ON THE DIOPHANTINE EQUATION $n^x + (5p)^y = z^2$

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Abstract

In this paper, the Diophantine equation $n^x + (5p)^y = z^2$, where n is a positive integer, p is a prime number, and x, y, z are non-negative integers, is investigated. We show that if $n \equiv 1 \pmod 4$ and $p \equiv 1 \pmod 4$, then the equation has no non-negative integer solution. If $n = 2$ and $p \equiv 3 \pmod 4$, then all non-negative integer solutions of the equation are $(p, x, y, z) \in \{(3, 6, 2, 17), (p, 3, 0, 3)\} \cup \left\{(p, 0, 1, \sqrt{5p+1}) : \sqrt{5p+1} \in \mathbb{Z}\right\}$. If $n \equiv 3 \pmod{60}$ or $n \equiv 27 \pmod{60}$, and $p \equiv 7 \pmod{12}$ with $\gcd(n, p) = 1$ and $\gcd(n, 5p-1) = 1$, then all non-negative integer solutions of the equation are $(x, y, z) \in \left\{(0, 1, \sqrt{5p+1}) : \sqrt{5p+1} \in \mathbb{Z}\right\} \cup \left\{(1, 0, \sqrt{n+1}) : \sqrt{n+1} \in \mathbb{Z}\right\}$.

1. Introduction

Recently, the Diophantine equations of the type $n^x + 5^y = z^2$, where n is a positive integer and x, y, z are non-negative integers, have been studied by many researchers. Some of these can be seen in [1], [3], [5], [6], [8], [10], [11], [12], [14], [15], [16], [20] and [21]. Furthermore, in 2014, Sroysang [17, 19] proved that the Diophantine equations $3^x + 85^y = z^2$ and $3^x + 45^y = z^2$ have the unique non-negative integer solution $(x, y, z) = (1, 0, 2)$. He [18] also showed that the Diophantine equation $4^x + 10^y = z^2$ has no non-negative integer solution. In 2019, Burshtein [4] established that the Diophantine equation $7^x + 10^y = z^2$ has no positive integer solution. In 2022, Biswas [2] showed that the Diophantine equation $3^x + 35^y = z^2$ has only two non-negative integer solutions $(x, y, z) \in \{(1, 0, 2), (0, 1, 6)\}$. In the same year, Thongnak, Chuayjan and Kaewong [22] proved that the Diophantine equation $2^x + 15^y = z^2$ has exactly three non-negative integer solutions $(x, y, z) \in \{(3, 0, 3), (0, 1, 4), (6, 2, 17)\}$.

In this paper, we will study the Diophantine equation

$$n^x + (5p)^y = z^2, \tag{1}$$

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where n is a positive integer, p is a prime number, and x, y, z are non-negative integers. Our main results will be stated and proved in Section 3.

2. Preliminaries

In the beginning of this section, we present an important theorem, which is the result of research by Euler [13, p. 118] and Chao [7].

Theorem 1. *The Diophantine equation $u^2 - 1 = v^q$ has the unique integer solution $(u, v, q) = (3, 2, 3)$, where $u, v,$ and q are positive integers with $q \geq 3$.*

Lemma 1. *Let $y = 0$. If $n = 2$, then Equation (1) has the unique non-negative integer solution $(x, y, z) = (3, 0, 3)$. If $n \neq 2$, then (x, y, z) is a non-negative integer solution of Equation (1) if and only if $(x, y, z) \in \{(1, 0, \sqrt{n+1}) : \sqrt{n+1} \in \mathbb{Z}\}$.*

Proof. Let x and z be non-negative integers such that $n^x + 1 = z^2$ or $z^2 - 1 = n^x$. It is easy to check that $z > 1$, $n > 1$, and $x \geq 1$. If $x = 1$, then $z^2 = n + 1$ and so $z = \sqrt{n+1}$. Thus $(x, y, z) = (1, 0, \sqrt{n+1})$, where $\sqrt{n+1}$ is an integer. If $x = 2$, then $z = 1$ and $n = 0$, a contradiction. If $x \geq 3$, then $n = 2$ and $(x, y, z) = (3, 0, 3)$, by Theorem 1. □

By Lemma 1, we have the following corollary.

Corollary 1. *If $x = 0$, then (x, y, z) is a non-negative integer solution of Equation (1) if and only if*

$$(x, y, z) \in \left\{(0, 1, \sqrt{5p+1}) : \sqrt{5p+1} \in \mathbb{Z}\right\}.$$

Lemma 2. *Let $n \equiv 2 \pmod 5$ or $n \equiv 3 \pmod 5$. If $y > 0$ and Equation (1) has a non-negative integer solution, then x is even.*

Proof. Let x, y, z be non-negative integers and (x, y, z) be a solution of Equation (1). Since $y > 0$, we get $(5p)^y \equiv 0 \pmod 5$. Assume that x is odd. Then $x = 2k+1$, for some non-negative integer k. Since $n \equiv 2 \pmod 5$ or $n \equiv 3 \pmod 5$, we have $n^2 \equiv -1 \pmod 5$ and so $n^x = n^{2k+1} \equiv 2(-1)^k \pmod 5$. Then $n^x \equiv 2 \pmod 5$ or $n^x \equiv -2 \pmod 5$. From Equation (1), it follows that $z^2 \equiv 2 \pmod 5$ or $z^2 \equiv -2 \pmod 5$. This is impossible since $z^2 \equiv 0 \pmod 5$ or $z^2 \equiv 1 \pmod 5$ or $z^2 \equiv 4 \pmod 5$. Hence, x is even. □

Lemma 3. *Let $n \equiv 3 \pmod 4$ and $p \equiv 3 \pmod 4$. If Equation (1) has a non-negative integer solution, then x and y have opposite parity.*

Proof. Let x, y and z be non-negative integers and (x, y, z) be a solution of Equation (1). Since $n \equiv 3 \pmod 4$ and $p \equiv 3 \pmod 4$, we have $z^2 = n^x + (5p)^y \equiv (-1)^x + (-1)^y \pmod 4$. Since n and $5p$ are odd, we get that z is even and so $z^2 \equiv 0 \pmod 4$. Then $(-1)^x + (-1)^y \equiv 0 \pmod 4$. Hence, x and y have opposite parity. $\square$

3. Main Results

Now, we prove our results.

Theorem 2. *If $n \equiv 1 \pmod 4$ and $p \equiv 1 \pmod 4$, then Equation (1) has no non-negative integer solution.*

Proof. Assume that x, y, z are non-negative integers and (x, y, z) is a solution of Equation (1). Since $n \equiv 1 \pmod 4$ and $p \equiv 1 \pmod 4$, we have $n^x \equiv 1 \pmod 4$ and $(5p)^y \equiv 1 \pmod 4$. Then $z^2 = n^x + (5p)^y \equiv 2 \pmod 4$. This is impossible since $z^2 \equiv 0 \pmod 4$ or $z^2 \equiv 1 \pmod 4$. $\square$

By Theorem 2, we have the following result of Moonchaisook et al. [9].

Corollary 2 ([9]).*The Diophantine equation $(5^n)^x + (4^m p + 1)^y = z^2$ has no non-negative integer solution, where p is an odd prime and m, n are positive integers.*

Theorem 3. *If $n = 2$ and $p \equiv 3 \pmod 4$, then (p, x, y, z) is a non-negative integer solution of Equation (1) if and only if*

$$(p, x, y, z) \in \{(3, 6, 2, 17), (p, 3, 0, 3)\} \cup \left\{(p, 0, 1, \sqrt{5p+1}) : \sqrt{5p+1} \in \mathbb{Z}\right\}.$$

Proof. Let x, y, z be non-negative integers and (x, y, z) be a solution of Equation (1). Then $2^x + (5p)^y = z^2$.

`Case 1`. $y = 0$. By Lemma 1, we get $(p, x, y, z) = (p, 3, 0, 3)$.

`Case 2`. $y > 0$. By Lemma 2, we get x is even. If $x = 0$, then we get $(p, x, y, z) = (p, 0, 1, \sqrt{5p+1})$, where $\sqrt{5p+1}$ is an integer, by Corollary 1. Next, we consider $x \geq 2$. Then $2^x + (5p)^y$ is odd and so $z^2 \equiv 1 \pmod 4$. Since $p \equiv 3 \pmod 4$, we get $z^2 = 2^x + (5p)^y \equiv 0 + (-1)^y \pmod 4$. Then $1 \equiv 0 + (-1)^y \pmod 4$ and so y is even. Then $y = 2h$, for some positive integer h. It follows that $z^2 - (5p)^{2h} = 2^x$ and so

$$(z - (5p)^h)(z + (5p)^h) = 2^x.$$

There exists a non-negative integer w such that

$$z - (5p)^h = 2^w \tag{2}$$

and

$$z + (5p)^h = 2^{x-w}. \tag{3}$$

From Equation (2) and Equation (3), we get $x > 2w$ and

$$2(5p)^h = 2^w(2^{x-2w} - 1).$$

Since p is a prime number with $p \equiv 3 \pmod 4$, we have $w = 1$ and $2^{x-2} - 1 = (5p)^h$. If $x = 2$, then $(5p)^h = 0$, a contradiction. Thus $x \geq 4$ and so $x = 2k$, for some positive integer $k \geq 2$. Then $\left(2^{k-1}\right)^2 - 1 = (5p)^h$. If $h = 2$, then $k = 1$ and $5p = 0$, a contradiction. Assume that $h \geq 3$. By Theorem 1, it follows that $5p = 2$. This is impossible. Thus $h = 1$. This implies that $y = 2$ and $(2^{k-1} - 1)(2^{k-1} + 1) = 5p$. Since p is a prime number with $p \equiv 3 \pmod 4$, we consider the following subcases:

`Case 2.1`. $2^{k-1} - 1 = 5$ and $2^{k-1} + 1 = p$. Then $p = 7$ and so $2^{k-1} = 6$, a contradiction.

`Case 2.2`. $2^{k-1} - 1 = p$ and $2^{k-1} + 1 = 5$. Then $p = k = 3$ and so $x = 6$. Then $z^2 = 2^6 + 15^2$ or $z = 17$. Thus $(p, x, y, z) = (3, 6, 2, 17)$.

`Case 2.3`. $2^{k-1} - 1 = 1$ and $2^{k-1} + 1 = 5p$. Then $5p = 3$, a contradiction.

`Case 2.4`. $2^{k-1} - 1 = 5p$ and $2^{k-1} + 1 = 1$. Then $5p < 1$, a contradiction. □

By Theorem 3, if $p = 3$, then we have the following result of Thongnak, Chuayjan and Kaewong [22].

Corollary 3 ([22]). *The Diophantine equation* $2^x + 15^y = z^2$ *has exactly three non-negative integer solutions* (x, y, z)*, namely,* $(3, 0, 3), (0, 1, 4)$ *and* $(6, 2, 17)$.

Theorem 4. *If* $n \equiv 3 \pmod{60}$ *or* $n \equiv 27 \pmod{60}$*, and* $p \equiv 7 \pmod{12}$ *with* $\gcd(n, p) = \gcd(n, 5p - 1) = 1$*, then* (x, y, z) *is a non-negative integer solution of Equation* (1) *if and only if*

$$(x, y, z) \in \left\{(0, 1, \sqrt{5p+1}) : \sqrt{5p+1} \in \mathbb{Z}\right\} \cup \left\{(1, 0, \sqrt{n+1}) : \sqrt{n+1} \in \mathbb{Z}\right\}.$$

Proof. Let x, y, z be non-negative integers and (x, y, z) be a solution of Equation (1). Since $n \equiv 3 \pmod{60}$ or $n \equiv 27 \pmod{60}$, it implies that $n \equiv 0 \pmod 3$, $n \equiv 3 \pmod 4$ and $n \equiv 2 \pmod 5$ or $n \equiv 3 \pmod 5$.

`Case 1`. $y = 0$. Since $n \neq 2$, we get $(x, y, z) = (1, 0, \sqrt{n+1})$, where $\sqrt{n+1}$ is an integer, by Lemma 1.

`Case 2`. $y > 0$. Since $n \equiv 2 \pmod 5$ or $n \equiv 3 \pmod 5$, it follows by Lemma 2 that $x = 2k$ for some non-negative integer k. If $k = 0$, then we get $(x, y, z) = (0, 1, \sqrt{5p+1})$, where $\sqrt{5p+1}$ is an integer, by Corollary 1. Next, we consider $k > 0$. Since $p \equiv 3 \pmod 4$, $n \equiv 3 \pmod 4$ and x is even, we obtain that y is odd, by Lemma 3. From Equation (1), we get $z^2 - n^{2k} = (5p)^y$ and so

$$(z - n^k)(z + n^k) = (5p)^y.$$

There exist non-negative integers u and v such that

$$z - n^k = 5^u \cdot p^v \tag{4}$$

and

$$z + n^k = 5^{y-u} \cdot p^{y-v}. \tag{5}$$

From Equation (4) and Equation (5), we get

$$2 \cdot n^k = 5^{y-u} \cdot p^{y-v} - 5^u \cdot p^v. \tag{6}$$

`Case 2.1`. $y - v = 0$. From Equation (6), we have

$$2 \cdot n^k = 5^{y-u} - 5^u \cdot p^y. \tag{7}$$

`Case 2.1.1`. $y - u = 0$. Then $2 \cdot n^k = 1 - (5p)^y < 0$, a contradiction.

`Case 2.1.2`. $u = 0$. Since $p > 5$, we have $2 \cdot n^k = 5^y - p^y < 0$, a contradiction.

`Case 2.1.3`. $y - u > 0$ and $u > 0$. From Equation (7), we get $5 \mid n$, a contradiction since $n \equiv 2 \pmod 5$ or $n \equiv 3 \pmod 5$.

`Case 2.2`. $v = 0$. From Equation (6), we have

$$2 \cdot n^k = 5^{y-u} \cdot p^y - 5^u. \tag{8}$$

`Case 2.2.1`. $y - u = 0$. From Equation (8), we get $2 \cdot n^k = p^y - 5^y$. Since $k > 0$, we have $n^k \equiv 0 \pmod 3$ and so $p^y - 5^y = 2 \cdot n^k \equiv 0 \pmod 3$. Then $p^y - (-1)^y \equiv 0 \pmod 3$. Since y is odd, we get $p^y + 1 \equiv 0 \pmod 3$. It is impossible since $p \equiv 1 \pmod 3$.

`Case 2.2.2`. $u = 0$. From Equation (8), we get

$$2 \cdot n^k = (5p)^y - 1 = (5p - 1)((5p)^{y-1} + (5p)^{y-2} + \cdots + 1).$$

Since $k > 0$ and $5p - 1 > 2$, there exists a prime q such that $q \mid n$ and $q \mid (5p - 1)$ which is impossible since $\gcd(n, 5p - 1) = 1$.

`Case 2.2.3`. $y - u > 0$ and $u > 0$. From Equation (8), we get $5 \mid n$ which is impossible since $n \equiv 2 \pmod 5$ or $n \equiv 3 \pmod 5$.

`Case 2.3`. $y - v > 0$ and $v > 0$. From Equation (6), we get $p \mid n$ which is impossible since $\gcd(n, p) = 1$. □

By Theorem 4, if $n = 3$ and $p = 7$, then we have the following result of Biswas [2].

Corollary 4 ([2]). *The Diophantine equation* $3^x + 35^y = z^2$ *has only two non-negative integer solutions* (x, y, z), *namely,* $(1, 0, 2)$ *and* $(0, 1, 6)$.

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