# ON MINIMUM-BASE PALINDROMIC REPRESENTATIONS OF POWERS OF 2

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**Abstract**

A positive integer $N$ is *palindromic in the base* $b$ when $N = \sum_{i=0}^{k} c_i b^i$, $c_k \neq 0$, and $c_i = c_{k-i},\ i = 0, 1, 2, \ldots, k$. Focusing on powers of 2, we investigate the smallest base $b$ when $N = 2^n$ is palindromic in the base $b$.

## 1. Introduction

Let $b > 1$ be an integer. Then every positive integer $N$ can be written uniquely in the form

$$N = \sum_{i=0}^{k} c_i b^i,$$

where $k \geq 0$ and $0 \leq c_i < b$ are integers with $c_k > 0$. We write

$$N = (c_k, c_{k-1}, c_{k-2}, \cdots, c_0)_b$$

and say $N$ has representation $(c_k, c_{k-1}, c_{k-2}, \ldots, c_0)$ in *radix* or *base* $b$. For example,

$$2023 = (2, 0, 2, 3)_{10} = (5, 6, 2, 0)_7 = (7, 14, 7)_{16} = (3, 18, 22)_{23}.$$

The coefficients $c_k, c_{k-1}, \ldots, c_0$ are called the *digits* of $N$ in base $b$ and $c_k$ is the *leading digit*.

If $N = (c_k, c_{k-1}, c_{k-2}, \cdots, c_0)_b$ and $c_j = c_{k-j}$ for all $j = 0, 1, 2, \ldots, k$, then we say $N$ is a *palindrome in the base* $b$ and that $(c_k, c_{k-1}, c_{k-2}, \cdots, c_0)$ is a $(k+1)$-*digit*

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*palindromic representation* of $N$ in the base $b$. For example, 2023 has a 3-digit palindromic representation in the base 16.

**Remark.** A palindromic representation that contains leading zeros can be reduced. If

$$N = (\underbrace{0, 0, \ldots, 0}_{z \text{ digits}}, c_{k-z}, c_{k-z-1}, c_{k-z-2}, \cdots, c_z, \underbrace{0, 0, \ldots, 0}_{z \text{ digits}})_b$$

then $N = b^{z-1}M$, where $M = (c_{k-z}, c_{k-z-1}, c_{k-z-2}, \ldots, c_z)_b$ is also a palindromic representation. Thus we only consider palindromic representation with a nonzero leading digit.

There are not many papers that discuss palindromic representations in different bases. Indeed, we were able to find only seven published results, but there may be others.

(i) In [4], the authors prove, for every base $b \geq 5$, that any positive integer can be written as a sum of three palindromes in base $b$.

(ii) In [5], the author proves that there exist exactly 203 positive integers $N$ such that $N$ is a palindrome in base 10 with $d \geq 2$ digits and $N$ is also a palindrome with $d$ digits in a base $b \neq 10$. The author of [1] extends this result. He shows

(a) If $k \geq 2$, there exists $d \geq 2$, $n \geq 0$ and a list of bases $\{b_1, b_2, \ldots, b_k\}$, such that for each $1 \leq i \leq k$, $n$ is a $d$-digit palindrome in base $b_i$.

(b) If $k \geq 2$ and $d \geq 2$, then there exists $n \geq 0$ and a list of bases $\{b_1, b_2, \ldots, b_k\}$, such that for each $1 \leq i \leq k$, $n$ is a $d$-digit palindrome in base $b_i$.

This settles a conjecture of J. Ernest Wilkins, see [5].

(iii) In [8, 9, 7, 10], for fixed base $b$, the authors investigate the number of positive integers up to $n$ that are palindromic in base $b$.

The only discussion pertaining to the problem that we investigate is the website [2].

It is not difficult to see that $N = (1, 1)_{N-1}$. Hence every positive integer has a palindromic representation. We define $b = b(N)$ to be the smallest base $b > 1$ such that $N$ has a palindromic representation in the base $b$. Table 1 enumerates $b(N)$ for all $N \leq 100$. The red entries in Table 1 are where $b(N) = N - 1$.

**Theorem 1** (K. Brown [2])**.** *If $b(N) = N - 1$, then $N = 3, 4, 6$ or $N > 6$ and is a prime.*

*Proof.* It is easy to check that $b(N) \neq N - 1$ when $N = 1, 2, 5$. Also, $b(N) = N - 1$ when $N = 3, 4, 6$. Hence, we suppose $N > 6$.

| $j \diagdown i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | 2 | 3 | 2 | 3 | 2 | 5 | 2 | 3 | 2 |
| 10 | 3 | 10 | 5 | 3 | 6 | 2 | 3 | 2 | 5 | 18 |
| 20 | 3 | 2 | 10 | 3 | 5 | 4 | 3 | 2 | 3 | 4 |
| 30 | 9 | 2 | 7 | 2 | 4 | 6 | 5 | 6 | 4 | 12 |
| 40 | 3 | 5 | 4 | 6 | 10 | 2 | 4 | 46 | 7 | 6 |
| 50 | 7 | 2 | 3 | 52 | 8 | 4 | 3 | 5 | 28 | 4 |
| 60 | 9 | 6 | 5 | 2 | 7 | 2 | 10 | 5 | 3 | 22 |
| 70 | 9 | 7 | 5 | 2 | 6 | 14 | 18 | 10 | 5 | 78 |
| 80 | 3 | 8 | 3 | 5 | 11 | 2 | 6 | 28 | 5 | 8 |
| 90 | 14 | 3 | 6 | 2 | 46 | 18 | 11 | 8 | 5 | 2 |

Table 1: $b(N)$, for $N = i + j$, where $i = 0, 10, 20, \ldots, 90$ and $j = 0, 1, 2, \ldots, 9$.

Suppose $N = ab$ with $a < b - 1$, where $b > 2$. Then $N = a(b-1) + a$, so $N$ has the palindromic representation $(a, a)_{b-1}$.

This covers all composites greater than 6 except for squared primes. For squares, we have

$$a^2 = (a-1)^2 + 2(a-1) + 1,$$

so every square $N = a^2 > 4$ has the palindromic representation $(1, 2, 1)_{a-1}$. □

**Remark.** Observe that $b(13) = 3$, because $13 = (1, 1, 1)_3$, so not every prime $N$ has $b(N) = N - 1$.

If $N = (\alpha c_k, \alpha c_{k-1}, \cdots \alpha c_0)_b$, then $N = \alpha(c_k, c_{k-1}, \cdots c_0)_b = \alpha M$, where $M = (c_k, c_{k-1}, \cdots c_0)_b$ and we say that the representation of $N$ in the base $b$ is a *multiple* of the representation of $M$ in the base $b$. Conversely if $N = \alpha M$ and $M = (c_k, c_{k-1}, \cdots c_0)_b$, where $\alpha c_i < b$, for $i = 0, 1, 2, \ldots, k$, then $N = \alpha(c_k, c_{k-1}, \cdots c_0)_b$.

For example, we have

$$2023 = (7, 14, 7)_{16} = 7(1, 2, 1)_{16} = 7 \cdot 289. \tag{1}$$

The representation of 2023 is a multiple of the representation of 289.

We say that the representation $(\alpha c_k, \alpha c_{k-1}, \cdots \alpha c_0)$ of $N$ in the base $b$ has *binomial form* or is a *multiple* $\alpha$ *of a binomial representation* if $c_i = \binom{k}{i}$, $i = 0, 1, 2, \ldots, k$. Every multiple of a binomial representation in the base $b$ is of course a palindromic representation.

The following easy results can be deduced from [2].

**Lemma 1.** *If* $\alpha\binom{k}{\lceil k/2 \rceil} < b$ *and* $N = \alpha(1+b)^k$, *then* $N$ *has binomial form*

$$\alpha\left(\binom{k}{k}, \binom{k}{k-1}, \binom{k}{k-2}, \ldots, \binom{k}{0}\right)_b.$$

**Corollary 1.** *If $n = xk + r$ with $r \geq 0$ and $2^r\binom{k}{\lceil k/2 \rceil} < 2^x - 1$, then $2^n$ has a representation in base $2^x - 1$ in binomial form.*

*Proof.* If $n = xk + r$, then, setting $b = 2^x - 1$ and $\alpha = 2^r$, we obtain $N = 2^n = \alpha(1+b)^k$. Because $2^r\binom{k}{\lceil k/2 \rceil} < b$, Lemma 1 applies and we have a representation of $2^n$ in base $2^x - 1$ in binomial form. □

Kevin Brown writes in his investigation [2]

> "This raises the question of whether the min-base representation for powers of 2 is always of the binomial form."

In the rest of this paper, we prove some partial results regarding this question and we also report some computational results.

## 2. New Results

**Lemma 2.** *If the representation of $2^n$ in the base $b$ is a multiple $\alpha$ of a binomial representation, then $b = 2^x - 1$ for some integer $x \leq n$ and $\alpha$ is a power of 2.*

*Proof.* We have

$$2^n = \sum_{i=0}^{k} \alpha \binom{k}{i} b^i$$

for some positive integer $\alpha$. However,

$$\sum_{i=0}^{k} \alpha \binom{k}{i} b^i = \alpha(b+1)^k.$$

Hence, $b + 1$ is a divisor of $2^n$ and thus $b = 2^x - 1$ for some $x \leq n$. Then

$$2^n = \alpha\, 2^{kx},$$

so $\alpha = 2^{n-kx}$ and therefore $\alpha$ is a power of 2. □

**Lemma 3.** *If $N = (c_{2k+1}, c_{2k}, \ldots, c_1, c_0)_b$ is a palindromic representation, then $b + 1$ divides $N$.*

*Proof.* Because $c_i = c_{2k+1-i}$, $i = 0, 1, 2, \ldots, k$, we have

$$N = \sum_{i=0}^{2k+1} c_i b^i = \sum_{i=0}^{k} c_i (b^{2k+1-i} + b^i) = \sum_{i=0}^{k} c_i (b^{2k+1-i} + b^i)$$

Then, because $b+1$ divides $b^{2k+1-i} + b^i$ for $i = 0, 1, \ldots, k$, we have that $b+1$ divides $N$. □

**Corollary 2.** *If $N = 2^n$ has a palindromic representation in base $b$ with an even number of digits, then $b = 2^x - 1$ for some $x$.*

*Proof.* Applying Lemma 3, we see that $b+1$ divides $2^n$. Therefore $b = 2^x - 1$, for some $x$. □

We now show that Corollary 2 does not necessarily hold if $k$ is even. We consider palindromic representations of $2^n$ with $k = 2$ (i.e., palindromic representations with three digits), which necessarily have the form $2^n = (c, d, c)_b$. We present examples of such representations where $b$ is not of the form $2^x - 1$ for some integer $x$.

**Theorem 2.** *$2^n$ has a 3-digit palindromic representation in the base $b$ if and only if $b^2 + 1 \leq 2^n \leq b^3 - 1$ and*

$$\frac{1}{b}\left\lfloor \frac{2^n}{b} \right\rfloor - \frac{b-1}{b} \leq c \leq \frac{1}{b}\left\lfloor \frac{2^n}{b} \right\rfloor, \tag{2}$$

*where $c = 2^n \bmod b \neq 0$.*

*Proof.* Suppose that

$$2^n = c + db + cb^2 \tag{3}$$

with $1 \leq c \leq b-1$ and $0 \leq d \leq b-1$. The smallest palindromic representation with 3 digits is $(1,0,1)_b$ and the largest is $(b-1, b-1, b-1)_b$, so we must have

$$b^2 + 1 \leq 2^n \leq b^3 - 1.$$

Reducing Equation (3) modulo $b$, we have $2^n \equiv c \pmod{b}$. Because $0 \leq c \leq b-1$, we have $c = 2^n \bmod b$.

Next, we can express $2^n = kb + c$, where $k = \lfloor \frac{2^n}{b} \rfloor$. We have

$$c(1 + b^2) + db = 2^n = kb + c,$$

from which it follows that $d = k - cb$. We require $0 \leq d = k - cb \leq b - 1$, so

$$c \leq \frac{k}{b} = \frac{1}{b}\left\lfloor \frac{2^n}{b} \right\rfloor$$

and

$$c \geq \frac{k - b + 1}{b} = \frac{1}{b}\left\lfloor \frac{2^n}{b} \right\rfloor - \frac{b-1}{b}.$$

These necessary conditions are also sufficient, so the desired result follows. □

**Theorem 3.** *The only 3-digit binomial form representations of $2^n$ are*

$$(2^i)(1,2,1)_{2^{(n-i)/2}-1},$$

*where $0 \leq 3i < n - 2$ and $n \equiv i \pmod{2}$.*

*Proof.* If $2^n$ has a 3-digit binomial form representation, then it is

$$\alpha(1, 2, 1)_b,$$

for some base $b$ and some multiplier $\alpha$. Hence,

$$2^n = \alpha(1 + 2b + b^2) = \alpha(1 + b)^2.$$

Thus $\alpha = 2^i$ and $(1 + b)^2 = 2^j$, where $i + j = n$ and $2^{i+1} < b = 2^{j/2} - 1$. Thus $i + 1 < j/2 = (n - i)/2$. Hence $3i + 2 < n$ and $n \equiv i \pmod 2$. □

For a given base $b$, is easy to check when the conditions of Theorem 2 are satisfied. Theorem 3 identifies the cases when they have binomial form. There are other palindromic representations as well; the smallest example is $2^{12} = (11, 6, 11)_{19}$. We list all such palindromic representations, for $n \leq 20$, in Table 2.

We observe from Table 2 that there are some palindromic representations having the form $2^n = (1, c, 1)_b$. It is perhaps of interest to consider these representations in more detail.

**Theorem 4.** *There is a representation $2^n = (1, c, 1)_b$ that is not of binomial form if and only if*

(i) *there is a factorization $2^n - 1 = kb$ with $b \leq k \leq 2b - 1$, and*

(ii) *it is not the case that $n$ is even and $b = 2^{n/2} - 1$.*

*Proof.* Suppose there is a representation $2^n = (1, c, 1)_b$. From Theorem 2 and its proof, we have $2^n \equiv 1 \bmod b$ and $2^n - 1 = kb$. Therefore, $\left\lfloor \frac{2^n}{b} \right\rfloor = k$. Then Inequality (2) is

$$\frac{k}{b} - \frac{b-1}{b} \leq 1 \leq \frac{k}{b},$$

which simplifies to

$$k - b + 1 \leq b \leq k,$$

or, equivalently,

$$b \leq k \leq 2b - 1,$$

where $c = 2^n \bmod b \neq 0$. Hence there is a factorization $2^n - 1 = kb$ with

$$b \leq k \leq 2b - 1.$$

We need to consider the possibility that this representation has binomial form. From Theorem 3 with $i = 0$, we see that $b = 2^{n/2} - 1$, so $n$ is even and $2^n = (1, 2, 1)_b$. □

**Example 1.** Suppose $n = 15$. We have the prime factorization $2^{15-1} = 7 \times 31 \times 151$. If we take $b = 151$ and $k = 7 \times 31 = 217$, then the conditions of Theorem 4 are satisfied. We obtain the palindromic representation $2^{15} = (1, 66, 1)_{151}$.

| $n$ | $b$ | $c$ | $d$ |
|---|---|---|---|
| 12 | 19 | 11 | 6 |
| 13 | 27 | 11 | 6 |
| 14 | 27 | 22 | 12 |
| 14 | 60 | 4 | 33 |
| 15 | 37 | 23 | 34 |
| 15 | 151 | 1 | 66 |
| 16 | 151 | 2 | 132 |
| 17 | 142 | 6 | 71 |
| 18 | 399 | 1 | 258 |
| 19 | 269 | 7 | 66 |
| 19 | 438 | 2 | 321 |
| 20 | 269 | 14 | 132 |
| 20 | 775 | 1 | 578 |
| 20 | 825 | 1 | 446 |

Table 2: Non-binomial palindromic representations $2^n = (c, d, c)_b$ with three digits for $n \leq 20$.

We now obtain some numerical conditions that guarantee that a palindromic representation has binomial form.

**Lemma 4.** *Suppose $b = 2^x - 1$ for some $x$. Let $n$ be a positive integer and let $k$ be the number of digits of $2^n$ in its $b$-adic expansion. Then the following hold.*

(1) *If $2^n$ has a palindromic representation in base $b$, then $n > k(x - 1)$.*

(2) *Suppose $r = n - kx \geq 0$. If $k \leq x - r$, or if $3 \leq k \leq x - r + 1$ and $x \geq 3$, then the palindromic representation of $2^n$ in the base $b$ is the binomial form given in Lemma 1, with $\alpha = 2^r$.*

*Proof.* We are assuming that $b = 2^x - 1$ for some $x$ and $2^n = (c_k, c_{k-1}, \ldots, c_0)_b$ is a palindromic representation. Hence

$$2^n = \sum_{i=0}^{k} c_i b^i$$

and $c_i = c_{k-i}$, for each $i$. In particular, because $c_0 = c_k \geq 1$, we have

$$2^n \geq b^k + 1 > (2^x - 1)^k.$$

However, $2^x - 1 > 2^{x-1}$ because $x \geq 2$. Hence $2^n > (2^{x-1})^k$ and therefore $n > (x - 1)k$, which proves (1).

Now assume that $r = n - kx \geq 0$ and $k \leq x - r$. We have $2^n = 2^r(b+1)^k$. It is well known (see [6, page 35]) that

$$\binom{2m}{m} = \frac{4^m}{\sqrt{\pi m}}\left(1 - \frac{1}{8m} + \frac{1}{128m^2} + \frac{5}{1024m^3} + O(m^{-4})\right) < \frac{4^m}{\sqrt{\pi m}}$$

and

$$\binom{2m-1}{m-1} = \frac{1}{2}\binom{2m}{m} < \frac{4^{m-\frac{1}{2}}}{\sqrt{\pi m}}.$$

Thus, when $k \geq 2$ is even:

$$\binom{k}{\lceil k/2 \rceil} = \binom{k}{k/2} < \frac{2^k}{\sqrt{k\pi/2}} \leq \frac{2^k}{\sqrt{\pi}} < \frac{2^k}{1.75} \leq \frac{2^{x-r}}{1.75},$$

since $\sqrt{\pi} > 1.75$ and $k \leq x - r$. Similarly, when $k \geq 1$ is odd:

$$\binom{k}{\lceil k/2 \rceil} = \binom{k}{(k+1)/2} = \binom{k}{(k-1)/2} < \frac{2^k}{\sqrt{(k+1)\pi/2}} \leq \frac{2^k}{\sqrt{\pi}} < \frac{2^k}{1.75} \leq \frac{2^{x-r}}{1.75}.$$

Consequently

$$2^r\binom{k}{\lceil k/2 \rceil} < \frac{2^x}{1.75} < 2^x - 1 = b,$$

since $x \geq 2$. Hence, Lemma 1 applies with $\alpha = 2^r$.

The proof is similar when $3 \leq k \leq x - r + 1$. When $k \geq 4$ is even:

$$\binom{k}{\lceil k/2 \rceil} < \frac{2^k}{\sqrt{k\pi/2}} \leq \frac{2^k}{\sqrt{2\pi}} < \frac{2^k}{2.5} \leq \frac{2^{x-r+1}}{2.5} = \frac{2^{x-r}}{1.25},$$

since $\sqrt{2\pi} > 2.5$ and $k \leq x - r + 1$. The same upper bound holds when $k \geq 5$ is odd.

Consequently

$$2^r\binom{k}{\lceil k/2 \rceil} < \frac{2^x}{1.25} < 2^x - 1 = b,$$

since $x \geq 3$. Hence, Lemma 1 applies with $\alpha = 2^r$. $\square$

We can also use the proof technique of Lemma 4 to show the following.

**Theorem 5.** *For all integers $n \geq 2$, $2^n$ has a base-$b$ binomial form representation for some $b \leq 2^y - 1$, where $y \leq \lceil \sqrt{2n} \rceil + 1$.*

*Proof.* Define $x = \lceil \sqrt{2n} \rceil$ and $k = \left\lfloor \frac{n}{x} \right\rfloor$. Then $n = xk + r$, where $0 \leq r \leq x - 1$. If $r \leq x - k$, then we have a binomial form representation of $2^n$ to the base $b$, where $b \leq 2^x - 1$, as shown in Lemma 4.

Thus we can suppose that $r \geq x-k+1$. We have $n = (x+1)k + r - k$, so $n = yk + r'$, where $y = x+1$ and $r' = r-k$. Since $r \leq x-1$, we have $r' \leq x-k-1 = y-k-2 < y-k$. So we will obtain a binomial form representation of $2^n$ to the base $b$, where $b \leq 2^y - 1$, provided that $r' \geq 0$, i.e., if $r \geq k$. We have $r \geq x-k+1$, so it is sufficient to show that $x-k+1 \geq k$, i.e., $x \geq 2k-1$.

We have $x \geq \sqrt{2n}$. Since $k \leq n/x$, we have $kx \leq n$. Hence $x \geq \sqrt{2n} \geq \sqrt{2kx}$. Squaring, we obtain $x^2 \geq 2kx$ and hence $x \geq 2k > 2k-1$, as desired. $\square$

In Tables 3 and 4, the minimum-base palindromic representation of $2^n$ is computed for all $n \leq 64$. In every case, the minimum base is of the form $2^x - 1$ for some integer $x$. The hypotheses of Lemma 4 are satisfied for each such $n$ except $n = 63$. Thus, for all $n \neq 63$, the minimum-base palindromic representation of $2^n$ is guaranteed to be of binomial form. However, even for $n = 63$, the hypotheses of Lemma 1 are satisfied.

| $n$ | $k$ | $x$ | $r$ | $b(2^n)=2^x-1$ | |
|---|---|---|---|---|---|
| 1 | 0 | 2 | 1 | $b(2^1)=3$ | $2^1=2\cdot(1)_3$ |
| 2 | 1 | 2 | 0 | $b(2^2)=3$ | $2^2=(1,1)_3$ |
| 3 | 1 | 2 | 1 | $b(2^3)=3$ | $2^3=2\cdot(1,1)_3$ |
| 4 | 2 | 2 | 0 | $b(2^4)=3$ | $2^4=(1,2,1)_3$ |
| 5 | 1 | 3 | 2 | $b(2^5)=7$ | $2^5=4\cdot(1,1)_7$ |
| 6 | 2 | 3 | 0 | $b(2^6)=7$ | $2^6=(1,2,1)_7$ |
| 7 | 2 | 3 | 1 | $b(2^7)=7$ | $2^7=2\cdot(1,2,1)_7$ |
| 8 | 2 | 4 | 0 | $b(2^8)=15$ | $2^8=(1,2,1)_{15}$ |
| 9 | 3 | 3 | 0 | $b(2^9)=7$ | $2^9=(1,3,3,1)_7$ |
| 10 | 3 | 3 | 1 | $b(2^{10})=7$ | $2^{10}=2\cdot(1,3,3,1)_7$ |
| 11 | 2 | 5 | 1 | $b(2^{11})=31$ | $2^{11}=2\cdot(1,2,1)_{31}$ |
| 12 | 4 | 3 | 0 | $b(2^{12})=7$ | $2^{12}=(1,4,6,4,1)_7$ |
| 13 | 3 | 4 | 1 | $b(2^{13})=15$ | $2^{13}=2\cdot(1,3,3,1)_{15}$ |
| 14 | 3 | 4 | 2 | $b(2^{14})=15$ | $2^{14}=4\cdot(1,3,3,1)_{15}$ |
| 15 | 3 | 5 | 0 | $b(2^{15})=31$ | $2^{15}=(1,3,3,1)_{31}$ |
| 16 | 4 | 4 | 0 | $b(2^{16})=15$ | $2^{16}=(1,4,6,4,1)_{15}$ |
| 17 | 4 | 4 | 1 | $b(2^{17})=15$ | $2^{17}=2\cdot(1,4,6,4,1)_{15}$ |
| 18 | 3 | 5 | 3 | $b(2^{18})=31$ | $2^{18}=8\cdot(1,3,3,1)_{31}$ |
| 19 | 3 | 6 | 1 | $b(2^{19})=63$ | $2^{19}=2\cdot(1,3,3,1)_{63}$ |
| 20 | 5 | 4 | 0 | $b(2^{20})=15$ | $2^{20}=(1,5,10,10,5,1)_{15}$ |
| 21 | 4 | 5 | 1 | $b(2^{21})=31$ | $2^{21}=2\cdot(1,4,6,4,1)_{31}$ |
| 22 | 4 | 5 | 2 | $b(2^{22})=31$ | $2^{22}=4\cdot(1,4,6,4,1)_{31}$ |
| 23 | 3 | 7 | 2 | $b(2^{23})=127$ | $2^{23}=4\cdot(1,3,3,1)_{127}$ |
| 24 | 4 | 6 | 0 | $b(2^{24})=63$ | $2^{24}=(1,4,6,4,1)_{63}$ |
| 25 | 5 | 5 | 0 | $b(2^{25})=31$ | $2^{25}=(1,5,10,10,5,1)_{31}$ |
| 26 | 5 | 5 | 1 | $b(2^{26})=31$ | $2^{26}=2\cdot(1,5,10,10,5,1)_{31}$ |

Table 3: Palindromic representations of $2^n$, where $1 \leq n = kx + r \leq 26$

| $n$ | $k$ | $x$ | $r$ | $b(2^n)=2^x-1$ | |
|---|---|---|---|---|---|
| 27 | 4 | 6 | 3 | $b(2^{27})=63$ | $2^{27}=8\cdot(1,4,6,4,1)_{63}$ |
| 28 | 4 | 7 | 0 | $b(2^{28})=127$ | $2^{28}=(1,4,6,4,1)_{127}$ |
| 29 | 4 | 7 | 1 | $b(2^{29})=127$ | $2^{29}=2\cdot(1,4,6,4,1)_{127}$ |
| 30 | 6 | 5 | 0 | $b(2^{30})=31$ | $2^{30}=(1,6,15,20,15,6,1)_{31}$ |
| 31 | 5 | 6 | 1 | $b(2^{31})=63$ | $2^{31}=2\cdot(1,5,10,10,5,1)_{63}$ |
| 32 | 5 | 6 | 2 | $b(2^{32})=63$ | $2^{32}=4\cdot(1,5,10,10,5,1)_{63}$ |
| 33 | 4 | 8 | 1 | $b(2^{33})=255$ | $2^{33}=2\cdot(1,4,6,4,1)_{255}$ |
| 34 | 4 | 8 | 2 | $b(2^{34})=255$ | $2^{34}=4\cdot(1,4,6,4,1)_{255}$ |
| 35 | 5 | 7 | 0 | $b(2^{35})=127$ | $2^{35}=(1,5,10,10,5,1)_{127}$ |
| 36 | 6 | 6 | 0 | $b(2^{36})=63$ | $2^{36}=(1,6,15,20,15,6,1)_{63}$ |
| 37 | 6 | 6 | 1 | $b(2^{37})=63$ | $2^{37}=2\cdot(1,6,15,20,15,6,1)_{63}$ |
| 38 | 5 | 7 | 3 | $b(2^{38})=127$ | $2^{38}=8\cdot(1,5,10,10,5,1)_{127}$ |
| 39 | 4 | 9 | 3 | $b(2^{39})=511$ | $2^{39}=8\cdot(1,4,6,4,1)_{511}$ |
| 40 | 5 | 8 | 0 | $b(2^{40})=255$ | $2^{40}=(1,5,10,10,5,1)_{255}$ |
| 41 | 5 | 8 | 1 | $b(2^{41})=255$ | $2^{41}=2\cdot(1,5,10,10,5,1)_{255}$ |
| 42 | 7 | 6 | 0 | $b(2^{42})=63$ | $2^{42}=(1,7,21,35,35,21,7,1)_{63}$ |
| 43 | 6 | 7 | 1 | $b(2^{43})=127$ | $2^{43}=2\cdot(1,6,15,20,15,6,1)_{127}$ |
| 44 | 6 | 7 | 2 | $b(2^{44})=127$ | $2^{44}=4\cdot(1,6,15,20,15,6,1)_{127}$ |
| 45 | 5 | 9 | 0 | $b(2^{45})=511$ | $2^{45}=(1,5,10,10,5,1)_{511}$ |
| 46 | 5 | 9 | 1 | $b(2^{46})=511$ | $2^{46}=2\cdot(1,5,10,10,5,1)_{511}$ |
| 47 | 5 | 9 | 2 | $b(2^{47})=511$ | $2^{47}=4\cdot(1,5,10,10,5,1)_{511}$ |
| 48 | 6 | 8 | 0 | $b(2^{48})=255$ | $2^{48}=(1,6,15,20,15,6,1)_{255}$ |
| 49 | 7 | 7 | 0 | $b(2^{49})=127$ | $2^{49}=(1,7,21,35,35,21,7,1)_{127}$ |
| 50 | 7 | 7 | 1 | $b(2^{50})=127$ | $2^{50}=2\cdot(1,7,21,35,35,21,7,1)_{127}$ |
| 51 | 6 | 8 | 3 | $b(2^{51})=255$ | $2^{51}=8\cdot(1,6,15,20,15,6,1)_{255}$ |
| 52 | 5 | 10 | 2 | $b(2^{52})=1023$ | $2^{52}=4\cdot(1,5,10,10,5,1)_{1023}$ |
| 53 | 5 | 10 | 3 | $b(2^{53})=1023$ | $2^{53}=8\cdot(1,5,10,10,5,1)_{1023}$ |
| 54 | 6 | 9 | 0 | $b(2^{54})=511$ | $2^{54}=(1,6,15,20,15,6,1)_{511}$ |
| 55 | 6 | 9 | 1 | $b(2^{55})=511$ | $2^{55}=2\cdot(1,6,15,20,15,6,1)_{511}$ |
| 56 | 8 | 7 | 0 | $b(2^{56})=127$ | $2^{56}=(1,8,28,56,70,56,28,8,1)_{127}$ |
| 57 | 7 | 8 | 1 | $b(2^{57})=255$ | $2^{57}=2\cdot(1,7,21,35,35,21,7,1)_{255}$ |
| 58 | 7 | 8 | 2 | $b(2^{58})=255$ | $2^{58}=4\cdot(1,7,21,35,35,21,7,1)_{255}$ |
| 59 | 5 | 11 | 4 | $b(2^{59})=2047$ | $2^{59}=16\cdot(1,5,10,10,5,1)_{2047}$ |
| 60 | 6 | 10 | 0 | $b(2^{60})=1023$ | $2^{60}=(1,6,15,20,15,6,1)_{1023}$ |
| 61 | 6 | 10 | 1 | $b(2^{61})=1023$ | $2^{61}=2\cdot(1,6,15,20,15,6,1)_{1023}$ |
| 62 | 6 | 10 | 2 | $b(2^{62})=1023$ | $2^{62}=4\cdot(1,6,15,20,15,6,1)_{1023}$ |
| 63 | 9 | 7 | 0 | $b(2^{63})=127$ | $2^{63}=(1,9,36,84,126,126,84,36,9,1)_{127}$ |
| 64 | 8 | 8 | 0 | $b(2^{64})=255$ | $2^{64}=(1,8,28,56,70,56,28,8,1)_{255}$ |

Table 4: Palindromic representations of $2^n$, where $27 \le n = kx + r \le 64$

We also have computed all palindromic representations of $2^n$ for every positive integer $n < 64$. Perhaps surprisingly, these computations show that every palindromic representation of $2^n$ is either of binomial form or has three digits.

## 3. Some Extensions

**Theorem 6.** *If $p$ is a prime and $N = p^n$ has a palindromic representation $N = (c_k, c_{k-1}, \ldots, c_0)_b$ with $k$ odd, then $b = p^x - 1$ for some $x$.*

*Proof.* Applying Lemma 3, we see that $b + 1$ divides $p^n$. Therefore $b = p^x - 1$, for some $x$. □

An obvious theorem whose proof we leave for the reader is Theorem 7.

**Theorem 7.** *For any positive integer $z$.*

(I) *$z^n$ has the $(n+1)$-digit representation $z = (1, 0, 0, \ldots, 0)_z$.*

(II) *$z^n + 1$ has the $(n+1)$-digit palindromic representation $z = (1, 0, 0, \ldots, 1)_z$.*

(III) *$z^n - 1$ has the $n$-digit palindromic representation $z = (1, 1, 1, \ldots, 1)_z$.*

Note that Theorem 7 implies $b(2^n \pm 1) = 2$.

A solution $(x, y, n, q)$ to the Nagell-Ljunggren Diophantine equation

$$\frac{x^n - 1}{x - 1} = y^q \tag{4}$$

is equivalent to the $n$-digit palindromic representation

$$y^q = (1, 1, 1, \ldots, 1)_z.$$

Bugeaud and Mihăilescu studied the Nagell-Ljunggren Diophantine equation and obtained the following theorem.

**Theorem 8** (Bugeaud and Mihăilescu [3])**.** *Apart from the solutions*

$$11^2 = (1, 1, 1, 1, 1)_3, \qquad 20^2 = (1, 1, 1, 1)_7 \quad \text{and} \quad 7^3 = (1, 1, 1)_{18},$$

*Equation (4) has no other solution $(x, y, n, q)$ if any of the following conditions are satisfied:*

(i) *$q = 2$,*

(ii) *3 divides $n$,*

(iii) *4 divides $n$,*

(iv) *$q = 3$ and $n \not\equiv 5 \pmod 6$.*

Some additional computational results are presented in Tables 5 and 6 in the Appendix.

## 4. Concluding Remarks

We have several conjectures:

(a) $b(2^n) = 2^x - 1$, for some $x$;

(b) The minimum palindromic representation of $2^n$ has binomial form;

(c) $b(2^{a^2}) = 2^a - 1$;

(d) For any base $b$ there are only finitely many integers $N = 2^n$ such that $b(N) = b$; and

(e) $b(2^n) = 3$ if and only if $n = 1, 2, 3$ or 4.

**Acknowledgement.** Every year on his birthday, the first author reports to Marco Buratti of Rome some amusing palindromic connection. (Marco is obsessed with palindromes.) This past year (2023) he reported that he turned 68 which is a palindrome in base 3 and also in base 16. The interlocution quickly turned to a discussion on what is the smallest base such that N is a palindrome in that base. Data was generated and on 22 September 2023, Marco wrote:

Dear Don,
thank you for your interesting message!
Is $b(2^n)$ always of the form $2^i - 1$?
Maybe my question is stupid.
Ciao,
Marco

It turns out that it is not so stupid after all.

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## Appendix

We provide in Tables 5 and 6 some additional computational results. Non-binomial forms are marked in red.

| | |
|---|---|
| $2^1 = 2 \cdot (1)_3$ | $2^{16} = (1,0,0,2,2,2,2,0,0,2,1)_3$ |
| $2^2 = (1,1)_3$ | $2^{17} = (2,0,1,2,2,2,1,0,1,1,2)_3$ |
| $2^3 = 2 \cdot (1,1)_3$ | $2^{18} = (1,1,1,0,2,2,1,2,1,0,0,1)_3$ |
| $2^4 = (1,2,1)_3$ | $2^{19} = (2,2,2,1,2,2,0,1,2,0,0,2)_3$ |
| $2^5 = (1,0,1,2)_3$ | $2^{20} = (1,2,2,2,0,2,1,1,0,1,0,1,1)_3$ |
| $2^6 = (2,1,0,1)_3$ | $2^{21} = (1,0,2,2,1,1,1,2,2,0,2,0,2,2)_3$ |
| $2^7 = (1,1,2,0,2)_3$ | $2^{22} = (2,1,2,2,0,0,0,2,1,1,1,1,2,1)_3$ |
| $2^8 = (1,0,0,1,1,1)_3$ | $2^{23} = (1,2,0,2,1,0,0,1,2,0,0,0,0,1,2)_3$ |
| $2^9 = 2 \cdot (1,0,0,1,1,1)_3$ | $2^{24} = (1,0,1,1,1,2,0,1,0,1,0,0,0,1,0,1)_3$ |
| $2^{10} = (1,1,0,1,2,2,1)_3$ | $2^{25} = (2,1,0,0,0,1,0,2,0,2,0,0,0,2,0,2)_3$ |
| $2^{11} = (2,2,1,0,2,1,2)_3$ | $2^{26} = (1,1,2,0,0,0,2,1,1,1,1,0,0,1,1,1,1)_3$ |
| $2^{12} = (1,2,1,2,1,2,0,1)_3$ | $2^{27} = (1,0,0,1,0,0,1,1,2,2,2,2,0,0,2,2,2,2)_3$ |
| $2^{13} = (1,0,2,0,2,0,1,0,2)_3$ | $2^{28} = (2,0,0,2,0,1,0,0,2,2,2,1,0,1,2,2,2,1)_3$ |
| $2^{14} = (2,1,1,1,1,0,2,1,1)_3$ | $2^{29} = (1,1,0,1,1,0,2,0,1,2,2,1,2,1,0,2,2,1,2)_3$ |
| $2^{15} = (1,1,2,2,2,2,1,1,2,2)_3$ | $2^{30} = (2,2,0,2,2,1,1,1,0,2,2,0,1,2,1,2,2,0,1)_3$ |

Table 5: Representations of $2^n$ in the base 3 for $n \leq 25$.
Non-palindromic forms are marked in red.

| | |
|---|---|
| $b(3^1) = 2$ | $3^1 = (1,1)_2$ |
| $b(3^2) = 2$ | $3^2 = (1,0,0,1)_2$ |
| $b(3^3) = 2$ | $3^3 = (1,1,0,1,1)_2$ |
| $b(3^4) = 8$ | $3^4 = (1,2,1)_8$ |
| $b(3^5) = 8$ | $3^5 = 3 \cdot (1,2,1)_8$ |
| $b(3^6) = 8$ | $3^6 = (1,3,3,1)_8$ |
| $b(3^7) = 24$ | $3^7 = (3,19,3)_{24}$ |
| $b(3^8) = 8$ | $3^8 = (1,4,6,4,1)_8$ |
| $b(3^9) = 26$ | $3^9 = (1,3,3,1)_{26}$ |
| $b(3^{10}) = 26$ | $3^{10} = 3 \cdot (1,3,3,1)_{26}$ |
| $b(3^{11}) = 80$ | $3^{11} = 27 \cdot (1,2,1)_{80}$ |
| $b(3^{12}) = 26$ | $3^{12} = (1,4,6,4,1)_{26}$ |
| $b(3^{13}) = 26$ | $3^{13} = 3 \cdot (1,4,6,4,1)_{26}$ |
| $b(3^{14}) = 80$ | $3^{14} = 9 \cdot (1,3,3,1)_{80}$ |
| $b(3^{15}) = 26$ | $3^{15} = (1,5,10,10,5,1)_{26}$ |
| $b(3^{16}) = 80$ | $3^{16} = (1,4,6,4,1)_{80}$ |
| $b(3^{17}) = 80$ | $3^{17} = 3 \cdot (1,4,6,4,1)_{80}$ |
| $b(3^{18}) = 26$ | $3^{18} = (1,6,15,20,15,6,1)_{26}$ |
| $b(5^1) = 2$ | $5^1 = (1,0,1)_2$ |
| $b(5^2) = 4$ | $5^2 = (1,2,1)_4$ |
| $b(5^3) = 4$ | $5^3 = (1,3,3,1)_4$ |
| $b(5^4) = 24$ | $5^4 = (1,2,1)_{24}$ |
| $b(5^5) = 24$ | $5^5 = 5 \cdot (1,2,1)_{24}$ |
| $b(5^6) = 24$ | $5^6 = (1,3,3,1)_{24}$ |
| $b(5^7) = 24$ | $5^7 = 5 \cdot (1,3,3,1)_{24}$ |
| $b(5^8) = 24$ | $5^8 = (1,4,6,4,1)_{24}$ |
| $b(5^9) = 124$ | $5^9 = (1,3,3,1)_{124}$ |
| $b(5^{10}) = 24$ | $5^{10} = (1,5,10,10,5,1)_{24}$ |
| $b(5^{11}) = 124$ | $5^{11} = 25 \cdot (1,3,3,1)_{124}$ |
| $b(5^{12}) = 24$ | $5^{12} = (1,6,15,20,15,6,1)_{24}$ |
| $b(7^1) = 2$ | $7^1 = (1,1,1)_2$ |
| $b(7^2) = 6$ | $7^2 = (1,2,1)_6$ |
| $b(7^3) = 6$ | $7^3 = (1,3,3,1)_6$ |
| $b(7^4) = 18$ | $7^4 = 7 \cdot (1,1,1)_{18}$ |
| $b(7^5) = 38$ | $7^5 = (11,24,11)_{38}$ |
| $b(7^6) = 18$ | $7^6 = (1,2,3,2,1)_{18}$ |
| $b(7^7) = 48$ | $7^7 = 7 \cdot (1,3,3,1)_{48}$ |
| $b(7^8) = 48$ | $7^8 = (1,4,6,4,1)_{48}$ |
| $b(7^9) = 18$ | $7^9 = (1,3,6,7,6,3,1)_{18}$ |
| $b(7^{10}) = 48$ | $7^{10} = (1,5,10,10,5,1)_{48}$ |
| $b(11^1) = 10$ | $11^1 = (1,1)_{10}$ |
| $b(11^2) = 3$ | $11^2 = (1,1,1,1,1)_3$ |
| $b(11^3) = 10$ | $11^3 = (1,3,3,1)_{10}$ |
| $b(11^4) = 10$ | $11^4 = (1,4,6,4,1)_{10}$ |
| $b(11^5) = 56$ | $11^5 = (51,19,51)_{56}$ |
| $b(11^6) = 35$ | $11^6 = (1,6,11,6,1)_{35}$ |
| $b(11^7) = 120$ | $11^7 = 11 \cdot (1,3,3,1)_{120}$ |
| $b(11^8) = 120$ | $11^8 = (1,4,6,4,1)_{120}$ |
| $b(13^1) = 3$ | $13^1 = (1,1,1)_3$ |
| $b(13^2) = 12$ | $13^2 = (1,2,1)_{12}$ |
| $b(13^3) = 12$ | $13^3 = (1,3,3,1)_{12}$ |
| $b(13^4) = 12$ | $13^4 = (1,4,6,4,1)_{12}$ |
| $b(13^5) = 12$ | $13^5 = (1,5,10,10,5,1)_{12}$ |
| $b(13^6) = 168$ | $13^6 = (1,3,3,1)_{168}$ |
| $b(13^7) = 168$ | $13^7 = 13 \cdot (1,3,3,1)_{168}$ |
| $b(13^8) = 168$ | $13^8 = (1,4,6,4,1)_{168}$ |
| $b(17^1) = 2$ | $17^1 = (1,0,0,0,1)_2$ |
| $b(17^2) = 4$ | $17^2 = (1,0,2,0,1)_4$ |
| $b(17^3) = 4$ | $17^3 = (1,0,3,0,3,0,1)_4$ |
| $b(17^4) = 16$ | $17^4 = (1,4,6,4,1)_{16}$ |
| $b(17^5) = 16$ | $17^5 = (1,5,10,10,5,1)_{16}$ |
| $b(17^6) = 63$ | $17^6 = (1,33,33,33,1)_{63}$ |
| $b(17^7) = 288$ | $17^7 = 17 \cdot (1,3,3,1)_{288}$ |
| $b(19^1) = 18$ | $19^1 = (1,1)_{18}$ |
| $b(19^2) = 15$ | $19^2 = (1,9,1)_{15}$ |
| $b(19^3) = 18$ | $19^3 = (1,3,3,1)_{18}$ |
| $b(19^4) = 18$ | $19^4 = (1,4,6,4,1)_{18}$ |
| $b(19^5) = 18$ | $19^5 = (1,5,10,10,5,1)_{18}$ |
| $b(19^6) = 360$ | $19^6 = (1,3,3,1)_{360}$ |
| $b(19^7) = 360$ | $19^7 = 19 \cdot (1,3,3,1)_{360}$ |
| $b(23^1) = 3$ | $23^1 = (2,1,2)_3$ |
| $b(23^2) = 22$ | $23^2 = (1,2,1)_{22}$ |
| $b(23^3) = 22$ | $23^3 = (1,3,3,1)_{22}$ |
| $b(23^4) = 22$ | $23^4 = (1,4,6,4,1)_{22}$ |
| $b(23^5) = 22$ | $23^5 = (1,5,10,10,5,1)_{22}$ |
| $b(23^6) = 22$ | $23^6 = (1,6,15,20,15,6,1)_{22}$ |
| $b(29^1) = 4$ | $29^1 = (1,3,1)_4$ |
| $b(29^2) = 21$ | $29^2 = (1,19,1)_{21}$ |
| $b(29^3) = 28$ | $29^3 = (1,3,3,1)_{28}$ |
| $b(29^4) = 28$ | $29^4 = (1,4,6,4,1)_{28}$ |
| $b(29^5) = 28$ | $29^5 = (1,5,10,10,5,1)_{28}$ |
| $b(29^6) = 28$ | $29^6 = (1,6,15,20,15,6,1)_{28}$ |

Table 6: Palindromic representations of $p^n < 2^{30}$, where $3 \le p \le 29$ is prime.