METRICAL PROPERTIES OF FUNCTIONS OF CONSECUTIVE DIGITS IN LÜROTH SERIES

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Abstract

Recently, the growth of the products of Lüroth quotients $d_i(x)$ in the Lüroth expansion of a real number x was studied in connection with improvements to Dirichlet's theorem. In this paper, for a non-decreasing positive measurable function $F(x_1, \dots, x_m)$ and a function $\phi : \mathbb{N} \to \mathbb{R}_{>0}$, we consider the following set:

$$\mathcal{E}_F(\phi) = \{x \in [0,1] : F(d_n(x), \dots, d_{n+m-1}(x)) \geq \phi(n) \text{ for infinitely many } n \in \mathbb{N}\},$$

and obtain its Lebesgue measure $\lambda(\mathcal{E}_F(\phi))$. As an application of our result, we consider the case when $F(x_1, \dots, x_m) = x_1 + \cdots + x_m$.

1. Introduction

Every $x \in (0,1]$ can be uniquely expressed in the form of a Lüroth series

$$x = \frac{1}{d_1(x)} + \frac{1}{d_1(x)(d_1(x)-1)d_2(x)} + \frac{1}{d_1(x)(d_1(x)-1)d_2(x)(d_2(x)-1)d_3(x)} + \dots,$$

where $d_n(x) \in \mathbb{N}$, $d_n(x) \geq 2$, $n = 1, 2, 3, \dots$. Lüroth series can be studied through the Lüroth map $\mathcal{L}$, whose definition will be addressed in Section 2.

Let λ denote the Lebesgue measure on $\mathbb{R}$. H. Jager and C. de Vroedt [3] showed that $\mathcal{L}$ is measure-preserving and ergodic. Also, they noted that the Lüroth quotients $d_1(x), d_2(x), \dots$ can be viewed as random variables and are independent.

Throughout this paper, let $\Psi : \mathbb{N} \to \mathbb{R}_{>0}$ be a positive function. We consider the Lebesgue measure of the following set:

$$\mathcal{E}(\Psi) := \big\{x \in (0,1] : d_n(x) \geq \Psi(n) \text{ for infinitely many } n \in \mathbb{N}\big\}.$$

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The following theorem is the Lüroth series analogue of the Borel-Bernstein Theorem for continued fractions.

Theorem 1 ([3])**.** *The Lebesgue measure of $\mathcal{E}(\Psi)$ is given by*

$$\lambda\big(\mathcal{E}(\Psi)\big) = \begin{cases} 0 & \text{if} \quad \sum_{n=1}^{\infty} \Psi(n)^{-1} < \infty \\ 1 & \text{if} \quad \sum_{n=1}^{\infty} \Psi(n)^{-1} = \infty. \end{cases}$$

Brown-Sarre, Robert, and Hussain [2] considered a weighted product of consecutive Lüroth digits: given $m \in \mathbb{N}$ and $\mathbf{t} = (t_0, \ldots, t_{m-1}) \in \mathbb{R}^m_{>0}$, they defined

$$\mathcal{E}_{\mathbf{t}}(\Psi) := \left\{ x \in (0,1] : \prod_{i=0}^{m-1} d_{n+i}^{t_i}(x) \geq \Psi(n) \text{ for infinitely many } n \in \mathbb{N} \right\}$$

and proved the following dichotomy statement for the Lebesgue measure of $\mathcal{E}_{\mathbf{t}}(\Psi)$.

Theorem 2 ([2])**.** *Let $m \in \mathbb{N}$, $\Psi : \mathbb{N} \to \mathbb{R}_{>0}$ and $\mathbf{t} = (t_0, \ldots, t_{m-1}) \in \mathbb{R}^m_{>0}$. If*

$$\liminf_{n \to \infty} \Psi(n) > 1, \tag{1}$$

then

$$\lambda\big(\mathcal{E}_{\mathbf{t}}(\Psi)\big) = \begin{cases} 0 & \text{if} \quad \sum_{n=1}^{\infty} \frac{(\log \Psi(n))^{\ell(\mathbf{t})-1}}{\Psi(n)^{1/T}} < \infty \\ 1 & \text{if} \quad \sum_{n=1}^{\infty} \frac{(\log \Psi(n))^{\ell(\mathbf{t})-1}}{\Psi(n)^{1/T}} = \infty \end{cases}$$

where

$$T := \max\{t_0, t_1, \ldots, t_{m-1}\}, \quad \ell(\mathbf{t}) := \#\{j \in \{0, \ldots, m-1\} : t_j = T\}. \tag{2}$$

A similar result for weighted products of multiple partial quotients in continued fractions was obtained earlier in [1].

In this paper, for a fixed $m \in \mathbb{N}$, we consider positive non-decreasing measurable m-variable function $F(x_1, x_2, \ldots, x_m)$ defined on $[2, \infty) \times [2, \infty) \times \ldots \times [2, \infty)$. The function F is said to be non-decreasing if for all fixed a_i, b_i $(a_i \geq 2, b_i \geq 2, 1 \leq i \leq m)$, the following condition holds:

$$a_i \leq b_i \text{ for all } i, 1 \leq i \leq m \Rightarrow F(a_1, a_2, \ldots, a_m) \leq F(b_1, b_2, \ldots, b_m).$$

We define

$$\mathcal{E}_F(\Psi) = \big\{ x \in [0,1) : F(d_n, \ldots, d_{n+m-1}) \geq \Psi(n) \text{ for infinitely many } n \in \mathbb{N} \big\}.$$

It is clear that $\mathcal{E}_F(\Psi)$ generalizes $\mathcal{E}_{\mathbf{t}}(\Psi)$ in Theorem 2, with

$$F(x_1, x_2, \ldots, x_m) = \prod_{i=1}^{m} x_i^{t_i}. \tag{3}$$

Our first theorem gives a general zero – one law for the sets $\mathcal{E}_F(\Psi)$. Let $\mathbb{D} = \mathbb{N}_{\geq 2}$, and define

$$H(x) = \sum_{\substack{F(x_1,x_2,\ldots,x_m)\geq x,\\ x_1,\ldots,x_m\in\mathbb{D}}} \prod_{i=1}^{n} \frac{1}{x_i(x_i-1)}. \tag{4}$$

Theorem 3. *Let $\Psi : \mathbb{N} \to \mathbb{R}_{>0}$. Suppose $F(x_1, \ldots, x_m) > 0$ for $x_1, \ldots, x_m \in \mathbb{D}$, and let $H(x)$ be defined as in (4). Then*

$$\lambda\big(\mathcal{E}_F(\Psi)\big) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} H\big(\Psi(n)\big) < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} H\big(\Psi(n)\big) = \infty. \end{cases}$$

Proof of Theorem 2 assuming Theorem 3. Let $t := \min\{t_0, t_1, \ldots, t_{m-1}\}$. According to [2, Lemma 3.4], if $m \in \mathbb{N}$, $\mathbf{t} \in \mathbb{R}^m_{>0}$, and $g \geq 2^{mt}$, then

$$\sum_{\substack{d_1^{t_0}\cdots d_m^{t_{m-1}}\geq g\\ d_1,\ldots,d_m\in\mathbb{D}}} \prod_{j=1}^{m} \frac{1}{d_j(d_j-1)} \asymp_{m,\mathbf{t}} \frac{\log^{\ell(\mathbf{t})-1} g}{g^{1/T}},$$

where T and $\ell(\mathbf{t})$ are as in (2). Thus, if we assume that $\Psi(n) > 2^{mt}$ for all large enough $n \in \mathbb{N}$ and take F as in (3), it follows that

$$H\big(\Psi(n)\big) \asymp_{m,\mathbf{t}} \frac{(\log \Psi(n))^{\ell(\mathbf{t})-1}}{\Psi(n)^{1/T}},$$

which makes the convergence/divergence conditions of Theorems 2 and 3 coincide.

It remains to consider the case when there are infinitely many $n \in \mathbb{N}$ such that $1 < \Psi(n) \leq 2^{mt}$. Then, in view of (1), we can choose a real number y and an increasing sequence of natural numbers $(n_j)_{j\geq 1}$ such that

$$1 < y < \Psi(n_j) \leq 2^{mt} \quad \text{for all} \quad j \in \mathbb{N}. \tag{5}$$

Then

$$\log^{\ell(\mathbf{t})-1} y < \log^{\ell(\mathbf{t})-1} \Psi(n_j) \quad \text{and} \quad 2^{-mt/T} \leq \Psi(n_j)^{-1/T} \quad \text{for all} \quad j \in \mathbb{N},$$

and therefore,

$$\sum_{j=1}^{\infty} \frac{\log^{\ell(\mathbf{t})-1} \Psi(n_j)}{\Psi(n_j)^{1/T}} = \infty.$$

Meanwhile, for any $x \in (0, 1]$ and any $n \in \mathbb{N}$, we have $\prod_{i=0}^{m-1} d_{n+i}^{t_i}(x) \geq 2^{mt}$, and thus from (5) we have $\mathcal{E}_{\mathbf{t}}(\Psi) = (0, 1]$. Hence Theorem 2 follows from Theorem 3. □

For computations it is convenient to replace the summation in (4) with integration. For that, one needs to impose some regularity conditions on F. Namely, suppose that

$$\sup_{x_i\geq 2, 1\leq i\leq m} \frac{F(x_1,x_2,\ldots,x_m)}{F(\lfloor x_1\rfloor,\lfloor x_2\rfloor,\ldots,\lfloor x_m\rfloor)} = M < \infty. \tag{6}$$

Set

$$G(x) = \int\cdots\int_{\substack{F(x_1,\ldots,x_m)\geq x,\\ x_1\geq 2,\ldots,x_m\geq 2}} \frac{1}{x_1^2\cdots x_m^2}\,dx_1\cdots dx_m. \tag{7}$$

Theorem 4. *Let $\Psi:\mathbb{N}\to\mathbb{R}_{>0}$. Suppose $F(x_1,\ldots,x_m)>0$ for $x_i\geq 2$, $1\leq i\leq m$, and that F is non-decreasing and satisfies* (6)*. Let $G(x)$ be defined as in* (7)*. Then*

$$\lambda\big(\mathcal{E}_F(\Psi)\big) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} G\big(\Psi(n)\big) < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} G\big(M\Psi(n)\big) = \infty, \end{cases}$$

where M is as in (6)*.*

As an example, this theorem can be applied to the function

$$F(x_1,\ldots,x_m) = x_1 + \ldots + x_m. \tag{8}$$

in the following way: if $\Psi:\mathbb{N}\to\mathbb{R}_{\geq m}$ and $F(x)$ is defined as in (8), then

$$\lambda\big(\mathcal{E}_F(\Psi)\big) = \begin{cases} 0 & \text{if } \sum\limits_{n=1}^{\infty} \frac{1}{\Psi(n)} < \infty \\ 1 & \text{if } \sum\limits_{n=1}^{\infty} \frac{1}{\Psi(n)} = \infty \end{cases} \tag{9}$$

Indeed, one can show that for any $m\in\mathbb{N}$ and $a\geq m$,

$$\int\cdots\int_{\substack{x_1+x_2+\ldots+x_m\geq a,\\ x_1\geq 1,\ldots,x_m\geq 1}} \frac{1}{x_1^2\cdots x_m^2}\,dx_1\cdots dx_m \asymp_m \frac{1}{a}$$

(the upper bound is derived in [6, (3.2)], and the lower bound can be established by a similar method).

We would like to remark though that (9) can be easily shown without using Theorem 4. Namely, if $\sum_{n\in\mathbb{N}}\frac{1}{\Psi(n)}$ converges, then the same is true for $\sum_{n\in\mathbb{N}}\frac{1}{\min(\Psi(n),\ldots,\Psi(n-m+1))}$. The set

$$\big\{x\in(0,1]: d_n + ... + d_{n+m-1} > \Psi(n)\big\} \text{ for infinitely many } n\}$$

is contained in

$$\{x\in(0,1]: d_n > \min\big(\Psi(n),\ldots,\Psi(n-m+1)\big) \text{ for infinitely many } n\},$$

which has measure 0 from Theorem 1. On the other hand, if $\sum_{n=1}^{\infty}\frac{1}{\Psi(n)}=\infty$, then by the divergence case of Theorem 1, for Lebesgue-a.e.x, one has $d_n(x)\geq\Psi(n)$ for infinitely many $n\in\mathbb{N}$, and hence the same is true for $d_n(x)+\ldots+d_{n+m-1}(x)$.

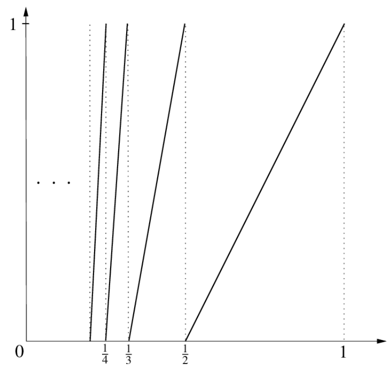

Figure 1: The Lüroth map

2. Lüroth Series

Let $d_1 : (0,1] \to \mathbb{D} = \mathbb{N}_{\geq 2}$ be defined as $d_1(x) = \left\lfloor \frac{1}{x} \right\rfloor + 1$. Then d_1 assigns to each $x \in (0,1]$ an integer greater than 1. *The Lüroth map* $\mathcal{L}(x) : [0,1] \to [0,1]$ is defined by

$$\mathcal{L}(x) = \begin{cases} d_1(x)\big(d_1(x)-1\big)x - \big(d_1(x)-1\big) & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}$$

Figure 1 shows the graph of the Lüroth map. It is constructed similarly to the Gauss map used to define continued fractions. However the Gauss map is piecewise fractional-linear, while the Lüroth map has linear branches. This makes the analysis of Lüroth series easier than that of continued fractions.

For $x \in (0,1]$ and $n \in \mathbb{D}$, we set $d_n(x) = d_1\big(\mathcal{L}^{n-1}(x)\big)$, where the exponent $n-1$ denotes the $(n-1)$-th iteration. We call $\mathcal{L}$ a shift operator. If x has Lüroth digits $(c_1, c_2, \ldots)$, then $\mathcal{L}(x)$ has Lüroth digits $(c_2, c_3, \ldots)$. For $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{D}^n$, we define *the cylinder of level* n *based at* $\mathbf{c}$ as

$$I_n(\mathbf{c}) := \{x \in (0,1] : d_1(x) = c_1, \ldots, d_n(x) = c_n\}.$$

We set $I_n(\mathbf{c})$ to be the set of all real numbers in $(0,1]$ whose Lüroth expansion begin with $(c_1, \ldots, c_n)$.

For each $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{D}^n$, set

$$\langle c_1, \ldots, c_n \rangle := \frac{1}{c_1} + \frac{1}{c_1(c_1-1)c_2} + \ldots + \frac{1}{c_1(c_1-1)c_2(c_2-1)\ldots c_{n-1}(c_{n-1}-1)c_n}.$$

Proposition 1 ([2]). *For every* $n \in \mathbb{N}$ *and* $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{D}^n$*, we have*

$$I_n(\mathbf{c}) = \left(\langle c_1, \ldots, c_n \rangle, \langle c_1, \ldots, c_n - 1 \rangle\right),$$

and therefore

$$\lambda\big(I_n(\mathbf{c})\big) = \prod_{j=1}^{n} \frac{1}{c_j(c_j-1)}. \tag{10}$$

3. Proof of Theorem 3

In this section, our arguments largely follow those of Huang, Wu, and Xu [4]. Fix an $m \in \mathbb{N}$. Let $n \in \mathbb{N}$. Suppose that F is a positive non-decreasing m-variable measurable function. Define

$$A_n = \{x \in (0,1] : F(d_1, \ldots, d_m) \geq \Psi(n)\}.$$

It is clear that $x \in \mathcal{E}_F(\Psi)$ if and only if $\mathcal{L}^{n-1}(x) \in A_n$ for infinitely many $n \in \mathbb{N}$. Clearly A_n can be written as a union of a collection of m-th order cylinders:

$$A_n = \bigcup_{\substack{F(d_1,\ldots,d_m) \geq \Psi(n) \\ d_1,\ldots,d_m \in \mathbb{D}}} I_m(d_1, \ldots, d_m).$$

Therefore,

$$\lambda(A_n) = \sum_{\substack{F(d_1,\ldots,d_m) \geq \Psi(n) \\ d_1,\ldots,d_m \in \mathbb{D}}} \lambda\big(I_m(d_1, \ldots, d_m)\big).$$

From (10), we have

$$\lambda(A_n) = \sum_{\substack{F(d_1,\ldots,d_m) \geq \Psi(n) \\ d_1,\ldots,d_m \in \mathbb{D}}} \prod_{k=1}^{m} \frac{1}{d_k(d_k-1)}. \tag{11}$$

The following theorem is Lemma 3.2 of [2].

Theorem 5. *Fix $m \in \mathbb{N}$. Let $(A_n)_{n\geq 1}$ be a sequence of at most countable unions of cylinders of level m. We have*

$$\lambda\left(\limsup_{n\to\infty} \mathcal{L}^{-n}[A_n]\right) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \lambda(A_n) < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} \lambda(A_n) = \infty \end{cases}$$

Proof of Theorem 3. By (11), the series

$$\sum_{n=1}^{\infty} H\big((\Psi(n)\big) \tag{12}$$

converges (respectively, diverges) if and only if so does the series $\sum_{n=1}^{\infty} \lambda(A_n)$. By Theorem 5, this implies that the series (12) converges (respectively, diverges) if and only if the set

$$\{x \in (0,1] : \mathcal{L}^n(x) \in A_n \text{ for infinitely many } n \in \mathbb{N}\}$$

has zero (respectively, full) measure. The latter, since $\mathcal{L}$ is measure-preserving, amounts to saying that $\lambda\big(\mathcal{E}_F\Psi\big)$ is equal to zero (respectively, one). □

In order to prove Theorem 4, we give the upper and lower bounds for $\lambda(A_n)$.

Theorem 6. *For $n \in \mathbb{N}$, we have $\lambda(A_n) \le 3^m G\big(\Psi(n)\big)$.*

Proof. It is clear that

$$\frac{1}{d_k(d_k-1)} \le \frac{3}{d_k(d_k+1)}, \quad d_k \ge 2.$$

From (11), we have

$$\begin{aligned}\lambda(A_n) &\le 3^m \sum_{\substack{F(d_1,\ldots,d_m)\ge\Psi(n)\\ d_1,\ldots,d_m\in\mathbb{D}}} \frac{1}{d_k(d_k+1)} = 3^m \sum_{\substack{F(d_1,\ldots,d_m)\ge\Psi(n)\\ d_1,\ldots,d_m\in\mathbb{D}}} \prod_{k=1}^{m} \int_{d_k}^{d_k+1} \frac{1}{x_k^2} dx_k \\ &= 3^m \sum_{\substack{F(d_1,\ldots,d_m)\ge\Psi(n)\\ d_1,\ldots,d_m\in\mathbb{D}}} \int_{d_1}^{d_1+1} \ldots \int_{d_m}^{d_m+1} \frac{1}{x_1^2\ldots x_m^2} dx_1 \ldots dx_m.\end{aligned}$$

The domain of the above integration is

$$Q_1 = \{[d_1, d_1{+}1)\times\cdots\times[d_m, d_m{+}1) : F(d_1,\ldots,d_m) \ge \Psi(n),\ d_1,\ldots,d_m \in \mathbb{D}\}, \tag{13}$$

which is a subset of

$$\{(x_1,\ldots,x_m) : x_1 \ge 2,\ldots,x_m \ge 2, F(x_1,\ldots,x_m) \ge \Psi(n)\}$$

since F is non-decreasing. Therefore,

$$\lambda(A_n) \le 3^m \int_{\substack{F(x_1,\ldots,x_m)\ge\Psi(n),\\ x_1\ge 2,\cdots,x_m\ge 2}} \cdots \int \frac{1}{x_1^2\cdots x_m^2}\, dx_1\cdots dx_m = 3^m G\big(\Psi(n)\big).$$

□

Theorem 7. *For $n \in \mathbb{N}$, we have $\lambda(A_n) \ge G\big(M\Psi(n)\big)$, where M is defined in (6).*

Proof. From (11) we have

$$\begin{aligned}\lambda(A_n) &= \sum_{\substack{F(d_1,\ldots,d_m)\ge\Psi(n)\\ d_1,\ldots,d_m\in\mathbb{D}}} \prod_{k=1}^{m} \frac{1}{d_k(d_k-1)} \ge \sum_{\substack{F(d_1,\ldots,d_m)\ge\Psi(n)\\ d_1,\ldots,d_m\in\mathbb{D}}} \prod_{k=1}^{m} \int_{d_k}^{d_k+1} \frac{1}{x_k^2} dx_k \\ &= \int\cdots\int_{Q_1} \frac{1}{x_1^2\cdots x_m^2}\, dx_1\cdots dx_m,\end{aligned} \tag{14}$$

where Q_1 is defined in (13). Let

$$Q_2 = \{(x_1, \dots, x_m) : x_1 \geq 2, \dots, x_m \geq 2, F(x_1, \dots, x_m) \geq M\Psi(n)\}.$$

Then $Q_2 \subset Q_1$. To see this, note that if $(x_1, \dots, x_m) \in Q_2$, then $F(x_1, \dots, x_m) \geq M\Psi(n)$. From (6), we have

$$F(\lfloor x_1 \rfloor, \dots, \lfloor x_m \rfloor) \geq \Psi(n).$$

Therefore,

$$(x_1, \dots, x_m) \in [\lfloor x_1 \rfloor, \lfloor x_1 \rfloor + 1) \times \cdots \times [\lfloor x_m \rfloor, \lfloor x_m \rfloor + 1) \subset Q_1.$$

From (14), we have

$$\mathcal{L}(A_n) \geq \int \cdots \int_{Q_2} \frac{1}{x_1^2 \cdots x_m^2} \, dx_1 \cdots dx_m.$$

From (7), we have $\lambda(A_n) \geq G\big(M\Psi(n)\big)$. □

Proof of Theorem 4. Set

$$\mathcal{R}_n = \big\{(d_1, \dots, d_m) \in \mathbb{D}^m : F(d_1, \dots, d_m) \geq \Psi(n)\big\},$$

and let $A_n = \bigcup_{\mathbf{d} \in \mathcal{R}_n} I_m(\mathbf{d})$.

If $\sum_{n=1}^{\infty} G\big(\Psi(n)\big) < \infty$, then by Theorem 6, we have

$$\sum_{n=1}^{\infty} \lambda(A_n) < \infty.$$

By Theorem 5, $\lambda\left(\{x \in (0,1] : \mathcal{L}^n(x) \in A_n \text{ for infinitely many } n \in \mathbb{N}\}\right) = 0$. Therefore,

$$\lambda\big(\mathcal{E}_F(\Psi)\big) = \lambda\left(\{x \in (0,1] : \mathcal{L}^{n-1}(x) \in A_n \text{ for infinitely many } n \in \mathbb{N}\}\right).$$

Since $\mathcal{L}$ is measure-preserving, we have $\lambda\big(\mathcal{E}_F(\Psi)\big) = 0$.

On the other hand, if $\sum_{n=1}^{\infty} G\big(M\Psi(n)\big) = \infty$, then by Theorem 7, we have

$$\sum_{n=1}^{\infty} \lambda(A_n) = \infty.$$

By Theorem 5, $\lambda\big(\{x \in (0,1] : \mathcal{L}^n(x) \in A_n \text{ for infinitely many } n \in \mathbb{N}\}\big) = 1$. From the fact that $\mathcal{L}$ is measure-preserving, we have $\lambda\big(\mathcal{E}_F(\Psi)\big) = 1$. □

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