tau}} \right). \end{aligned} \tag{18}$$

We focus on the two following cases. For $0 < \tau \le 1$, $w = \exp(\frac{c}{\tau})$, with $c \ge 1$, and

$$l(c, \tau, x) := \exp\left(-\sqrt{\frac{c + 2\pi k}{6.455\tau}} \right),$$

we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} l(c, \tau, k) &\le \sum_{k=0}^{k_1} l(c, \tau, k) + \int_{k_1}^{\infty} l(c, \tau, x) dx \\ &\le \sum_{k=0}^{k_1} l(c, \tau, k) + l(c, \tau, k_1) \frac{\sqrt{6.455}\sqrt{2\pi k_1 + c} + 6.455}{\pi} = O_{k_1, c}(1), \end{aligned} \tag{19}$$

where $O_{k_1,c}(1)$ will be computed later, optimizing on k_1 and c. For $\tau \ge 1$,

$$\log w = (1 + \epsilon) 6.455 \log^2(\tau + 3), \tag{20}$$

$$h(\epsilon,\tau,x) := \exp\left(-\sqrt{(1+\epsilon)\log^2(\tau+3)+\frac{2\pi x}{6.455\tau}}\right),$$

with $\epsilon > 0$, we obtain

$$\sum_{k=0}^{\infty} h(\epsilon,\tau,k) \leq \sum_{k=0}^{k_2} h(\epsilon,\tau,k) + \int_{k_2}^{\infty} h(\epsilon,\tau,x)dx \leq O_{\epsilon,k_2}(1), \tag{21}$$

with

$$O_{\epsilon,k_2}(1) =$$
$$\sum_{k=0}^{k_2} h(\epsilon,\tau,k) + h(\epsilon,\tau,k_2)\cdot\frac{\sqrt{6.455\tau}\sqrt{2\pi k_2+\tau(1+\epsilon)6.455\log^2(\tau+3)}+6.455\tau}{\pi},$$

where the $O_{\epsilon,k_2}(1)$ will be computed later optimizing on k_2 and ϵ. The above upper bounds Equation (19) and Equation (21) will be used in the next section to prove an explicit version of Lemma III.4.13 of [20]. It is interesting to note that within this non-explicit result, a stronger version of Equation (11) was used, to assure that Equation (19) and Equation (21) would converge for any $w \geq 0$. As there is no explicit version of this stronger PNT, we have that the two series converge only for certain values of w. This will come with a loss in a term in Lemma 7, and therefore balancing it with the above sums will be fundamental.

4.2. Some Useful Lemmas

The bulk of the proof of Theorem 4 can be contained in the following lemmas, which encapsulate explicit versions of Lemma III.4.13 of [20].

Lemma 6. *Let f be a 2π-periodic function of bounded variation on $[0, 2\pi]$ with mean value*

$$\overline{f} := \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt.$$

For all real numbers τ, w, z such that $\tau \neq 0$, $1 < w < z$, we have

$$\sum_{w<p\leq z} \frac{1}{p} f(\tau\log p) = \overline{f}\log\left(\frac{\log z}{\log w}\right) + E_\tau(w).$$

Writing $S(f) := \sup_t|f(t)|$ and $V(f) := \int_0^{2\pi}|d\{f(t)\}|$, for $0 < |\tau| \leq 1$, $w = \exp(\frac{c}{\tau})$

$$|E_\tau(w)| \leq \left(\frac{\pi}{2c} + 0.1522 O_{k_1,c}(1)\right)V(f) + \frac{2.3391}{c}S(f), \tag{22}$$

with $O_{k_1,c}(1)$ *defined in Equation* (19)*, while for* $|\tau| \geq 1$*,* $w = \exp((1+\epsilon)6.455\log^2(\tau+3))$*, with* $\epsilon > 0$*,*

$$|E_\tau(w)| \leq \left(\frac{\pi}{2}\frac{1}{\tau\log w} + 0.1522O_{k_2,\epsilon}(1)\right)V(f) + \frac{2.3391}{(1+\epsilon)6.455\log^2(\tau+3)}S(f), \tag{23}$$

with $O_{k_2,\epsilon}(1)$ *defined in Equation* (21)*.*

Proof. It is sufficient to prove this for $\tau > 0$. Define $R(t) := \pi(t) - \mathrm{li}(t)$. By partial summation, we have

$$\begin{aligned}\sum_{w<p\leq z}\frac{1}{p}f(\tau\log p) &= \int_w^z \frac{f(\tau\log t)}{t\log t}\,dt + \left.\frac{R(t)f(\tau\log t)}{t}\right|_w^z - \int_w^z R(t)\,d\left(\frac{f(\tau\log t)}{t}\right)\\ &= \overline{f}\log\left(\frac{\log z}{\log w}\right) + \int_{\tau\log w}^{\tau\log z}\left(f(t)-\overline{f}\right)\frac{dt}{t} + \left.\frac{R(t)f(\tau\log t)}{t}\right|_w^z\\ &\quad - \int_w^z \frac{R(t)}{t}d\{f(\tau\log t)\} + \int_w^z \frac{R(t)f(\tau\log t)}{t^2}dt. \end{aligned} \tag{24}$$

For the second term of Equation (24), we have from Equation 3.6 of [11] that, for any real a and b,

$$\left|\int_a^b \left(f(t)-\overline{f}\right)\,dt\right| \leq \frac{\pi}{4}V(f). \tag{25}$$

By the second mean value theorem for integrals, there exists a $c \in (\tau\log w, \tau\log z]$ so that

$$\int_{\tau\log w}^{\tau\log z}\left(f(t)-\overline{f}\right)\frac{dt}{t} = \frac{1}{\tau\log w}\int_{\tau\log w}^{c}(f(t)-\overline{f})dt + \frac{1}{\tau\log z}\int_c^{\tau\log z}(f(t)-\overline{f})dt. \tag{26}$$

Combining Equation (25) and Equation (26) we determine

$$\left|\int_{\tau\log w}^{\tau\log z}\left(f(t)-\overline{f}\right)\frac{dt}{t}\right| \leq \frac{\pi}{2}\frac{V(f)}{\tau\log w}.$$

The third term of Equation (24) was previously estimated in Equation (16), the fourth in Equation (18), Equation (19) and Equation (21), and the fifth in Equation (17). Combining these results together, we have Equation (22) and Equation (23). □

Recall Mertens' second theorem in the following forms. Proposition 1 is given as a corollary of [19, Theorem 1].

Proposition 1. *Let* $x > 1$*. We have*

$$\sum_{p\leq x}\frac{1}{p} = \log\log x + M + M'(x),$$

where $M \approx 0.2614\ldots$ *and*

$$|M'(x)| \leq \frac{1}{\log^2(x)}.$$

Proposition 2. *Let* $x \geq 2$. *We have*

$$\log\log x + 0.2614 \leq \sum_{p\leq x} \frac{1}{p} \leq \log\log x + 0.8666.$$

Proof. The bounds follow from Theorem 5 in [19] and some simple computations. Note that the upper bound is optimal, with equality occurring at $x = 2$. $\square$

We also introduce a helpful estimate.

Proposition 3. *Let* $x > 1$. *We have*

$$\sum_{p\leq x} \frac{\log^2 p}{p} \leq (1+10^{-8})\frac{\log^2 x}{2}.$$

Proof. For $1 < x < 355991$, one may verify that

$$\sum_{p\leq x} \frac{\log^2 p}{p} \leq \frac{\log^2 x}{2}.$$

When $x \geq 355991$, we begin by applying partial summation to the sum in question:

$$\begin{aligned}\sum_{p\leq x} \frac{\log^2 p}{p} =&\pi(x)\frac{\log^2 x}{x} + \int_2^{355991} \pi(t)\left(\frac{\log^2 t - 2\log t}{t^2}\right) dt \\ &+ \int_{355991}^{x} \pi(t)\left(\frac{\log^2 t - 2\log t}{t^2}\right) dt.\end{aligned} \tag{27}$$

One may compute the first integral exactly and find that it is bounded by 65.204. For the other instances of $\pi(t)$, it is suitable to use [6, Theorem 1.10.7], which states that, for $t \geq 355991$,

$$\pi(t) \leq \frac{t}{\log t}\left(1 + \frac{1}{\log t} + \frac{2.51}{\log^2 t}\right). \tag{28}$$

Taking Equation (28) in Equation (27) and simplifying, one arrives at

$$\sum_{p\leq x} \frac{\log^2 p}{p} \leq \frac{\log^2 x}{2}\left(1+\epsilon(x)\right),$$

where

$$\epsilon(x) \leq \frac{1.02\log\log x}{\log^2 x} - \frac{8.808}{\log^2 x} + \frac{15.06}{\log^3 x}.$$

We observe that $\epsilon(x) < 0$ until $x > e^{e^{8.634}}$, and then $\epsilon(x)$ takes a maximum at $x_0 \approx e^{e^{9.134}}$. At this maximum, $\epsilon(x_0) \leq 10^{-8}$, establishing the result. $\square$

We can now obtain an important explicit estimate.

Lemma 7. *Define $f(t) := |\cos(t) - K|$, where K is defined in Theorem 4. Uniformly for $0 < \alpha \le 1, \tau \in \mathbb{R}$, we have*

$$\sum_{p \le \exp(1/\alpha)} \frac{f(\tau \log p)}{p} \le (1-K)\log\frac{1}{\alpha} + (2+2K)\log\log(|\tau|+3) + C_0,$$

where $C_0 = 7.28$.

Proof. We may assume $\tau > 0$. Start by considering $\tau \le \alpha$. We observe that the Taylor expansion of $\cos x$ yields

$$|f(\tau \log p) - (1-K)| \le \frac{1}{2}(\tau \log p)^2.$$

Hence,

$$\sum_{p \le w} \frac{f(\tau \log p)}{p} \le (1-K)\sum_{p \le w}\frac{1}{p} + \frac{\tau^2}{2}\sum_{p \le w}\frac{\log^2 p}{p}. \tag{29}$$

Applying Propositions 1 and 3 to Equation (29), we obtain

$$\sum_{p \le \exp(1/\alpha)} \frac{f(\tau \log p)}{p} \le (1-K)\left(\log\frac{1}{\alpha} + M + M'(\exp(1/\alpha))\right) + \frac{(1+10^{-8})}{4}\left(\frac{\tau}{\alpha}\right)^2.$$

Let $c > 1$ be a constant that will be chosen later. When $\frac{\tau}{c} \le \alpha \le 1$, we have

$$\sum_{p \le \exp(1/\alpha)} \frac{f(\tau \log p)}{p} \le (1-K)\log\frac{1}{\alpha} + (1-K)(M+1) + \frac{(1+10^{-8})c^2}{4}. \tag{30}$$

Now, we consider $\alpha < \frac{\tau}{c} \le 1$. If $w = \exp(\frac{c}{\tau})$, then Equation (29) yields

$$\begin{aligned}\sum_{p \le w} \frac{f(\tau \log p)}{p} &\le (1-K)\left(\log\log w + M + M'(w)\right) + \frac{(1+10^{-8})}{4} \\ &\le (1-K)\log\log w + (1-K)(M+1) + \frac{(1+10^{-8})c^2}{4}.\end{aligned} \tag{31}$$

Noting that $\overline{f} = 1-K$, $S(f) = 1+K$, and $V(f) = 4$, we can now take $z = \exp(\alpha^{-1})$ in Lemma 6. This yields

$$\sum_{w < p \le z} \frac{f(\tau \log p)}{p} \le (1-K)\left(\log\frac{1}{\alpha} - \log\log w\right) + |E_\tau(w)|, \tag{32}$$

where $E_\tau(w)$ is taken from Equation (22). Combining Equation (31) and Equation (32) gives

$$\sum_{p \le \exp(1/\alpha)} \frac{f(\tau \log p)}{p} \le (1-K)\log\frac{1}{\alpha} + (1-K)(M+1) + \frac{(1+10^{-8})c^2}{4} + |E_\tau(w)|. \tag{33}$$

For our choice of c, we focus on minimizing $\frac{(1+10^{-8})c^2}{4} + |E_\tau(w)|$. Taking $k_1 = 0$, we find that the best choice of c is 2.67 and this leads to Equation (33) becoming

$$\sum_{p \leq \exp(\alpha^{-1})} \frac{f(\tau \log p)}{p} \leq (1-K) \log \frac{1}{\alpha} + 7.28. \tag{34}$$

If $|\tau| > 1$, we first consider the case that $(1+\epsilon)6.455 \log^2(|\tau|+3) \leq \frac{1}{\alpha}$. Taking w as in Equation (20) and $z = \exp(\alpha^{-1})$ in Lemma 6, we obtain

$$\sum_{w < p \leq z} \frac{f(\tau \log p)}{p} \leq (1-K)\left(\log({}^1\!/\!_\alpha) - \log\log w\right) + E_\tau(w), \tag{35}$$

where $|E_\tau(w)|$ is bounded in Equation (23) (and therefore depends on choices of ϵ and k_2). It follows trivially from Proposition 1 and $f(\tau \log p) \leq 1 + K$ that

$$\sum_{p \leq w} \frac{f(\tau \log p)}{p} \leq (1+K)\left(\log\log w + M + \frac{1}{((1+\epsilon)6.455)^2 \log^4(|\tau|+3)}\right). \tag{36}$$

Taking Equation (35) and Equation (36) together, with our choice for w, yields

$$\begin{aligned}\sum_{p \leq \exp(\alpha^{-1})} \frac{f(\tau \log p)}{p} \leq& (1-K) \log \frac{1}{\alpha} + 4K \log\log(|\tau|+3) + 2K \log((1+\epsilon)6.455) \\ &+ (1+K)\left(M + \frac{1}{((1+\epsilon)6.455)^2 \log^4(|\tau|+3)}\right) + E_\tau(w).\end{aligned} \tag{37}$$

We need to optimize the last three terms of Equation (37) with respect to ϵ and k_2. For fixed ϵ and τ, it appears that $O_{k_2,\epsilon}(1)$ as defined in Equation (21) is decreasing in k_2, but the savings are slight for large enough k_2. Therefore, in the interest of simpler computations, we choose $k_2 = 3 \cdot 10^5$. Some rough optimization over the terms involving ϵ in Equation (37) shows that $\epsilon = 3.61$ gives a relatively small maximum over these terms as a function of τ. Making this choice of ϵ and bounding the terms by their maximum in τ, we determine that

$$\sum_{p \leq \exp(\alpha^{-1})} \frac{f(\tau \log p)}{p} \leq (1-K) \log \frac{1}{\alpha} + 4K \log\log(|\tau|+3) + 3.25. \tag{38}$$

The final case to consider is $|\tau| > 1$ and $(1+\epsilon)6.455 \log^2(|\tau|+3) > \frac{1}{\alpha}$. In this case the sum in question is bounded by the sum estimated in Equation (36), given our choice of ϵ, this gives

$$\sum_{p \leq \exp(\alpha^{-1})} \frac{f(\tau \log p)}{p} \leq (2+2K) \log\log(|\tau|+3) + 4.87. \tag{39}$$

Taking Equation (34) as the worst case between Equation (30), Equation (34), Equation (38) and Equation (39), completes the proof. □

Here is interesting to note that, as it will be clear from the following results, the constant C_0 is the main contributor to the size of the constant in Theorem 4, and thus of $\delta(c)$. Thus, reducing C_0 would be a good starting point to improve Theorem 2.

Lemma 8. *For $\alpha \in [0,1]$, define $\lambda := \lambda(\alpha)$ to be the real number satisfying*

$$\sum_{p \leq \exp(\alpha^{-1})} \frac{1-g(p)}{p} = \lambda \sum_{p \leq \exp(\alpha^{-1})} \frac{1}{p}.$$

Then,

$$\begin{aligned}\Re \sum_{p \leq \exp(\alpha^{-1})} \frac{g(p)}{p^{1+i\tau}} \leq & (1-K\lambda)\log(\frac{1}{\alpha}) + (2+2K)(\log\log|\tau|+3) \\ & + C_0 + (K-K\lambda)(M+M'(\exp(1/\alpha))),\end{aligned}$$

for any $\tau \in \mathbb{R}$.

Proof. Consider the identity

$$\begin{aligned}\Re\left(\frac{g(p)}{p^{i\tau}}\right) &= g(p)(\cos(\tau\log p)-K) + Kg(p) \\ &\leq |\cos(\tau\log p)-K| = f(\tau\log p) + Kg(p).\end{aligned}$$

The definition of λ implies that

$$\sum_{p \leq \exp(\alpha^{-1})} \frac{g(p)}{p} = (1-\lambda)\left(\log\frac{1}{\alpha} + M + M'(\exp(1/\alpha))\right). \tag{40}$$

Therefore,

$$\Re\left(\sum_{p \leq \exp(\alpha^{-1})} \frac{g(p)}{p^{1+i\tau}}\right) \leq \sum_{p \leq \exp(\alpha^{-1})} \frac{f(\tau\log p)}{p} + K \sum_{p \leq \exp(1/\alpha)} \frac{g(p)}{p}.$$

The result follows by applying Lemma 7 and Equation (40) to the terms above. □

Let $F(s)$ be the Dirichlet series corresponding to $g(n)$. We have the following bound.

Lemma 9. *For $\Re(s) > 1$, we have*

$$|F(s)| \leq \exp(\nu_2) \cdot \exp\left\{\Re\left(\sum_p \frac{g(p)}{p^s}\right)\right\},$$

where $\nu_2 = \gamma - M \leq 0.316$.

Proof. Since $g(n)$ is completely multiplicative, we have that

$$F(s) = \prod_p \left(1 - \frac{g(p)}{p^s}\right)^{-1}.$$

Therefore,

$$|F(s)| = \left|\exp\left(-\sum_p \log\left(1 - \frac{g(p)}{p^s}\right)\right)\right|.$$

Applying the Taylor expansion of $\log(1-x)$ to the inside of the above sum, gives

$$\begin{aligned} |F(s)| &= \left|\exp\left(\sum_p \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{g(p)}{p^s}\right)^k\right)\right| \\ &= \left|\exp\left(\sum_p \frac{g(p)}{p^s}\right)\right| \cdot \left|\exp\left(\sum_{k=2}^{\infty} \frac{1}{k}\sum_p \frac{g(p)^k}{p^{ks}}\right)\right|. \end{aligned} \tag{41}$$

The sum in the right-most term can be bounded above by the "prime" zeta function

$$P(s) := \sum_p \frac{1}{p^s},$$

which converges for $\Re(s) > 1$. Therefore, we have

$$\left|\sum_{k=2}^{\infty} \frac{1}{k}\sum_p \frac{g(p)^k}{p^{ks}}\right| \le \left|\sum_{k=2}^{\infty} \frac{P(ks)}{k}\right| \le \sum_{k=2} \frac{P(k)}{k} = \gamma - M. \tag{42}$$

The equality in Equation (42) follows from the definition of B, since

$$\gamma - M = \sum_p -\log\left(1 - \frac{1}{p}\right) - \frac{1}{p} = \sum_p \sum_{k=2}^{\infty} \frac{1}{kp^k} = \sum_{k=2}^{\infty} \frac{P(k)}{k}.$$

Inserting Equation (42) into Equation (41) yields the desired result. □

The following result will be used in bounding the sum over primes in Lemma 9.

Lemma 10. *Uniformly for* $0 < \alpha \le 1$, *we have*

$$\sum_{\exp(\alpha^{-1})}^{\infty} \frac{1}{p^{1+\alpha}} \le 0.9235 =: v_1.$$

Proof. By partial summation we have

$$\sum_{\exp(\alpha^{-1})}^{\infty} \frac{1}{p^{1+\alpha}} = -\pi(\exp(\alpha^{-1}))\exp(-(1+\alpha^{-1})) + (1+\alpha)\int_{\exp(\alpha^{-1})}^{\infty} \frac{\pi(x)}{x^{2+\alpha}}dx.$$

Using (3.6) from [19], we then obtain

$$\sum_{\exp(\alpha^{-1})}^{\infty} \frac{1}{p^{1+\alpha}} \leq (1+\alpha)1.2551 \int_{\exp(\alpha^{-1})}^{\infty} \frac{1}{x^{1+\alpha}\log x} dx \leq \frac{(1+\alpha)1.2551}{e},$$

the result now follows taking the maximum over $\alpha \in (0, 1]$. □

Note that using a better explicit version of the PNT could improve the above result, as this improvement appears to be minor we decided, for the sake of simplicity, for the above version.

4.3. Proof of Theorem 4

We are now able to prove Theorem 4.

Proof of Theorem 4. Consider $F(1 + \alpha + it)$, where $0 < \alpha \leq 1$ and $t \in \mathbb{R}$. By Lemma 9, we have

$$|F(1+\alpha+it)| \leq \exp(\nu_2) \cdot \exp\left\{ \Re \left(\sum_p \frac{g(p)}{p^{1+\alpha+it}} \right) \right\}. \tag{43}$$

We break the sum over primes in Equation (43) at $\exp(\alpha^{-1})$, yielding the bound

$$\begin{aligned} \left| \Re \left(\sum_p \frac{g(p))}{p^{1+\alpha+it}} \right) \right| = & \quad \left| \Re \left(\sum_{p \leq \exp(\alpha^{-1})} \frac{g(p)}{p^{1+\alpha+it}} \right) + \Re \left(\sum_{p > \exp(\alpha^{-1})} \frac{g(p)}{p^{1+\alpha+it}} \right) \right| \\ \leq & \quad \left| \sum_{p \leq \exp(\alpha^{-1})} \frac{g(p)}{p^{1+\alpha+it}} \right| + \left| \sum_{p > \exp(\alpha^{-1})} \frac{1}{p^{1+\alpha}} \right|. \end{aligned} \tag{44}$$

Simply ignoring p^{α} in the first sum on the right of Equation (44) and applying Lemma 10 to the second sum, we obtain

$$\left| \Re \left(\sum_p \frac{g(p)}{p^{1+\alpha+it}} \right) \right| \leq \left| \sum_{p \leq \exp(\alpha^{-1})} \frac{g(p)}{p^{1+it}} \right| + \nu_1. \tag{45}$$

Now, we may apply Lemma 8 to the remaining sum in Equation (45) and place this estimate in Equation (43) to establish

$$|F(1+\alpha+i\tau)| \leq \exp(C_0 + \nu_1 + \nu_2 + (K - K\lambda)(M+1)). \tag{46}$$

Write $C := C_0 + \nu_1 + \nu_2 + K(M+1)$. Recalling Theorem 5, we see that Equation (46) implies

$$H^2(\alpha) \leq \exp(2C)\alpha^{2K\lambda-2} \sum_{k \in \mathbb{Z}} \frac{\log^{2+2K}(|k|+4)}{(k - 1/2)^2 + 1}.$$

The integer sum above is a computable constant. Calling its square root ν_3, gives

$$H(\alpha) \leq \nu_3 \exp(C)\alpha^{K\lambda-1}, \tag{47}$$

and we can note that $\nu_3 \leq 4.36$. Now, if $\Lambda := \Lambda(x)$ is defined by

$$\sum_{p\leq x} \frac{1-g(p)}{p} = \Lambda \sum_{p\leq x} \frac{1}{p}, \tag{48}$$

then, for $1/\log x \leq \alpha \leq 1$,

$$\begin{aligned} \sum_{p\leq \exp(\alpha^{-1})} \frac{1-g(p)}{p} &\geq \sum_{p\leq x} \frac{1-g(p)}{p} - \sum_{\exp(\alpha^{-1})<p\leq x} \frac{2}{p} \\ &\geq (\Lambda - 2)\sum_{p\leq x}\frac{1}{p} + 2\sum_{p\leq \exp(\alpha^{-1})} \frac{1}{p}. \end{aligned}$$

Recalling the definition of λ in Lemma 8 and using Proposition 2, we easily obtain

$$\alpha^{\lambda} \leq \alpha^2 (\log x)^{2-\Lambda} e^{3(0.867-0.261)} \leq \alpha^2 (\log x)^{2-\Lambda} e^{1.82},$$

which, when applied to Equation (47), implies

$$H(\alpha) \leq \nu_3 \exp(C) e^{1.82K} \alpha^{2K-1} (\log x)^{(2-\Lambda)K}.$$

Taking this estimate for $H(\alpha)$ in Theorem 5, we find that $|G(x)|$ is at most

$$\begin{aligned} &\left(3.14\nu_3 \exp(C)e^{1.82K} + o(1)\right) \frac{x(\log x)^{(2-\Lambda)K}}{\log x} \int_{1/\log x}^{1} \alpha^{2K-2}\, d\alpha + \mathcal{O}_{x_0}\left(\frac{x}{\sqrt{\log x}}\right) \\ &\quad = \left(\frac{3.14\nu_3 \exp(C)e^{1.82K}}{1-2K} + o(1)\right) \frac{x(\log x)^{(2-\Lambda)K}}{\log x} \left(\log x^{1-K\Lambda} - \log x^{2-\Lambda}\right) \\ &\qquad + \mathcal{O}_{x_0}\left(\frac{x}{\sqrt{\log x}}\right). \end{aligned}$$

Collecting all the constants up to this point and calling them $\mathbf{a}$ ($\approx 5.5 \cdot 10^5$), we arrive at the bound

$$\begin{aligned} |G(x)| &\leq (\mathbf{a} + o(1)) \frac{x}{\log x} (\log x)^{1-2K+(2-\Lambda)K} + \mathcal{O}_{x_0}\left(\frac{x}{\sqrt{\log x}}\right) \\ &= (\mathbf{a} + o(1)) x (\log x)^{-\Lambda K} + \mathcal{O}_{x_0}\left(\frac{x}{\sqrt{\log x}}\right). \end{aligned} \tag{49}$$

It follows from Equation (48) that

$$\Lambda = \frac{\sum_{p\leq x} \frac{1-g(p)}{p} - \Lambda M - \Lambda M'(x)}{\log\log x},$$

and that for $x < 2$, we can take $\Lambda = 0$. For $x \geq 2$, we have $\Lambda \leq 2$. Furthermore, one may verify that $|M'(x)| < 0.6051$ for $2 \leq x < 4$ and $|M'(x)| \leq \frac{1}{\ln^2 4} < 0.6051$ for $x > 4$. Therefore, we may write

$$\begin{aligned}\mathbf{a}(\log x)^{-\Lambda K} &= \mathbf{a}\exp(K\Lambda M)\exp\left(K\Lambda M'(x)\right)\exp\left(-K\sum_{p\leq x}\frac{1-g(p)}{p}\right)\\ &\leq \mathbf{a}\exp\left(2KM+1.21K\right)\exp\left(-K\sum_{p\leq x}\frac{1-g(p)}{p}\right) \qquad (50)\\ &= 9.75\cdot 10^{5}\exp\left(-K\sum_{p\leq x}\frac{1-g(p)}{p}\right).\end{aligned}$$

Taking Equation (50) in Equation (49) completes the proof. □

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