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$F_4 F_5^6 F_7 F_8^4 F_9 F_{10}^2 F_{12}^5 F_{14} F_{15}^{-1} F_{18} F_{24} F_{30} = 36!$

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Abstract
In this paper, we show that 36! is the largest factorial which belongs to the multiplicative group generated by the Fibonacci numbers.

1. Introduction

During the author's fellowship at STIAS in Fall of 2023, his colleague, E. Ntakadzeni Madala, reminded us of the paper [4] and wondered what its title meant as there is no word in it. In fact, in that joint paper with P. Stănică, we investigated the Diophantine equation

$$F_{n_1} \cdots F_{n_k} = m_1! \cdots m_t!$$

where $1 \le n_1 < n_2 < \cdots < n_k$ and $1 \le m_1 \le \cdots \le m_t$ are integers. The formula appearing in the title of [4] ends up being the "largest solution" where largest means with the largest left-hand (or right-hand) sides among all solutions. Note that $F_1 = F_2 = 1$ can be added or omitted as factors to the left-hand side of the product and they were added to create the impression of a solution which is "longer" (has more Fibonacci factors in it) than its most compact representation. Here we decided to study a variant of this equation where now we allow Fibonacci numbers to repeat on the left-hand side but ask for $t = 1$ in the right-hand side. It turns out that (one of) the largest solution then is

$$F_3^2 F_4 F_5^5 F_7 F_8^3 F_{10}^2 F_{12}^4 F_{14} F_{18} F_{24} = 30!$$

The representation of 30! as a product of Fibonacci numbers is not unique as $F_6 = 8 = 2^3 = F_3^3$ and $F_{12} = 144 = 2^4 \cdot 3^2 = F_3 F_4^2 F_6 = F_3^4 F_4^2$. But we can even allow

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negative exponents on the left and then we therefore ask, what are the factorials which belong to the multiplicative group generated by the Fibonacci numbers? Let us denote this group by $\mathcal{G}_F$. In the paper [3], we coined the terminology *Fibonacci integer* to mean a positive integer which belongs to $\mathcal{G}_F$ and we studied upper and lower bounds for the counting function of the Fibonacci integers. Here, we prove the following result.

Theorem 1. *The largest factorial which is a Fibonacci integer is* 36!.

The title formula gives a representation of 36! as a member of $\mathcal{G}_F$. It was already remarked in [3] that if x is such that every prime number $p \le x$ is a Fibonacci integer then all integers whose primes factors are at most x are also Fibonacci integers. It was also remarked that 37 is the first prime which is not a Fibonacci integer, which gives an immediate proof of the fact that 36! is a Fibonacci integer (save for writing down an actual representation of 36! as a Fibonacci integer), but 37! is not a Fibonacci integer. Yet, we need to prove that no $n!$ for $n \ge 37$ is a Fibonacci integer, which is what we do in this paper. Let us first begin with some preliminaries.

2. Preliminaries

Let $(\alpha, \beta) := ((1+\sqrt{5})/2, (1-\sqrt{5})/2)$ be the roots of the characteristic equation $x^2 - x - 1 = 0$ of the Fibonacci sequence. It is well-known that

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{holds for all} \qquad n \ge 0.$$

For a positive integer m write

$$\Phi_m(X) := \prod_{\substack{1 \le k \le m \\ (k,m)=1}} (X - \exp(2\pi i k/m)) \in \mathbb{Z}[X]$$

for the mth cyclotomic polynomial, and let

$$\Phi_m(X,Y) := \prod_{\substack{1 \le k \le m \\ (k,m)=1}} (X - \exp(2\pi i k/m)Y) \in \mathbb{Z}[X,Y]$$

be its homogenization. Following [3], let Φ_m stand for $\Phi_m(\alpha, \beta)$. We then have

$$F_m = \prod_{\substack{d \mid m \\ d > 1}} \Phi_d,$$

which gives, by the Möbius inversion formula, for $m > 1$

$$\Phi_m = \prod_{d|m} F_{m/d}^{\mu(d)}.$$

The formulas show that $\{\Phi_m\}_{m>1}$ generate the same multiplicative group as the Fibonacci numbers. It turns out that $\mathcal{G}_F$ is almost freely generated by $\{\Phi_m\}_{m\geq 2}$. This is not exactly so because of the exceptions

$$\Phi_2 = 1, \qquad \Phi_6 = \frac{F_6}{F_2F_3} = 2^2 = \Phi_3^2, \qquad \Phi_{12} = \frac{F_{12}F_2}{F_4F_6} = 6 = \Phi_3\Phi_4. \tag{1}$$

The number Φ_m for $m > 1$ captures the *primitive divisors* of F_m which are prime factors $p \mid F_m$ which do not divide any Fibonacci number F_n for $1 \leq n < m$. Let us recall the following theorem of Carmichael [2].

Theorem 2 (Primitive Divisor Theorem)**.** *If $m > 12$, then F_m has a prime factor p such that $p \nmid F_n$ for any positive integer $n < m$. All such primes p have the property that $p \equiv \pm 1 \pmod{m}$.*

One verifies by hand that F_m has primitive divisors for $m \leq 12$ except when $m \in \{2, 6, 12\}$ (the index $m = 5$ is exceptional as for it $p = 5$ is the only primitive prime factor and the congruence condition on p modulo m from the statement of the above theorem is replaced by $p \equiv 0 \pmod{m}$). Hence, the only multiplicative relations among the Φ_m's are obtained from the three ones appearing in (1). Let Ψ_m be the product of the primitive prime factors of F_m with the exponent they appear in the factorization of F_m. Clearly, $\Psi_m \mid \Phi_m$. The quotient $\delta_m := \Phi_m/\Psi_m$ is also well understood. To write it down, let for a positive integer k, $z(k)$ denote the smallest positive integer m such that $k \mid F_m$. Then the definition of Ψ_m becomes

$$\Psi_m = \prod_{\substack{p^{a_p} \| F_m \\ z(p)=m}} p^{a_p}.$$

Further, $\delta_m = 1$ except if

$$m = p^k z(p) \qquad \text{for some integer} \quad k \geq 1 \quad \text{and prime} \quad p,$$

in which case if additionally $m \notin \{6, 12\}$, we have $\delta_m = p$. For $m \in \{6, 12\}$, we have $6 = 2z(2)$ and $\delta_6 = 2^2$ and $12 = 2^2 z(2) = 3z(3)$ and $\delta_{12} = 2 \cdot 3$. Furthermore, since for prime p for which $z(p) = m$ we have

$$p \equiv 0, \pm 1 \pmod{m}, \tag{2}$$

it follows that if $p \neq 2, 5$ then p is larger than the largest prime factor of m. Further, if $m > 12$ and $m = p^k z(p)$ for some prime $p > 2$ and integer $k \geq 1$, then $p = P(m)$

is the largest prime factor of m, and k is also uniquely determined as $\nu_p(m)$ except if $p = 5$ in which case it is $\nu_p(m) - 1$. Here, we use the notation $\nu_p(m)$ for the exact exponent of the prime p in the factorization of the integer m.

Let $\mathcal{M} = \mathbb{N}\backslash\{1, 2, 6, 12\}$. Then $\mathcal{G}_F$ is freely generated by Φ_m for $m \in \mathcal{M}$. This group is also freely generated by $(F_m)_{m\in\mathcal{M}}$. Furthermore, the numbers $\{\Phi_m\}_{m\in\mathcal{M}}$ almost freely generate the multiplicative semigroup generated by the Fibonacci numbers. However, there are Fibonacci integers which are not in the semigroup generated by $\{\Phi_m\}_{m\in\mathcal{M}}$ such as

$$\frac{\Phi_{24}}{\Phi_3}, \qquad \frac{\Phi_{25}}{\Phi_5}, \qquad \frac{\Phi_{37\cdot 19} \cdot \Phi_{113\cdot 19}}{\Phi_{19}}.$$

The following lemma is Lemma 1 in [3].

Lemma 1. *Assume that $\mathcal{I}, \mathcal{J}$ are finite multisets of indices with $n_i,\ m_j \in \mathcal{M}$, $n_i \neq m_j$ for all $i \in \mathcal{I},\ j \in \mathcal{J}$ and such that*

$$\prod_{i\in\mathcal{I}} \Phi_{n_i} \prod_{j\in\mathcal{J}} \Phi_{m_j}^{-1} \in \mathbb{N}.$$

There exists an injection f from the multiset of prime factors of

$$\prod_{j\in\mathcal{J}} \Psi_{m_j} = p_1 \cdots p_k$$

into the multiset $\{n_i\}_{i\in\mathcal{I}}$, where $f(p_l) = p_l^{k_l} z(p_l)$.

Remark 1 in [3] points out that the above lemma does not tell the entire story as it does not address δ_{m_j} for $j \in \mathcal{J}$. We will overcome this difficulty for our problem in the next section.

3. The Proof of Theorem 1

We assume that $n \geq 37$ and that

$$n! = \prod_{i\in\mathcal{I}} \Phi_{n_i} \prod_{j\in\mathcal{J}} \Phi_{m_j}^{-1}. \tag{3}$$

Let $\mathcal{J}^*$ be the multiset of prime factors of $\prod_{j\in\mathcal{J}} \Psi_{m_j}$, and let $\mathcal{I}^*$ be the multiset $\{n_i\}_{i\in\mathcal{I}}$. Using Lemma 1 we write

$$\prod_{j\in\mathcal{J}} \Phi_{m_j} = \prod_{j\in\mathcal{J}} \delta_{m_j} \prod_{j\in\mathcal{J}} \Psi_{m_j} = \left(\prod_{j\in\mathcal{J}} \delta_{m_j}\right) p_1 \ldots p_k$$

and let f be the injection from $\mathcal{J}^*$ into $\mathcal{I}^*$ given by $f(p_l) = p_l^{k_l} z(p_l)$ for $l = 1, \ldots, k$. Let $\mathcal{I}_1$ be a sub-multiset consisting of $i \in \mathcal{I}$ such that n_i in the multiset $\mathcal{I}^*$ is in the image of f as a multiset. Then

$$n! = \prod_{i \in \mathcal{I} \setminus \mathcal{I}_1} \Phi_{n_i} \left(\prod_{l=1}^{k} \frac{\Phi_{f(p_l)}}{p_l} \right) \prod_{j \in \mathcal{J}} \delta_{m_j}^{-1}. \tag{4}$$

Let p be a prime with $z(p) > 12$. In the left-hand side,

$$\frac{n}{p-1} - \frac{\log(n+1)}{\log p} \le \nu_p(n!) < \frac{n}{p-1} \tag{5}$$

(see Lemma 1 in [1]). In the right-hand side of (1), all factors Φ_{n_i} for $i \in \mathcal{I} \setminus \mathcal{I}_1$, and $\frac{\Phi_{f(p_l)}}{p_l}$ for $l = 1, \ldots, k$, are integers, so ν_p of such factors is non-negative. Further,

$$\nu_p(\delta_{m_j}) = \begin{cases} 1 & \text{if} \quad m_j = p^{u_j} z(p) \quad \text{for some} \quad u_j \ge 1; \\ 0 & \text{otherwise.} \end{cases}$$

Putting

$$o_p := \nu_p(F_{z(p)}),$$

we get

$$\nu_p(n!) \le s_0(p) o_p + \sum_{u \ge 1} s_i(p) - N_p, \tag{6}$$

where for $u \ge 0$ we put

$$s_u(p) = \#\{i \in \mathcal{I} : n_i = p^u z(p)\}, \qquad \text{and} \qquad N_p = \#\{j \in \mathcal{J} : \nu_p(\delta_{m_j}) = 1\}.$$

The $\le$ sign in (6) is given by the (possible) contributions of p in $\Phi_{f(p_l)}$. Namely, note that if $p = p_l$ for some $l = 1, \ldots, k$ then $f(p_l) = n_i$ contributes 1 in the count $s_{k_l}(p)$ but $\nu_p(\Phi_{f(p_l)}/p_l) = 0$. Furthermore,

$$\nu_p(n!) \ge s_0(p) o_p - N_p. \tag{7}$$

Indeed, the above follows from the fact that if $z(p) = n_i$ for some i, then either $i \in \mathcal{I} \setminus \mathcal{I}_1$, or $n_i \in f(\mathcal{J}^*)$, so $z(p) = p_l^{k_l} z(p_l)$ for some $k_l \ge 1$, which shows that $p_l \ne p$, and therefore $\nu_p(\Phi_{z(p)}/p_l) = \nu_p(\Phi_{z(p)}) = o_p$.

Lemma 2. *For $z(p) > 12$, we have*

$$N_p < \frac{2n}{z(p) p(p-2)}. \tag{8}$$

Assume that Lemma 2 is proved, and let us show how we finish. We use that $z(37) = z(113) = 19$ and $o_{37} = o_{113} = 1$. Thus, using (5), (7) and (8) for $p = q = 113$, we get that

$$\frac{n}{112} + \frac{2n}{19 \cdot 111 \cdot 113} = \frac{n}{q-1} + \frac{4n}{z(q)q(q-2)} > s_0(113) = s_0(37). \tag{9}$$

For each $i \geq 1$, let q_i be a prime number such that $z(q_i) = 19 \cdot 37^i$, and the existence of such primes is guaranteed by the Primitive Divisor Theorem. Then $q_i \geq 2 \cdot 19 \cdot 37^i - 1$ by (2). Inequality (7) applied to q_i gives

$$\begin{aligned} \frac{n}{2 \cdot 19 \cdot 37^i - 2} & + \frac{2n}{19 \cdot 37^i (2 \cdot 19 \cdot 37^i - 1)(2 \cdot 19 \cdot 27^i - 3)} \\ & \geq \frac{n}{q_i - 1} + \frac{2n}{z(q_i) q_i (q_i - 2)} \geq s_0(q_i) = s_i(37). \end{aligned} \tag{10}$$

Putting together (9), (10), (6) and (5), we get

$$\begin{aligned} \frac{n}{36} & - \frac{\log(n+1)}{\log 37} \leq \nu_{37}(n!) \leq s_0(37) + \sum_{i \geq 1} s_i(37) \\ & \leq n \Bigg(\frac{1}{112} + \frac{4}{19 \cdot 111 \cdot 113} \\ & \quad + \sum_{i \geq 1} \left(\frac{1}{2 \cdot 19 \cdot 37^i - 2} + \frac{2}{19 \cdot 37^i (2 \cdot 19 \cdot 37^i - 1)(2 \cdot 19 \cdot 37^i - 3)} \right) \Bigg). \end{aligned}$$

The above inequality gives $n < 2000$.

It remains to cover the range $n \in [37, 2000)$. By the Primitive Divisor Theorem, it follows that $\max\{n_i : i \in \mathcal{I}\} \leq 2000$ thanks to (2).

Since $n \geq 37$, it follows that $37 \mid n!$, and hence, $37 \mid \Phi_{n_i}$ for some $i \in \mathcal{I}$. Since $z(37) = 19$ and $19 \cdot 37^2 > 2000$, it follows that $n_i \in \{19, 19 \cdot 37\}$ for some $i \in \mathcal{I}$. Recall the congruence (2). The case $n_i = 19 \cdot 37$ together with the fact that none of $2 \cdot 19 \cdot 37 \pm 1$ is prime yields that Φ_{n_i} is divisible by a prime $q \geq 4 \cdot 19 \cdot 37 - 1 > 2000$. Such a prime q cannot divide $n!$; therefore q divides $\prod_{j \in \mathcal{J}} \Phi_{m_j}$. Lemma 1 now shows that for some $i' \in \mathcal{I}$, $n_{i'}$ is of the form $f(q) = q^k z(q) > 2000 \cdot 19 \cdot 37 > 2000$, a contradiction. Thus, $n_i = 19$.

Since $113 \mid \Phi_{19}$, Inequality (7) tells us that $113 \mid n!$, unless 113 divides δ_{m_j} for some $j \in \mathcal{J}$. This makes m_j a multiple of $113 \cdot 19 = 2147 > 2000$, and since the Primitive Divisor Theorem implies $\max\{n_i : i \in \mathcal{I}\} \geq \max\{m_j : j \in \mathcal{J}\}$, we get a contradiction. Since $n \geq 113$, it follows that $73 \mid n!$ and since $z(73) = 37$, we get that there is $i' \in \mathcal{I}$ such that $n_{i'} \in \{37 \cdot 73^u : u \geq 0\}$. Since $37 \cdot 73 > 2000$, we conclude that $n_{i'} = 37$. Since $\Phi_{37} = 73 \cdot 149 \cdot 2221$, the above argument on δ_{m_j} with the prime 2221 shows that $2221 \mid n!$. So, $n \geq 2221$, which is a contradiction. Thus, $n \leq 36$.

4. Proof of Lemma 2

Proof of Lemma 2. Assume $z(p) > 12$ and write

$$n! = \prod_{i\in\mathcal{I}} \Phi_{n_i} \prod_{j\in\mathcal{J}} \Phi_{m_j}^{-1}.$$

We need to count the number of $j \in \mathcal{J}$ with $\nu_p(\delta_{m_j}) = 1$. Let $v_p(\delta_{m_j}) = 1$ and write $m_j = p^{i_1}z(p)$. Notice that (2) implies that p is the smallest prime factor of Φ_{m_j}. Since $v_p(\Phi_{m_j}) = 1$, we may write

$$\Phi_{m_j} = p_1 \cdots p_k, \qquad p_1 < p_2 < \cdots < p_k$$

where $p = p_1 < p_2 \le \cdots \le p_k$ are primes. Let $q = p_2$ be the minimal primitive prime factor of Φ_{m_j}/p. Lemma 1 shows that there is $i \in \mathcal{I}$ such that $n_i = q^{k_1}z(q)$, and let r be the minimal primitive prime factor of Φ_{n_i}. Note that

$$r \ge q^{k_1}z(q) - 1 \quad \ge \quad (p^{i_1}z(p) - 1)^{k_1}p^{i_1}z(p) - 1 \ge (p^{i_1}z(p)/2)^{k_1}p^{i_1}z(p) - 1.$$

In particular,

$$r - 1 > p^{i_1k_1+i_1}z(p)^{k_1+1}/2^{k_1} - 2 > p^{i_1k_1+i_1}z(p), \tag{11}$$

where the last inequality is equivalent to

$$p^{i_1k_1+i_1}z(p)\left(\left(\frac{z(p)}{2}\right)^{k_1} - 1\right) > 2,$$

which holds since $k_1 \ge 1$ and $z(p) > 12$.

Recall Equation (3). The fraction

$$\prod_{l=2}^{k} \frac{\Phi_{f(p_l)}}{p_l} \tag{12}$$

has numerator divisible by a prime r which is at least as large as shown at (11). Formally, $r := r_{i_1,k_1}$ as it depends on both exponents i_1 and k_1. Assume for now that $\nu_r(\delta_{m_j}) = 0$ for all $j \in \mathcal{J}$. Then every time p appears in δ_{m_j} for some j, then $r = r_{i_1,k_1}$ appears in the numerator of the number (12), where i_1 can be read from m_j (it is $\nu_p(m_j)$) and k_1 can be read from the function $f(p_2)$. Since $\nu_r(n!) < n/(r-1)$, we obtain

$$N_p < n\sum_r \frac{1}{r-1},$$

where r is the number of such distinct r's. Notice that the inequality (11) implies $r - 1 > p^{i_1+k_1}z(p)$. From the point of view of $r = r_{i_1,k_1}$, distinct primes r have distinct pairs (i_1, k_1). Fixing the value $L := i_1 + k_1$, the values of the pair (i_1, k_1) can be fixed in at most $L - 1$ ways. Thus,

$$N_p < n \sum_{L \geq 2} \frac{L-1}{p^L z(p)} = \frac{n}{z(p)(p-1)^2},$$

which is much better than desired for the case of $v_r(\delta_{m_j}) = 0$ for all $j \in \mathcal{J}$.

Assume next that there is $j' \in \mathcal{J}$ such that $\nu_r(\delta_{m_{j'}}) = 1$. Then $m_{j'} = r^{i_2} z(r)$. Note that $P(m_{j'}) = r > P(m_j)$ so in particular $m_{j'} \neq m_j$. Write

$$\Phi_{m_{j'}} = p'_1 p'_2 \cdots p'_{k'}, \qquad p'_1 < p'_2 \leq \cdots \leq p'_{k'}$$

with $p'_1 = r$ and let $q' = p'_2$ be the minimal primitive prime factor of $\Phi_{m'_j}/p'_1$. Lemma 1 shows that there is i' such that $n_{i'} = (q')^{k_2} z(q')$. Note that $P(n_{i'}) > P(n_i)$ so in particular $n_{i'} \neq n_i$. Let r' be the minimal primitive prime factor of $\Phi_{n'_i}$. At this step, we note that in fact we used the same construction as previously (obtaining r out of p assuming $\nu_p(\delta_{m_j}) = 1$ for some $j \in \mathcal{J}$) using the two exponents (i_1, k_1) and then inequality (11) holds, except that now we started with $p' = r$ and obtained r' and the pair of exponents (i_1, k_1) has been replaced by (i_2, k_2). Thus, we have

$$r' - 1 \geq (p')^{i_2 k_2 + i_2} z(p') \geq (p')^{i_2 + k_2} z(p) \geq p^{(i_1+k_1)(i_2+k_2)} z(p) \geq p^{i_1+k_1+i_2+k_2} z(p).$$

If $\nu_{r'}(\delta_{m_j}) = 0$ for all $j \in \mathcal{J}$, we stop. If not, we continue and create (p'', r'') starting with $p'' = r'$, and so on. Note that at every stage we use new m_j's and n_i's, distinct from all the previous ones used, just because their largest prime factors form an increasing sequence. The increasing property of the minimal primitive prime factors and the finiteness of $\mathcal{J}$ guarantee that the algorithm stops, and we achieve $v_{r^*}(\delta_{m_j}) = 0$ for all $j \in \mathcal{J}$, where r^* is the minimal prime factor of some $\Phi_{n_{i^*}}$. Let $\mathcal{J}_p := \{j \in \mathcal{J} : v_p(\delta_{m_j}) = 0\}$, and let $r_j := r^*$ for $j \in \mathcal{J}_p$ described above. Then, we have

$$N_p < n \sum_r \frac{1}{r-1},$$

where the sum is over all the possible distinct primes $r \in \{r_j : j \in \mathcal{J}_p\}$. We need to lower bound the values of such r and upper bound the number of them. For this, let s be the length of the chain for r and assume that the intermediary steps have pairs of parameters $(i_1, k_1), \ldots, (i_s, k_s)$. Then put

$$i_1 + j_1 + i_2 + j_2 + \cdots + i_s + j_s = L.$$

For a fixed s, the number of such compositions is

$$\binom{L}{2s-1}.$$

Summing up over all s we get a bound of 2^{L-1}. As in the earlier case, the distinct primes r in the sum have distinct sequences of pairs $(i_1, k_1), \ldots, (i_s, k_s)$, and it follows that

$$N_p \le n \sum_{L \ge 2} \frac{2^{L-1}}{p^L z(p)} = \frac{2}{pz(p)(p-2)},$$

which is what we wanted to prove. □

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