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AN ANALOGUE TO THE INFINITARY HALES-JEWETT THEOREM

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Abstract

Recently, N. Hindman, D. Strauss, and L. Q. Zamboni proved an extension of the Hales–Jewett theorem, which is consistent with some sufficiently well-behaved homomorphism. Their results deal with a finite set of alphabets. In this article, we show that similar types of results are true for any increasing sequence of finite alphabets.

1. Introduction

Throughout the article we will use the notation ω to denote $\mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of all positive integers, and $\mathcal{P}_f(X)$ to denote the set of all nonempty finite subsets of a set X. For any nonempty set $\mathbb{A}$ (or alphabet), let $w(\mathbb{A})$ be the set of all finite words $w = a_1 a_2 \ldots a_n$ with $n \in \mathbb{N}$ and $a_i \in \mathbb{A}$. The quantity n is called the *length* of w and is denoted by $|w|$. The set $w(\mathbb{A})$ naturally becomes a semigroup under the operation of concatenation of words. We will use the symbol θ to denote the *empty word*. For each $u \in w(\mathbb{A})$ and $a \in \mathbb{A}$, let $|u|_a$ be the number of occurrences of a in u. We will identify the elements of $\mathbb{A}$ with the words over $\mathbb{A}$ whose length is one. Let v (a variable) be a letter not belonging to $\mathbb{A}$. By a *variable word* over $\mathbb{A}$ we mean a word w over $\mathbb{A} \cup \{v\}$ with $|w|_v \geq 1$. Let $S_1(\mathbb{A})$ be the set of variable words over $\mathbb{A}$. If $w \in S_1(\mathbb{A})$ and $a \in \mathbb{A}$, then $w(a)$ is the word obtained by replacing each occurrence of v by a. A *finite coloring* of a set A is a function

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from A to the finite set $\{1, 2, \ldots, n\}$. A subset B of A is said to be *monochromatic* if the function is constant on B. Let us recall the Hales–Jewett theorem.

Theorem 1.1 ([4]). *If $\mathbb{A}$ is finite, then for any finite coloring of $w(\mathbb{A})$ there exists a variable word w such that $\{w(a) : a \in \mathbb{A}\}$ is monochromatic.*

The following theorem is known as the infinitary Hales–Jewett theorem.

Theorem 1.2 ([3]). *For every finite coloring of $w(\mathbb{A})$, there exists a sequence of variable words over $\mathbb{A}$, say $(w_n(x))_{n=0}^{\infty}$, such that for every $n \in \mathbb{N}$ and every $m_1 < m_2 < \ldots < m_n$, the words of the form $w_{m_0}(a_0) w_{m_1}(a_1) \ldots w_{m_n}(a_n)$ with $a_i \in \mathbb{A}$, $i \in \{i, 2, \ldots, n\}$, are of the same color.*

Now, we need to recall some definitions from [7].

Definition 1.3. Let $n \in \mathbb{N}$ and $v_1, v_2, \ldots, v_n$ be distinct variables which are not members of $\mathbb{A}$.

(a) An *n-variable word* over $\mathbb{A}$ is a word w over $\mathbb{A} \cup \{v_1, v_2, \ldots, v_n\}$ such that $|w|_{v_i} \geq 1$ for each $i \in \{1, 2, \ldots, n\}$.

(b) If w is an n-variable word over $\mathbb{A}$ and $\vec{x} = (x_1, x_2, \ldots, x_n)$, then $w(\vec{x})$ is the result of replacing each occurrence of v_i in w by x_i for each $i \in \{1, 2, \ldots, n\}$.

(c) If w is an n-variable word over $\mathbb{A}$ and $u = a_1 a_2 \ldots a_n$ is a length n word, then $w(u)$ is the result of replacing each occurrence of v_i in w by a_i for each $i \in \{1, 2, \ldots, n\}$.

(d) The set $S_n(\mathbb{A})$ contains all n-variable words over $\mathbb{A}$ and $S_0(\mathbb{A}) = w(\mathbb{A})$.

The following particular homomorphism is very useful to us.

Definition 1.4. Let $n \in \mathbb{N}$, and let $\mathbb{A}$ be an alphabet of finite symbols and $\vec{a} \in \mathbb{A}^n$. Then the homomorphism $h_{\vec{a}} : S_n(\mathbb{A}) \to w(\mathbb{A})$ is defined by $h_{\vec{a}}(w) = w(\vec{a})$ for all $w \in S_n(\mathbb{A})$.

Definition 1.5. If S, T, and R are semigroups (or partial semigroups) such that $S \cup T$ is a semigroup (or partial semigroup), and T is an ideal of $S \cup T$, then a homomorphism $\tau : T \to R$ is said to be *S-independent* if, for every $w \in T$ and every $u \in S$,

$$\tau(uw) = \tau(w) = \tau(wu).$$

Example 1.6. If T is a semigroup with identity e, then for any $n \geq 1$, a homomorphism $\tau : S_n(\mathbb{A}) \cup S_0(\mathbb{A}) \to T$ is $S_0(\mathbb{A})$-independent if $\tau[S_0(\mathbb{A})] = \{e\}$.

The following theorem is the multi-variable extension of the Hales–Jewett theorem.

Theorem 1.7. *If $n \in \mathbb{N}$ and $\mathbb{A}$ is finite, then for any finite coloring of $w(\mathbb{A})$ there exists $w \in S_n(\mathbb{A})$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\}$ is monochromatic.*

Let $\mathbb{A}_1 \subseteq \mathbb{A}_2 \subseteq \mathbb{A}_3 \subseteq \ldots$ be an increasing sequence of alphabets. Then for each $n, i \in \mathbb{N}$, let us denote by $S_n(\mathbb{A}_i)$, the set of all n-variable words over $\mathbb{A}_i$. Let $S_n = \bigcup_{i=1}^{\infty} S_n(\mathbb{A}_i)$ be the set of all n-variable words over $\mathbb{A} = \bigcup_{i=1}^{\infty} \mathbb{A}_i$ and $S_0 = \bigcup_{i=1}^{\infty} w(\mathbb{A}_i)$ be the set of all words of finite length over $\mathbb{A} = \bigcup_{i=1}^{\infty} \mathbb{A}_i$. The following theorem is due to N. Karagiannis [8], which is a stronger version of the infinitary Hales–Jewett theorem.

Theorem 1.8 ([8])**.** *Let $(\mathbb{A}_i)_{i=1}^{\infty}$ be an increasing sequence of finite alphabets, and $\mathbb{A} = \bigcup_{i=1}^{\infty} \mathbb{A}_i$. Then for any finite coloring of S_0 there exists a sequence $(w_n(x))_{n=0}^{\infty}$ of variable words over $\mathbb{A}$ such that for every $n \in \mathbb{N}$ and every $m_1 < m_2 < \ldots < m_n$, the words of the form $w_{m_0}(a_0)\, w_{m_1}(a_1) \ldots w_{m_n}(a_n)$ with $a_i \in \mathbb{A}_{m_i}$, $i \in \{1, 2, \ldots, n\}$, are of the same color.*

In [8], N. Karagiannis proved a version of the infinitary Hales–Jewett theorem replacing the alphabet $\mathbb{A}$ by an increasing chain of finite alphabets $\mathbb{A}_1 \subseteq \mathbb{A}_2 \subseteq \ldots \subseteq \mathbb{A}_n \subseteq \ldots$. In this article, we will use the algebra of the Stone-Čech compactification of discrete semigroups and partial semigroups to provide the combined extension of Karagiannis's version of the infinitary Hales–Jewett theorem with some 'suitable' homomorphisms.

The remainder of the paper is organized as follows. In Section 2, first, we will recall the algebra of the Stone-Čech compactification of $\mathbb{N}$, and then some relative notions of largeness in the Ramsey theory. In Section 3, we will improve all the results of [7], by proving that we can replace a single alphabet set by an infinitely increasing sequence of alphabets. We will also show that we can combine the Hales–Jewett theorem with some "well-behaved homomorphisms"[2] defined from the free semigroup S_0 to some other semigroups.

2. Preliminaries

Now we pause our attention to introduce the algebra of the Stone-Čech compactification briefly.

2.1. Algebra of the Stone-Čech Compactification of Discrete Semigroups

Let $(S, \cdot)$ be a discrete semigroup and denote its Stone-Čech compactification by βS. It can be shown that βS is the set of all ultrafilters on S, where the points of S are identified with the principal ultrafilters. The basis for the topology over βS is $\{\bar{A} : A \subseteq S\}$, where $\bar{A} = \{p \in \beta S : A \in p\}$. The operation of S can be extended

[2] we will see later in Section 3

to βS, which makes $(\beta S, \cdot)$ a compact right topological semigroup with S contained in its topological center. More precisely, for $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous, and for $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous. For $p, q \in \beta S$ and for $A \subseteq S$, $A \in p \cdot q$ if and only if $\left\{x \in S : x^{-1}A \in q\right\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$. Since $(\beta S, \cdot)$ is a compact Hausdorff right topological semigroup, it has the smallest two-sided ideal $K(\beta S)$, which is the union of all minimal right ideals of S, as well as the union of all minimal left ideals of S. Every left ideal of βS contains a minimal left ideal and every right ideal of βS contains a minimal right ideal. The intersection of any minimal left ideal and any minimal right ideal is a group, and any two such groups are isomorphic. Any idempotent p in βS is said to be minimal if and only if $p \in K(\beta S)$. A subset A of S is central if and only if there is a minimal idempotent p such that $A \in p$. For more details, the reader can see [6].

We need to use some elementary structure of partial semigroups. Here we recall some definitions from [5].

2.2. Partial Semigroup

Definition 2.1. A *partial semigroup* is defined as a pair $(G, *)$, where $*$ is an operation defined on a subset X of $G \times G$ such that for all x, y, z in G,

$$(x * y) * z = x * (y * z)$$

in the sense that if either side is defined, so is the other, and they are equal.

If $(G, *)$ is a partial semigroup, we will denote it by G when the operation $*$ is clear from the context. The following example of partial semigroup will be useful in our context.

Example 2.2. For any sequence $\langle x_n \rangle_{n=1}^{\infty}$ in ω, let

$$G = \mathrm{FS}\left(\langle x_n \rangle_{n=1}^{\infty}\right) = \left\{ \sum_{j \in H} x_j : H \in \mathcal{P}_f(\mathbb{N}) \right\}$$

and

$$X = \left\{ \left(\sum_{j \in H_1} x_j, \sum_{j \in H_2} x_j \right) : H_1 \cap H_2 = \emptyset \right\}.$$

Define $* : X \to G$ by

$$\left(\sum_{j \in H_1} x_j, \sum_{j \in H_2} x_j \right) \longrightarrow \sum_{j \in H_1} x_j + \sum_{j \in H_2} x_j.$$

It is easy to check that G is a commutative partial semigroup. One can similarly check the same for

$$G = \mathrm{FP}\left(\langle x_n \rangle_{n=1}^{\infty}\right) = \left\{ \prod_{j \in H} x_j : H \in \mathcal{P}_f\left(\mathbb{N}\right) \right\}.$$

Definition 2.3. Let $(G, *)$ be a partial semigroup. For any $g \in G$, define $\varphi\left(g\right) = \{h \in G : g * h \text{ is defined}\}$. For any $H \in \mathcal{P}_f\left(G\right)$, define $\sigma\left(H\right) = \bigcap_{h \in H} \varphi\left(h\right)$. We say that $(G, *)$ is *adequate* if and only if $\sigma\left(H\right) \neq \emptyset$ for all $H \in \mathcal{P}_f\left(G\right)$.

For a semigroup $(S, \cdot)$, denote by ${}^{\mathbb{N}}S$, the set of all sequences in S, and let

$$\mathcal{J}_m = \left\{ t = (t_1, t_2, \ldots, t_m) \in \mathbb{N}^m : t_1 < t_2 < \ldots < t_m \right\}.$$

Definition 2.4. Let $(S, \cdot)$ be a semigroup.

(1) Denote by ${}^{\mathbb{N}}S$, the set of all sequences in S.

(2) Denote
$$\mathcal{J}_m = \left\{ t = (t_1, t_2, \ldots, t_m) \in \mathbb{N}^m : t_1 < t_2 < \ldots < t_m \right\}.$$

(3) A set $A \subseteq S$ is said to be a *J-set* if and only if for each $F \in \mathcal{P}_f\left({}^{\mathbb{N}}S\right)$, there exist $m \in \mathbb{N}$, an element $a = (a_1, a_2, \ldots, a_{m+1}) \in S^{m+1}$ and $t = (t_1, t_2, \ldots, t_m) \in \mathcal{J}_m$ such that for each $f \in F$,
$$\left(\prod_{j=1}^{m} a_j \cdot f\left(t_j\right) \right) \cdot a_{m+1} \in A.$$
We define $J(S)$ to be the set $J\left(S\right) = \{p \in \beta S : \text{ for all } A \in p,\ A \text{ is a } J\text{-set}\}$. A set $A \subseteq S$ is called a *C-set* if it is an element of some idempotent in $J\left(S\right)$.

Now, we need to recall the definition of J-sets for partial semigroups. Let $\mathcal{F}$ be the set of all adequate sequences in S.

Definition 2.5. For any adequate partial semigroup G, a subset $A \subseteq G$ is said to be a *J-set* if and only if for all $F \in \mathcal{P}_f\left(\mathcal{F}\right)$ and $L \in \mathcal{P}_f\left(S\right)$, there exist $m \in \mathbb{N}$, an element $a = (a_1, a_2, \ldots, a_{m+1}) \in S^{m+1}$ and $t = (t_1, t_2, \ldots, t_m) \in \mathcal{J}_m$ such that for every $f \in F$,
$$\left(\prod_{i=1}^{m} a_i * f\left(t_i\right) \right) * a_{m+1} \in A \cap \sigma\left(L\right).$$

3. Main Results

The main purpose of this article is to generalize the results obtained in [7] by replacing a finite alphabet by an infinite sequence of finite alphabets. Let $(\mathbb{A}_n)_{n=0}^{\infty}$ be an increasing sequence of finite alphabets, and $\mathbb{A} = \bigcup_{n=0}^{\infty} \mathbb{A}_n$.

We will use the following fact later in our proofs.

Fact 3.1. Let $\tau : S_0 \cup S_1 \to \omega$ be a homomorphism defined by $\tau(w) = |w|_v$. Hence $\tau(w) = 0$ if and only if $w \in S_0$. Let $D \subseteq S_0$ be a J-set, and $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$. Choose $y_1 = 0$, and for every $n \in \mathbb{N}$, $y_{n+1} = x_n$. Then $A = \mathrm{FS}(\langle y_n \rangle_{n=1}^{\infty})$ is an adequate partial semigroup. One can easily check that $\tau^{-1}[A]$ is also an adequate partial semigroup. Let T be an adequate partial semigroup, and S be any partial subsemigroup of T. Let F be a nonempty finite set of partial semigroup homomorphisms from T to S, which are S-independent. The proof of the fact that for any J-set $D \subseteq S$, the set $\bigcap_{\nu \in F} \nu^{-1}[D]$ is a J-set in T is similar to the proof of [2, Lemma 2.1]. So we leave it to the reader. Hence $\tau^{-1}[A]$ is a J-set in $T = S_0 \cup S_1$. Also note that S_0 is not a J-set in $S_n \cup S_0$.

Let us recall a special homomorphism from [7].

Definition 3.2. Let $n \in \mathbb{N}$ and $\vec{a} \in \mathbb{A}^n$. We define $h_{\vec{a}} : S_0 \bigcup S_n \to S_0$ by

$$h_{\vec{a}}(w) = \begin{cases} w(\vec{a}) & \text{if } w \in S_n \\ w & \text{if } w \in S_0 \end{cases}.$$

The following result is a version of the Hales–Jewett theorem.

Lemma 3.3. *Let $m, n \in \mathbb{N}$, and $F = \{h_{\vec{a}} : \vec{a} \in \mathbb{A}_m^n\}$ be a finite nonempty set of homomorphisms from $T = S_n \cup S_0$ to S_0 which are the identity maps on S_0, i.e. $h_{\vec{a}}(s) = s$ for all $s \in S_0$, and $h_{\vec{a}} \in F$. If $D \subseteq S_0$ is a piecewise syndetic set in S_0, then D contains the set $\{w_n(\vec{a}) : \vec{a} \in \mathbb{A}_m^n\}$ for some $w_n \in S_n(\mathbb{A})$.*

Proof. If $n \in \mathbb{N}$, then from [7, Corollary 6], $\bigcap_{h_{\vec{a}} \in F} h_{\vec{a}}^{-1}[D]$ is a piecewise syndetic set in T. Since S_0 is not a piecewise syndetic set in T, the set $S_n \cap \bigcap_{h_{\vec{a}} \in F} h_{\vec{a}}^{-1}[D]$ is a piecewise syndetic set in T. Now, choose $N > m$ such that $w_n \in S_n(\mathbb{A}_N) \cap \bigcap_{h_{\vec{a}} \in F} h_{\vec{a}}^{-1}[D]$. Hence $w_n(\vec{a}) \in D$ for all $\vec{a} \in \mathbb{A}_m^n$. □

The proof of the following lemma is similar to the proof of the above lemma.

Lemma 3.4. *Let $m, n \in \mathbb{N}$, and $F = \{h_{\vec{a}} : \vec{a} \in \mathbb{A}_m^n\}$ be a finite nonempty set of homomorphisms from $T = S_n \cup S_0$ to S_0 which are the identity maps on S_0. If $D \subseteq S_0$ is a J-set in S_0, then D contains the set $\{w_n(\vec{a}) : \vec{a} \in \mathbb{A}_m^n\}$ for some $w_n \in S_n(\mathbb{A})$.*

Proof. This result follows immediately from [2, Theorem 2.3]. □

The following theorem shows that we can find a similar configuration to [8, Theorem 3] in the central sets.

Theorem 3.5. *Let $l \in \mathbb{N}$, and $D \subseteq S_0$ be a central set. Then there exists an infinite sequence $(w_i(x))_{i=1}^{\infty}$ of variable words over $\mathbb{A}$ such that for every $l \in \mathbb{N}$, and*

$$m_1 < m_2 < \ldots < m_l,$$

we have

$$w_{m_1}(\vec{a_1})\, w_{m_2}(\vec{a_2}) \ldots w_{m_l}(\vec{a_l}) \in D,$$

where $\vec{a_1} \in \mathbb{A}^n_{m_1}, \vec{a_2} \in \mathbb{A}^n_{m_2}, \ldots, \vec{a_l} \in \mathbb{A}^n_{m_l}$.

Proof. Choose a minimal idempotent $p \in \beta S_0$ such that $D \in p$. Then $D^* = \left\{x \in D : x^{-1}D \in p\right\} \in p$. Let $F_1 = \{h_{\vec{a_1}} : \vec{a_1} \in \mathbb{A}^n_1\}$. So from Lemma 3.3, there exists $w_1 \in S_n(\mathbb{A}_1)$ such that $w_1 \in S_n \cap \bigcap_{\vec{a_1} \in \mathbb{A}^n_1} h^{-1}_{\vec{a_1}}[D^*]$, and this implies $w_1(\vec{a_1}) \in D^*$ for all $\vec{a_1} \in \mathbb{A}^n_1$. Hence $D^* \cap \bigcap_{\vec{a_1} \in \mathbb{A}^n_1} w_1(\vec{a_1})^{-1}[D^*] \in p$. Suppose for $k \in \mathbb{N}$, we have a sequence $(w_i(x))_{i=1}^{k}$ such that for all $m_1 < m_2 < \ldots < m_l \leq k$, we have $w_{m_1}(\vec{a_1}) \ldots w_{m_l}(\vec{a_l}) \in D^*$. Hence,

$$E = D^* \cap \bigcap_{m_1 < m_2 < \ldots < m_l \leq k} w_{m_1}(\vec{a_1}) \ldots w_{m_l}(\vec{a_l})^{-1}[D^*] \neq \emptyset.$$

Choose $w_{k+1} \in S_n \cap \bigcap_{\overrightarrow{a_{k+1}} \in \mathbb{A}^n_{k+1}} h^{-1}_{\overrightarrow{a_{k+1}}}[E]$. So, for $m_1 < m_2 < \ldots < m_s \leq k+1$, we have $w_{m_1}(\vec{a_1}) \ldots w_{m_s}(\vec{a_s}) \in D^*$. Now by induction, we can choose such an infinite sequence $(w_i(x))_{i=1}^{\infty}$. □

Using the fact that $C^* \in p$, whenever $C \in p$, one can prove the following generalization of the above theorem.

Corollary 3.6. *Let $l \in \mathbb{N}$, and $D \subseteq S_0$ be a C-set. Then there exists a sequence $(w_n(x))_{n=1}^{\infty}$ of variable words over $\mathbb{A}$ such that for every $l \in \mathbb{N}$, and every*

$$m_1 < m_2 < \ldots < m_l,$$

we have

$$w_{m_1}(\vec{a_1})\, w_{m_2}(\vec{a_2}) \cdots w_{m_l}(\vec{a_l}) \in D,$$

where $\vec{a_1} \in \mathbb{A}^n_{m_1}, \vec{a_2} \in \mathbb{A}^n_{m_2}, \ldots, \vec{a_l} \in \mathbb{A}^n_{m_l}$.

Proof. The proof is similar to the proof of the Theorem 3.5. □

For any sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$, define $y_1 = 0$ and $y_{n+1} = x_n$. Note that $A = \mathrm{FS}(\langle y_n \rangle_{n=1}^{\infty})$ is an adequate partial semigroup.

Theorem 3.7. *Let $\tau : T = S_0 \cup S_1 \to \omega$ be a homomorphism defined by $\tau(w) = |w|_{v_1}$. If $D \subseteq S_0$ is a J-set, and $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in $\mathbb{N}$, then there exists a sequence $\langle w_n \rangle_{n=1}^{\infty}$ such that for all $n \in \mathbb{N}$, and for each $G \in \mathcal{P}_f(\mathbb{N})$*

1. $\{w_n(a_n) : a_n \in \mathbb{A}_n\} \subseteq D$, *and*
2. $\sum_{n \in G} \tau(w_n) \in FS(\langle x_n \rangle_{n=1}^{\infty})$.

Proof. For the given sequence $\langle x_n \rangle_{n \in \mathbb{N}}$, let us construct a sequence $\langle y_n \rangle_{n \in \mathbb{N}}$ as mentioned above. Then $A_1 = \mathrm{FS}(\langle y_n \rangle_{n=1}^{\infty})$ is an adequate partial semigroup. Let $F_1 = \{h_{a_1} : a_1 \in \mathbb{A}_1\}$ be a finite set of partial semigroup homomorphisms from $\tau^{-1}(A_1)$ to S_0. Let $D \subseteq S_0$ be a J-set. One can easily verify that $\bigcap_{a_1 \in \mathbb{A}_1} h_{a_1}^{-1}[D]$ is a J-set in $\tau^{-1}(A_1)$. As S_0 is not a J-set in $\tau^{-1}(A_1)$, there exists $w_1 \in S_1(\mathbb{A}_m) \cap \tau^{-1}(A_1)$ for some $m \geq 1$ such that $w_1 \in \bigcap_{a_1 \in \mathbb{A}_1} h_{a_1}^{-1}[D]$. Hence

1. for every $a_1 \in \mathbb{A}_1$, we have $w_1(a_1) \in D$; and
2. $\tau(w_1) \in \mathrm{FS}(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A_1 = \mathrm{FS}(\langle y_n \rangle_{n=1}^{\infty})$.

Now by induction, assume that for some $k \in \mathbb{N}$,

1. $\{w_n(a_n) : a_n \in \mathbb{A}_n\} \subseteq D$ for all $n \in \{1, 2, \ldots, k\}$; and
2. $\sum_{n \in G} \tau(w_n) \in \mathrm{FS}(\langle x_n \rangle_{n=1}^{\infty})$ for $G \subseteq \{1, 2, \ldots, k\}$.

Now choose a sequence $\langle z_n \rangle_{n=2}^{\infty}$ such that $z_1 = 0$ and

$$A_{k+1} = \mathrm{FS}(\langle z_n \rangle_{n=1}^{\infty}) \subseteq \cap_{x \in \left\{\sum_{n \in G} \tau(w_n) : G \subseteq \{1,2,\ldots,k\}\right\}} \left(-x + \mathrm{FS}(\langle x_n \rangle_{n=1}^{\infty})\right).$$

Then $\tau^{-1}[A_{k+1}]$ is an adequate partial semigroup. Now, using a similar argument, used in the first step of this proof, we have $m \geq k+1$, and an element $w_{k+1} \in S_1(\mathbb{A}_m) \cap \tau^{-1}[A_{k+1}]$ such that $w_{k+1} \in \bigcap_{a_{k+1} \in \mathbb{A}_{k+1}} h_{a_{k+1}}^{-1}[D]$. Hence $\tau(w_{k+1}) \in A_{k+1}$ and $\{w_{k+1}(a_{k+1}) : a_{k+1} \in \mathbb{A}_{k+1}\}$. This proves the result. □

The following corollary is an extension of Theorem 3.5 combined with the homomorphism τ defined above.

Corollary 3.8. *Let $\tau : T = S_0 \cup S_1 \to \omega$ be a homomorphism defined by $\tau(w) = |w|_{v_1}$. If $D \subseteq S_0$ is a C-set, and $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in $\mathbb{N}$, then there exists $\langle w_n \rangle_{n=1}^{\infty}$ such that for any $l \in \mathbb{N}$, $G = \{n_1, n_2, \ldots, n_l\} \in \mathcal{P}_f(\mathbb{N})$,*

1. $\{w_{n_1}(a_{n_1}) w_{n_2}(a_{n_2}) \ldots w_{n_l}(a_{n_l}) : a_{n_i} \in \mathbb{A}_{n_i}, n_i \in G\} \subseteq D$, *and*
2. $\sum_{n_i \in G} \tau(w_{n_i}) \in FS(\langle x_n \rangle_{n=1}^{\infty})$.

Proof. As $D \subseteq S_0$ is a C-set, choose an idempotent $p \in \beta S_0$ such that $D \in p$. Then, $D^* \in p$. Now, proceeding as the proof of Theorem 3.5, assume

$$E = D^* \cap \cap_{G=\{n_1,n_2,\ldots,n_l\}\subseteq\{1,2,\ldots,k\}} w_{n_1}(a_{n_1})\, w_{n_2}(a_{n_2}) \ldots w_{n_l}(a_{n_l})^{-1}[D^*] \in p,$$

and A_{k+1} as in Theorem 3.5. As $\bigcap_{a_{k+1}\in\mathbb{A}_{k+1}} h_{a_{k+1}}^{-1}[E]$ is a J-set, we have our desired result. □

In [2, Theorem 2.5], the authors were able to prove that the conclusion of [7, Corollary 16] holds for J-sets. In the following theorem, we will show that, instead of a finite alphabet set, we can choose an increasing sequence of finite alphabets in the assumption of [7, Corollary 16].

Theorem 3.9. *Let $k, n \in \mathbb{N}$ with $k < n$, and let T be the set of all words over $\{v_1, v_2, \ldots, v_k\}$ in which v_i occurs for each $i \in \{1, 2, \ldots, k\}$. Given $w \in S_n$, let $\tau(w)$ be obtained from w by deleting all occurrence of elements of $\mathbb{A}$ as well as occurrences of v_i, $k < i \leq n$. Let $\langle y_t\rangle_{t=1}^{\infty}$ be a sequence in T, and let $D \subseteq S_0$ be a J-set. Then there exists a sequence of variable words $\langle w_t\rangle_{t=1}^{\infty}$ over $\mathbb{A}$ such that*

1. *for any $m \in \mathbb{N}$, $\{w_m(a_m) : a_m \in \mathbb{A}_m^n\} \subseteq D$, and*
2. *for every $G = \{m_1 < m_2 < \ldots < m_l\} \in \mathcal{P}_f(\mathbb{N})$,*

$$\prod_{i=1}^{l} \tau(w_{m_i}) \in FP(\langle y_t\rangle_{t=1}^{\infty}).$$

Proof. The proof is similar to the proof of Theorem 3.7. Let $T^* = T \cup \{\theta\}$, where $\theta \notin T$ and $\tau^* : S_n \cup S_0 \to T^*$ be defined by

$$\tau^*(w) = \begin{cases} \tau(w), \text{if } w \in S_n \\ \theta, \text{ if } w \in S_0 \end{cases}.$$

Then $(\tau^*)^{-1}[\mathrm{FP}(\langle y_t\rangle_{t=1}^{\infty}) \cup \theta]$ is a partial subsemigroup of $S_n \cup S_0$. Let $A_1 = (\tau^*)^{-1}[\mathrm{FP}(\langle y_t\rangle_{t=1}^{\infty}) \cup \theta]$, and $F_1 = \{h_{\vec{a_1}} : \vec{a_1} \in \mathbb{A}_1^n\}$. Now $\bigcap_{\vec{a_1}\in\mathbb{A}_1^n} h_{\vec{a_1}}^{-1}[D]$ is a J-set in $S_n \cup S_0$, and so in S_n. Hence there exists $w_1 \in S_n \cap \bigcap_{\vec{a_1}\in\mathbb{A}_1^n} h_{\vec{a_1}}^{-1}[D]$ such that $\tau(w_1) \in \mathrm{FP}(\langle y_t\rangle_{t=1}^{\infty})$. In other words for every $\vec{a_1} \in \mathbb{A}_1^n$ we have $w_1(\vec{a_1}) \in D$ and $\tau(w_1) \in \mathrm{FP}(\langle y_t\rangle_{t=1}^{\infty})$. Now proceeding along the lines of the proof of Theorem 3.8, we have our desired result. □

The following corollary shows that in the above theorem if we replace J sets with C sets, then we have much stronger configurations.

Corollary 3.10. *Let $k, n \in \mathbb{N}$ with $k < n$ and let T be the set of words over $\{v_1, v_2, \ldots, v_k\}$ in which v_i occurs for each $i \in \{1, 2, \ldots, k\}$. Given $w \in S_n$, let*

$\tau(w)$ be obtained from w by deleting all occurrence of elements of A as well as occurrences of v_i, $k < i \leq n$. Let $\langle y_t \rangle_{t=1}^{\infty}$ be a sequence in T, and $C \subseteq S_0$ be a C-set. Then there exists a sequence of variable words $\langle w_t \rangle_{t=1}^{\infty}$ over $\mathbb{A}$ such that for all $G = \{m_1 < m_2 < \ldots < m_l\} \in \mathcal{P}_f(\mathbb{N})$,

1. *$\left\{ w_{m_1}(a_{m_1}) w_{m_2}(a_{m_2}) \ldots w_{m_l}(a_{m_l}) : a_{m_i} \in \mathbb{A}_{m_i}^n, m_i \in G \right\} \subseteq C$, and*
2. *$\prod_{i=1}^{l} \tau(w_{m_i}) \in FP(\langle y_t \rangle_{t=1}^{\infty})$.*

Proof. The proof is similar to the proof of Theorem 3.7. So we leave it to the reader. □

One of the most important roles played by IP sets is in Ramsey theory. So let us recall the definition of it.

Definition 3.11. For any commutative semigroup S, $A \subseteq S$ is said to be an IP *set* if there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ such that

$$A = FS(\langle x_n \rangle_{n=1}^{\infty}) = \left\{ \sum_{t \in H} x_t : H \in \mathcal{P}_f(\mathbb{N}) \right\}.$$

It can be shown that A contains an IP set if and only if it is a member of some idempotent element $p \in \beta S$.

In the following theorem, let us assume that the semigroup T and the matrix M satisfy all the appropriate hypotheses for matrix multiplication to make sense and be distributive over addition, as it was assumed in [7, Theorem 17].

Theorem 3.12. *Let $(T, +)$ be a commutative semigroup with identity element 0. Let $k, m, n \in \mathbb{N}$, and M be a $k \times m$ matrix. For $i \in \{1, 2, \ldots, m\}$, let τ_i be an S_0-independent homomorphism from S_n to T. Define a function ψ on S_n by*

$$\psi(w) = \begin{pmatrix} \tau_1(w) \\ \tau_2(w) \\ \vdots \\ \tau_m(w) \end{pmatrix},$$

with the property that for any collection of IP-sets $\{C_i : i \in \{1, 2, \ldots, k\}\}$ in T, there exists $a \in S_n$ such that $M\psi(a) \in \times_{i=1}^{k} C_i$. Let $B_i = FS\left(\left\langle x_n^{(i)} \right\rangle_{n=1}^{\infty}\right)$ for $1 \leq i \leq k$ be k-IP sets in T, and $D \subseteq S_0$ be a C-set in S_0. Then there exists $\langle w_i \rangle_{i=1}^{\infty} \subseteq S_n$ such that for each $G = \{m_1, m_2, \ldots, m_l\} \in \mathcal{P}_f(\mathbb{N})$, we have

1. *$w_{m_1}(\overrightarrow{a_{m_1}}) w_{m_2}(\overrightarrow{a_{m_2}}) \cdots w_{m_l}(\overrightarrow{a_{m_l}}) \in D$, and*
2. *$M \circ \psi(w_{m_1} w_{m_2} \ldots w_{m_l}) \in \times_{i=1}^{k} B_i$.*

Proof. Let $\phi : S_n \cup S_0 \to B = \times_{i=1}^{k} (B_i \cup \{0\})$, where $\phi(w) = M\psi(w)$. Hence from [7, Lemma 17], ϕ is a homomorphism. Let us choose an idempotent $p \in \beta S_0$ such that $D \in p$, and so $D^* \in p$. Now for each $\vec{a_1} \in \mathbb{A}_1^n$, $h_{\vec{a_1}} : \phi^{-1}[B] \to S_0$ is a partial semigroup homomorphism, which is fixed on S_0. As S_0 is not a J set, $\bigcap_{\vec{a_1} \in \mathbb{A}_1^n} h_{\vec{a_1}}^{-1}[D^*] \setminus S_0$ is a J-set in $\phi^{-1}[B]$. Hence there exists $w_1 \in \phi^{-1}[B] \setminus S_0$ such that $w_1(\vec{a_1}) \in D^*$, for all $\vec{a_1} \in \mathbb{A}_1^n$, and $\phi(w_1) = M\psi(w_1) \in B$. Choose an IP set $B_1 \subseteq -M\psi(w_1) + B$, and clearly $(0, 0, \ldots, 0) \in B_1$. So, $S_0 \subseteq \phi^{-1}[B_1]$. Hence $\phi^{-1}(B_1)$ is a partial semigroup. Now for $\vec{a_2} \in \mathbb{A}_2^n$ the map $h_{\vec{a_2}}$ defined by

$$h_{\vec{a_2}} : \phi^{-1}[B_1] \to E = D^* \cap \bigcap_{\vec{a_1} \in \mathbb{A}_1^n} w_1(\vec{a_1})^{-1} D^*,$$

is again a partial semigroup homomorphism fixed on S_0. So, there exists $w_2 \in \bigcap_{\vec{a_2} \in \mathbb{A}_2^n} h_{\vec{a_2}}^{-1}[E] \cap \left(\phi^{-1}[B_1] \setminus S_0\right)$. Thus for every $\vec{a_2} \in \mathbb{A}_2^n$,

$$h_{\vec{a_2}}(w_2) \in E = D^* \cap \bigcap_{\vec{a_1} \in \mathbb{A}_1^n} w_1(\vec{a_1})^{-1} D^*.$$

Thus, $w_2(\vec{a_2}) \in D^*$, $w_1(\vec{a_1}) = w_2(\vec{a_2}) \in D^*$ and $\phi(w_2) = M\psi(w_2) \in B_1$. Hence we have

1. $M\psi(w_1) \in B$;
2. for every $\vec{a_1} \in \mathbb{A}_1^n$, we have $w_1(\vec{a_1}) \in D^*$;
3. $M\psi(w_2) \in -M\psi(w_1) + B$;
4. for every $\vec{a_2} \in \mathbb{A}_1^n$, we have $w_2(\vec{a_2}) \in D^*$; and
5. for every $\vec{a_1}, \vec{a_2} \in \mathbb{A}_1^n$, we have $w_1(\vec{a_1}) w_2(\vec{a_2}) \in D^*$.

Moreover, $M \circ \psi(w_1 w_2) = M(\psi(w_1) + \psi(w_2)) \in B$.

Now proceeding iteratively, we have our desired result. □

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