WEIGHTED SUMMATION OF DIRICHLET CONVOLUTIONS AND APPLICATIONS TO SOME DIVISOR PROBLEMS OVER k–FREE AND B–FREE INTEGERS

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Abstract

We express functions of the type $\sum_{j\le x} w(j)(u*v)(j)$, for generic weight functions w, in terms of the summatory functions $\sum_{j\le x}(u*v)(j)$ of the Dirichlet convolution of u and v, when u and v are both completely multiplicative. This information is used to analyze some divisor problems over k-free and B-free integers. In particular, we prove a conjecture about an asymptotic formula for the sum of the divisors of all k-free integers less than or equal to a given threshold, and we extend this analysis to certain classes of B-free integers. Some analogues of these results for B-free integers in reduced residue classes are also obtained.

1. Introduction

A recurring problem in number theory is understanding how a weighted average $\sum_{j\le x} w(j)a(j)$ of a given arithmetic function a relates to its original summatory function $\sum_{j\le x} a(j)$. For instance, [15, 16, 19] deal with the case when $a = \mu$ is the Möbius function, while [9, 10, 13] handle certain sums over arithmetic progressions. In the same vein, we recently engaged in the study of the functions

$$\tilde{D}_k(x) = \sum_{n\le x} q_k(n)\sum_{d|n} 1, \; k\ge 2, \; x\ge 1, \tag{1}$$

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where

$$q_k(n) \;=\; \begin{cases} 1, & \text{if } n \text{ is } k\text{-free}, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

is the characteristic function of k-free integers (a positive integer n is *k-free* if n is not divisible by the k-th power of any prime number). Just for a brief history, Jakimczuk and Lalín [8] proved that, for every $k \geq 2$, there exists $\beta_k \in \mathbb{R}$ such that, for every $\epsilon > 0$,

$$\tilde{D}_k(x) \;=\; \prod_{p \text{ prime}} \left[1 - \frac{k+1}{p^k} \;+\; \frac{k}{p^{k+1}}\right] x\log(x) \;+\; \beta_k x \;+\; O_\epsilon\left(x^{3/4+\epsilon}\right). \tag{3}$$

At about the same time, we independently formulated the following unpublished conjecture (which we proved for $k = 2$ in [4]).

Conjecture 1. For each $k \geq 2$, there is a constant β_k such that, for every $\epsilon > 0$,

$$\tilde{D}_k(x) \;=\; \frac{1}{\zeta^2(k)} \left(\prod_{p \text{ prime}} \phi_k(p) \right) x\log(x) \;+\; \beta_k x \;+\; O_{k,\epsilon}(x^{1/2+\epsilon}), \tag{4}$$

with

$$\phi_k(n) \;=\; \left[1 \;+\; \frac{1}{n^2}\left(\sum_{j=0}^{k-3} \frac{j+1}{n^j}\right) \frac{\left[1 - \left(\frac{n^{k-1}-1}{n^k-1}\right)\right]}{\left[1 + \left(\frac{n^{k-1}-1}{n^k-1}\right)\right]}\right] \left[1 - \left(\frac{n^{k-1}-1}{n^k-1}\right)^2\right],\; n \geq 1 \tag{5}$$

(ζ is the Riemann zeta function).

Remark 1. For the sake of clarity, we remark that the expressions for the coefficients of the leading terms of $\tilde{D}_k$ in (3) and in (4) are identical. Using the Euler product for ζ, this assertion reduces to an identity between two polynomials in p. In fact, note that, for fixed $k \geq 3$, the sum $\sum_{j=0}^{k-3} \frac{j+1}{n^j}$ that appears in $\phi_k(n)$ is the derivative of

$$\sum_{j=0}^{k-3} x^{j+1} \;=\; \frac{x^{k-2}-1}{x-1}$$

at $x \;=\; \frac{1}{n}$.

Motivated by Conjecture 1, in this note we address the problem of expressing weighted summatory functions of the form

$$D[w,u,v](x) \;=\; \sum_{j \leq x} w(j)(u * v)(j),\; x \geq 1, \tag{6}$$

in terms of

$$D[u,v](x) \ = \ \sum_{j\le x}(u*v)(j), \tag{7}$$

where u, v and w are arithmetic functions and $u*v$ is the Dirichlet convolution of u and v. In several cases of interest, it is sufficient to find $f:\mathbb{N}\to\mathbb{C}$ such that

$$D[w,u,v](x) \ = \ \sum_{j\le x} f(j)D[u,v](x/j).$$

However, finding useful expressions for f may be tricky depending on the structure of w. As an additional tool, we present an alternative strategy that works for general weight functions w, provided that u and v are both completely multiplicative and information about the convolution of w with the Möbius function μ is available.

Theorem 1. Let u and v be completely multiplicative arithmetical functions. For any arithmetical function w, and $x\ge 1$,

$$D[w,u,v](x) \ = \ \sum_{r\le x} g(r)\sum_{d|r} u(d)v(r/d)\sum_{m|d}\mu(m)v(m)D[u,v]\left(\frac{x}{mr}\right), \tag{8}$$

where $g = w*\mu$ is the Dirichlet convolution of w and μ.

Applying Theorem 1 for $w=q_k$ and $u=v\equiv 1$ and using available estimates [3]

$$\Delta(x) = O(x^{\eta}), \tag{9}$$

for the error term $\Delta(x)$ of the classical Dirichlet function:

$$D(x) \ := \ \sum_{n\le x}\sum_{d|n} 1 \ = \ x\log(x) + (2\gamma-1)x + \Delta(x), \tag{10}$$

we prove an improved Jakimczuk–Lalín version of Conjecture 1.

Corollary 1. For $k\ge 2$, there exists a constant β_k such that, for every $\epsilon>0$,

$$\tilde{D}_k(x) \ = \ \left(\prod_{p \text{ prime}}\left[1-\tfrac{k+1}{p^k} \ + \ \tfrac{k}{p^{k+1}}\right]\right)x\log(x) \ + \ \beta_k\, x \ + O\left(x^{\eta_k+\epsilon}\right),$$

$$\text{with } \eta_k \ = \ \begin{cases} 1/k, & k=2,3,\\ \text{any } \eta\ge 1/4 \text{ such that } \Delta(x) \ = \ O\left(x^{\eta}\right), & k\ge 4,\end{cases}$$

for Δ defined[2] in (10).

[2]Hardy showed that η can not be smaller than 1/4 [5].

There is also a nice extension of Corollary 1 for certain types of B-free integers. Let $B : b_1, b_2, b_3 \dots$ be a sequence of pairwise distinct positive integers such that $b_j > 1$ for all j,

$$\sum_j \frac{1}{b_j} < \infty \quad \text{and} \quad \gcd(b_i, b_j) \;=\; 1 \text{ for } i \neq j.$$

A positive integer n is B*-free* if n is not divisible by any of the b_j's [1]. For example, if B_k is the set of the kth powers of all prime numbers, then the set of k-free numbers may be referred to as the set of B_k-free numbers.

The analogue of (1) for B-free integers is

$$\tilde{D}_B(x) \;=\; \sum_{n\le x} q_B(n) \sum_{d|n} 1, \tag{11}$$

where q_B is the characteristic function of B-free integers. Let B be formed by powers of the prime numbers with possibly varying exponents, that is,

$$b_i \;=\; p_i^{\kappa_i}, i = 1, 2, \dots \tag{12}$$

(here and throughout the paper, $p_1, p_2, \dots, p_\ell, \dots$ will denote the sequence of the prime numbers in ascending order). We shall refer to B_κ for the b's in (12).

Corollary 2. Let $\kappa = (\kappa_1, \kappa_2, \dots) \;\in\; \mathbb{N}^{\mathbb{N}}$. If $\kappa_{min} := \min_{i\ge1}\{\kappa_i\} \geq 2$, there exists a constant β_κ such that, for every $\epsilon > 0$,

$$\tilde{D}_{B_\kappa}(x) \;=\; \left(\prod_{i=1}^{\infty}\left[1 - \frac{\kappa_i + 1}{p_i^{\kappa_i}} \;+\; \frac{\kappa_i}{p_i^{\kappa_i+1}}\right]\right) x\log(x) \;+\; \beta_\kappa\, x \;+\; O\left(x^{\eta_\kappa+\epsilon}\right),$$

$$\text{with } \eta_\kappa = \begin{cases} 1/\kappa_{min}, & \kappa_{min} = 2, 3, \\ \text{any } \eta \geq 1/4 \text{ such that } \Delta(x) \;=\; O\left(x^{\eta}\right), & \kappa_{min} \geq 4, \end{cases}$$

for Δ defined in (10).

Theorem 1 can also be useful for the analysis of some problems over reduced residue classes, due to its tight connection with completely multiplicative functions (Dirichlet characters, in particular). For instance, Pongsriian and Vaugham and others (see [14, 10] and the references therein) estimated the Dirichlet divisor function D for n in residue classes:

$$D(x, a, \xi) \;=\; \sum_{\substack{n \,\le\, x \\ n \equiv a \ (mod\ \xi)}} \sum_{d|n} 1, \quad 0 \leq a < \xi. \tag{13}$$

In this regard, one could also consider the problem of estimating the analogue of (13) for $\tilde{D}_{B_\kappa}$:

$$\tilde{D}_{B_\kappa}(x,a,\xi) \;=\; \sum_{\substack{n\,\leq\,x \\ n\,\equiv\,a\ (mod\ \xi)}} q_{B_\kappa}(n)\sum_{d|n}1, \quad 0\leq a<\xi. \tag{14}$$

Corollary 3. Let $\kappa=(\kappa_1,\kappa_2,\dots)\ \in\ \mathbb{N}^{\mathbb{N}}$ and $\kappa_{min}:=\min_{i\geq 1}\{\kappa_i\}$. If $\kappa_{\min}\geq 2$ and ξ and a are coprime, there is a constant $\tilde{c}_{\kappa,a,\xi}$ such that

$$\begin{aligned}\tilde{D}_{B_\kappa}(x,a,\xi) \;&=\; \frac{\varphi(\xi)}{\xi^2}\Bigg(\prod_{\chi_\xi(p_i)=1}\left[1-\frac{\kappa_i+1}{p_i^{\kappa_i}}\;+\;\frac{\kappa_i}{p_i^{\kappa_i+1}}\right]\Bigg)x\log(x)\;+\;\tilde{c}_{\kappa,a,\xi}\,x\\ &\quad + O\left(x^{\eta_\kappa+\epsilon}\right),\quad \text{with}\ \ \eta_\kappa \;=\; \begin{cases}1/2, & \kappa_{min}=2,\\ 1/3 & \kappa_{min}\geq 4,\end{cases}\end{aligned} \tag{15}$$

(χ_ξ is the principal Dirichlet character with modulus ξ).

1.1. Organization of the Paper

In Section 2, we state and prove some lemmas about multiplicative and completely multiplicative functions. In Section 3, we prove the main results of the paper. In Section 4, we make a few remarks about the work of Jakimczuk and Lalín [8] and show that Corollary 1 could also be obtained with their methods.

2. A Few Lemmas About Multiplicative and Completely Multiplicative Functions

In the following statements, void sums must be interpreted as zero and $u(n)^0:=1, v(n)^0:=1$.

Lemma 1. Let $u,v:\mathbb{N}\to\mathbb{C}$. For non-negative integers τ, ξ and $n\geq 1$,

$$\begin{aligned}\sum_{j=0}^{\tau+\xi}u(n)^j v(n)^{\tau+\xi-j} \;&=\; \left(\sum_{j=0}^{\tau}u(n)^j v(n)^{\tau-j}\right)\left(\sum_{i=0}^{\xi}u(n)^i v(n)^{\xi-i}\right)\\ &\quad -u(n)v(n)\left(\sum_{j=0}^{\tau-1}u(n)^j v(n)^{\tau-1-j}\right)\left(\sum_{i=0}^{\xi-1}u(n)^i v(n)^{\xi-1-i}\right).\end{aligned} \tag{16}$$

Proof. For $A=\left(\sum\limits_{j=0}^{\tau}u(n)^j v(n)^{\tau-j}\right)\left(\sum\limits_{i=0}^{\xi}u(n)^i v(n)^{\xi-i}\right)$, we have

$$\begin{aligned} A &= \left(\sum_{j=0}^{\tau-1} u(n)^j v(n)^{\tau-j} + u(n)^\tau\right)\left(\sum_{i=1}^{\xi} u(n)^i v(n)^{\xi-i} + v(n)^\xi\right) \\ &= \left(\sum_{j=0}^{\tau-1} u(n)^j v(n)^{\tau-j}\right)\left(\sum_{i=1}^{\xi} u(n)^i v(n)^{\xi-i}\right) \\ &\quad + u(n)^\tau \left(\sum_{i=1}^{\xi} u(n)^i v(n)^{\xi-i}\right) \; + \; v(n)^\xi \left(\sum_{j=0}^{\tau} u(n)^j v(n)^{\tau-j}\right) \\ &= u(n)v(n)\left(\sum_{j=0}^{\tau-1} u(n)^j v(n)^{\tau-1-j}\right)\left(\sum_{i=0}^{\xi-1} u(n)^i v(n)^{\xi-1-i}\right) \\ &\quad + \left(\sum_{\ell=\tau+1}^{\tau+\xi} u(n)^\ell v(n)^{\tau+\xi-\ell}\right) \; + \; \left(\sum_{\ell=0}^{\tau} u(n)^\ell v(n)^{\tau+\xi-\ell}\right). \end{aligned}$$

□

Lemma 2. If $u, v : \mathbb{N} \to \mathbb{C}$ are completely multiplicative for positive integers r and s,

$$(u * v)(rs) \; = \sum_{m \mid \gcd(r,s)} \mu(m)u(m)v(m)(u*v)\left(\frac{r}{m}\right)(u*v)\left(\frac{s}{m}\right). \tag{17}$$

Proof. The proof is by induction on the number of distinct prime factors of rs. Note that (17) holds for $r = s = 1$. Assume that $rs > 1$ and let p be a prime factor of rs, with

$$r = p^\tau r', \quad s = p^\xi s', \quad \text{and } \gcd(p, r') \; = \; \gcd(p, s') \; = \; 1,$$

with τ and ξ not both vanishing, and assume that (17) holds for r' and s'. Hence,

$$\begin{aligned} (u*v)(rs) \; &= \sum_{\ell=0}^{\tau+\xi} u\left(p^\ell\right) v(p^{\tau+\xi-\ell})(u*v)(r's') \\ &= \sum_{\ell=0}^{\tau+\xi} u\left(p^\ell\right) v(p^{\tau+\xi-\ell}) \sum_{m \mid \gcd(r',s')} \mu(m)u(m)v(m)(u*v)\left(\frac{r'}{m}\right)(u*v)\left(\frac{s'}{m}\right). \end{aligned} \tag{18}$$

If $\tau = 0$, then $r' = r$, $\gcd(r, s) \; = \; \gcd(r', s')$ and we get

$$
\begin{aligned}
(u*v)(rs) &= \sum_{m|\gcd(r,s)} \mu(m)u(m)v(m)(u*v)\left(\frac{r}{m}\right)\left(\sum_{\ell=0}^{\xi} u\left(p^{\ell}\right)v(p^{\xi-\ell})\right)(u*v)\left(\frac{s'}{m}\right)\\
&= \sum_{m|\gcd(r,s)} \mu(m)u(m)v(m)(u*v)\left(\frac{r}{m}\right)(u*v)\left(\frac{s}{m}\right),
\end{aligned}
$$

that is, (17) holds for r and s (the same conclusion holds if $\xi = 0$).

If both τ and ξ are non-vanishing, then

$$
\gcd(r,s) \;=\; p^{\nu}\gcd(r',s'),
$$

for some $\nu \geq 1$. By Lemma 1 and by (18), we obtain

$$
\begin{aligned}
(u*v)(rs) \;\;&=\;\; \left(\sum_{\ell=0}^{\tau} u\left(p^{\ell}\right)v(p^{\tau-\ell})\right)\left(\sum_{\ell=0}^{\xi} u\left(p^{\ell}\right)v(p^{\xi-\ell})\right)A\\
&\quad - u(p)v(p)\left(\sum_{\ell=0}^{\tau-1} u\left(p^{\ell}\right)v(p^{\tau-1-\ell})\right)\left(\sum_{\ell=0}^{\xi-1} u\left(p^{\ell}\right)v(p^{\xi-1-\ell})\right)A,
\end{aligned}
$$

with

$$
A \;=\; \sum_{m|\gcd(r',s')} \mu(m)u(m)v(m)(u*v)\left(\frac{r'}{m}\right)(u*v)\left(\frac{s'}{m}\right).
$$

Hence,

$$
\begin{aligned}
(u*v)(rs) \;\;&=\;\; \sum_{m|\gcd(r',s')} \mu(m)u(m)v(m)(u*v)\left(\frac{r}{m}\right)(u*v)\left(\frac{s}{m}\right)\\
&\quad + \sum_{m|\gcd(r',s')} \mu(mp)u(mp)v(mp)(u*v)\left(\frac{r/p}{m}\right)(u*v)\left(\frac{s/p}{m}\right).
\end{aligned}
$$

Since every divisor d of $\gcd(r,s)$ with $\mu(d) \neq 0$ is of the form $d = m$ or $d = pm$ with $m \mid \gcd(r',s')$, we obtain (17) for r and s. □

For $u : \mathbb{N} \to \mathbb{C}$, $r \in \mathbb{N}$ and $x \geq 1$, let

$$
S_r[u,v](x) \;=\; \sum_{j \leq \frac{x}{r}} (u*v)(rj). \tag{19}
$$

Lemma 3. If $u, v : \mathbb{N} \to \mathbb{C}$ are completely multiplicative, for $x \geq 1$,

$$S_r[u,v](x) \ = \ \sum_{d|r} u(d)v(r/d) \sum_{m|d} \mu(m)v(m)D[u,v]\left(\frac{x}{mr}\right), \tag{20}$$

for $D[u,v]$ defined by (7).

Proof. Let us rewrite (17) in a different way:

$$(u*v)(rs) \ = \ \sum_{d|r} u(d)v(r/d) \sum_{m|\gcd(d,s)} \mu(m)v(m)(u*v)\left(\frac{s}{m}\right). \tag{21}$$

Summing (21) in s, we get

$$\begin{aligned} S_r[u,v](x) &= \sum_{s\leq \frac{x}{r}} \left(\sum_{d|r} u(d)v(r/d) \sum_{m|\gcd(d,s)} \mu(m)v(m)(u*v)\left(\frac{s}{m}\right)\right) \\ &= \sum_{d|r} u(d)v(r/d) \sum_{m|d} \mu(m)v(m) \sum_{j\leq \frac{x}{mr}} (u*v)\left(\frac{jm}{m}\right) \\ &= \sum_{d|r} u(d)v(r/d) \sum_{m|d} \mu(m)v(m)D[u,v]\left(\frac{x}{mr}\right). \end{aligned}$$

□

Remark 2. Lemma 3 was obtained previously by Pongsriiam and Vaugham for the constant function $u = v \equiv 1$ [14, p.8].

Lemma 4. For generic functions $w, u, v : \mathbb{N} \to \mathbb{C}$ and for $x \geq 1$,

$$D[w,u,v](x) \ = \ \sum_{r\leq x} g(r)S_r[u,v](x),$$

where $g \ = \ w * \mu$ is the Dirichlet convolution of w and μ.

Proof. We have

$$\begin{aligned} D[w,u,v](x) &= \sum_{j\leq x} w(j)(u*v)(j) \ = \ \sum_{j\leq x} (g*1)(j)(u*v)(j) \\ &= \sum_{j\leq x} \left(\sum_{r|j} g(r)\right)(u*v)(j) \ = \ \sum_{r\leq x} g(r) \left(\sum_{j\leq \frac{x}{r}} (u*v)(jr)\right) \\ &= \sum_{r\leq x} g(r)S_r[u,v](x). \end{aligned}$$

□

2.1. Some Special Convolutions

Given a sequence of real numbers $a = (a_1, a_2, \dots)$ and a positive integer $n \geq 1$, we define

$$n^a \;=\; \prod_{i=1}^{\ell} (p_i^{\alpha_i})^{a_i}\,,$$

where $n \;=\; \prod_{i=1}^{\ell} p_i^{\alpha_i}$ is the decomposition of n into prime numbers (note that some of the exponents α_i might be zero).

Lemma 5. For $n \geq 1$ and $\kappa = (\kappa_1, \kappa_2, \dots,)$,

$$(q_{B_\kappa} * \mu)(n) \;=\; \begin{cases} \mu\left(n^{1/\kappa}\right), & \text{if } n^{1/\kappa} \in \mathbb{N}, \\ 0, & \text{otherwise}, \end{cases} \tag{22}$$

where $1/\kappa := (1/\kappa_1, 1/\kappa_2, \dots,)$.

Proof. Let $n \;=\; \prod_{i=1}^{\ell} p_i^{\alpha_i}$ be the decomposition of n into prime numbers. Note that, if $\alpha_j > \kappa_j$ for some j and $n = rs$,

$$\text{either } p_j^{\kappa_j} \mid r \quad \text{or } p_j^2 \mid s.$$

Hence,

$$(q_{B_\kappa} * \mu)(n) \;=\; \sum_{rs=n} q_{B_\kappa}(r)\mu(s) \;=\; 0.$$

Assume now that $\alpha_i \leq \kappa_i$ for all i and write $\{1, 2, \dots, \ell\} \;= \mathcal{K} \sqcup \mathcal{W}$, where

$$\mathcal{K} \;=\; \{i : \alpha_i = \kappa_i\}, \quad \mathcal{W} \;=\; \{i : \alpha_i < \kappa_i\}.$$

Let

$$\theta = \prod_{i \,\in\, \mathcal{W}} p_i^{\alpha_i}.$$

Note that if $n = rs$ and $q_{B_\kappa}(r)\mu(s) \neq 0$, then

$$r = r' \prod_{i \,\in\, \mathcal{K}} p_i^{\alpha_i - 1} \quad \text{and } s = s' \prod_{i \,\in\, \mathcal{K}} p_i$$

with $r's' = \theta$. Therefore,

$$(q_{B_\kappa} * \mu)(n) \;=\; \sum_{s' \mid \theta} \mu\left(\prod_{i \,\in\, \mathcal{K}} p_i\right) \mu(s') \;=\; \begin{cases} \mu\left(\prod_{i \,\in\, \mathcal{K}} p_i\right), & \theta = 1, \\ 0, & \theta > 1. \end{cases}$$

This tells us that, if,

$$(q_{B_\kappa} * \mu)(n) \neq 0, \text{ then } n = \prod_{i \in \mathcal{K}} p_i^{\kappa_i}$$

for some $\mathcal{K} \subset \mathbb{N}$ and, in this case,

$$(q_{B_\kappa} * \mu)(n) = \mu\left(n^{1/\kappa}\right).$$

To complete the proof, note that if $n^{1/\kappa} \in \mathbb{N}$, then

$$n = \prod_{i \in \mathcal{K}} p_i^{r_i \kappa_i}$$

for some $\mathcal{K} \subset \mathbb{N}$ and $r_i \in \mathbb{N}$. If $r_i > 1$ for some i, we concluded above that $(q_{B_\kappa} * \mu)(n) = 0$. However, in this case, $\mu\left(n^{1/\kappa}\right)$ is also vanishing and (22) still holds. □

Lemma 6. Let $w : \mathbb{N} \to \mathbb{N}$ be defined by

$$w(n) = q_{B_\kappa}(n)\chi_\xi(n) \text{ for all } n \geq 1,$$

where χ_ξ is the principal Dirichlet character modulus ξ and B_κ is defined in (12). The Dirichlet convolution of $(w * \mu)(n)$ is non-vanishing only for n of the form $n = \ell r^\kappa$, with $\ell \mid \xi$ and for some r such that $\gcd(r, \xi) = 1$. In this case,

$$(w * \mu)(n) = \mu(\ell)\mu(r). \tag{23}$$

Proof. Every integer $n \geq 1$ can be written uniquely as $n = \alpha\ell$, with $\ell \mid \xi^s$ for some s and $\gcd(\alpha, \xi) = 1$. Hence,

$$\begin{aligned}(w * \mu)(n) &= \sum_{\alpha' \mid \alpha} \sum_{\ell' \mid \ell} \chi_\xi(\ell'\alpha') q_{B_\kappa}(\alpha'\ell') \mu\left(\frac{\alpha\ell}{\alpha'\ell'}\right) \\ &= \sum_{\alpha' \mid \alpha} q_{B_\kappa}(\alpha') \mu\left(\frac{\alpha}{\alpha'}\right) \mu(\ell).\end{aligned}$$

By (22), we obtain

$$(w * \mu)(n) = \begin{cases} \mu(r)\mu(\ell), & \alpha = r^\kappa,\ r \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that, given that $\ell \mid \xi^s$, if $\ell \nmid \xi$, then $\mu(\ell) = 0$. □

3. Proofs

In this section, we give proofs of Theorem 1, Corollary 2, and Corollary 3.

Proof of Theorem 1. This follows immediately by Lemmas 3 and 4. □

Proof of Corollary 2. Theorem 1 for $u = v \equiv 1$ and Lemma 5 give

$$\tilde{D}_{B_\kappa}(x) \;=\; \sum_{r^\kappa \le x} \mu(r) \sum_{d \mid r^\kappa} \sum_{m \mid d} \mu(m) D\left(\frac{x}{r^\kappa m}\right). \tag{24}$$

Note that, by (22), $(q_{B_\kappa} * \mu)(n)$ is non-vanishing only when n is of the form r^κ for some integer r.

Combining (24) with (10), we obtain

$$\begin{aligned} \tilde{D}_{B_\kappa}(x) \;=\; & \sum_{r^\kappa \le x} \mu(r) \sum_{d \mid r^\kappa} \sum_{m \mid d} \mu(m) \left(\frac{x}{r^\kappa m}\right) \log\left(\frac{x}{r^\kappa m}\right) \\ & + (2\gamma - 1) \sum_{r^\kappa \le x} \mu(r) \sum_{d \mid r^\kappa} \sum_{m \mid d} \mu(m) \left(\frac{x}{r^\kappa m}\right) \\ & + \sum_{r^\kappa \le x} \mu(r) \sum_{d \mid r^\kappa} \sum_{m \mid d} \mu(m) \Delta\left(\frac{x}{r^\kappa m}\right). \end{aligned} \tag{25}$$

It is sufficient to show that the quantities

$$\begin{aligned} a_\kappa(x) &:= \sum_{r^\kappa \le x} \frac{\mu(r)}{r^\kappa} \sum_{d \mid r^\kappa} \sum_{m \mid d} \frac{\mu(m)}{m} \\ b_\kappa(x) &:= \sum_{r^\kappa \le x} \frac{\mu(r)}{r^\kappa} \sum_{d \mid r^\kappa} \sum_{m \mid d} \frac{\mu(m)}{m} \log(r^\kappa m) \\ c_\kappa(x) &:= \sum_{r^\kappa \le x} \frac{\mid \mu(r) \mid}{r^{\kappa\, \eta_k}} \sum_{d \mid r^\kappa} \sum_{m \mid d} \frac{\mid \mu(m) \mid}{m^{\eta_k}} \end{aligned}$$

satisfy:

(i) $a_\kappa(x)$ and $b_\kappa(x)$ converge as $x \to \infty$ and, for $a_\kappa := \lim_{x\to\infty} a_\kappa(x)$ and $b_\kappa := \lim_{x\to\infty} b_\kappa(x)$ we have

$a_\kappa - a_\kappa(x) = O_{\kappa,\epsilon}\left(x^{-1+1/\kappa_{min}+\epsilon}\right)$ and $b_\kappa - b_\kappa(x) = O_{\kappa,\epsilon}\left(x^{-1+1/\kappa_{min}+\epsilon}\right)$ for all $\epsilon > 0$;

(ii) $c_\kappa(x) \;=\; O_{\kappa,\epsilon}(x^\epsilon)$ for every $\epsilon > 0$.

In fact, By (25) and (i) and (ii),

$$\tilde{D}_\kappa(x) \;=\; a_\kappa x\log(x) \;+\; (b_\kappa + [2\gamma-1]a_\kappa)x \;+\; O_{\kappa,\epsilon}\left(x^{\eta_k+\epsilon}\right). \tag{26}$$

The proofs of (i) for $a_\kappa(x)$ and $b_\kappa(x)$ are quite similar to each other. Let us prove (i) for $a_\kappa(x)$. Denote by $\mathbf{1}$ the constant function $\mathbf{1}(j) = 1$ for all $j \geq 1$. Because

$$r^\kappa = \prod_{p_i \,\mid\, r} p_i^{\kappa_i} \tag{27}$$

when $\mu(r) \neq 0$, we get

$$a_\kappa(x) \;=\; \sum_{r^\kappa \leq x} \frac{\mu(r)}{r^\kappa} \sum_{d|r^\kappa} \mathbf{1}(d) \prod_{\substack{p \,\mid\, d \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right). \tag{28}$$

Because $\sum_{d|y} 1 \;=\; O_\epsilon(y^\epsilon)$ for every $\epsilon > 0$, we have

$$\sum_{r^\kappa \leq x} \frac{\mu(r)}{r^\kappa} \sum_{d|r^\kappa} 1 \;\ll_\epsilon\; \sum_{r^\kappa \leq x} \frac{1}{r^{\kappa(1-\epsilon)}} \;\leq\; \sum_{r^\kappa \leq x} \frac{1}{r^{\kappa_{min}(1-\epsilon)}}. \tag{29}$$

This shows that $a_\kappa(x)$ converges absolutely (as $x \to \infty$) for every $\epsilon < 1/2$. In addition,

$$|\, a_\kappa - a_\kappa(x) \,| \ll_\epsilon \sum_{r^{\kappa_{min}} > x} \frac{1}{r^{\kappa_{min}(1-\epsilon)}} \;=\; O_{\kappa,\epsilon}\left(x^{\frac{-\kappa_{min}+\kappa_{min}\epsilon+1}{\kappa_{min}}}\right). \tag{30}$$

By (28), we also have

$$\begin{aligned} a_\kappa \;&=\; \sum_{r^\kappa \leq x} \frac{\mu(r)}{r^\kappa} \prod_{p_i \,\mid\, r} \left[1 + \sum_{\ell=1}^{\kappa_i} \mathbf{1}\left(p_i^\ell\right) \left(1 - \frac{1}{p_i}\right)\right] \\ &=\; \sum_{r=1}^{\infty} \frac{\mu(r)}{r^\kappa} \prod_{p_i \,\mid\, r} \left[\kappa_i + 1 - \frac{\kappa_i}{p_i}\right] \;=\; \prod_{i\,=\,1}^{\infty} \left(1 - \frac{1}{p_i^{\kappa_i}}\left[\kappa_i + 1 - \frac{\kappa_i}{p_i}\right]\right). \end{aligned}$$

The proof of (ii) is analogous to the proof of (i) and follows by using the Van Der Corput estimate

$$\Delta(x) \;=\; O\left(x^{33/100}\right) \tag{31}$$

for $\Delta(x)$ in the cases $\kappa_{min} = 2$ and $\kappa_{min} = 3$ (see Hölder's approach to the k-free divisor problem for similar arguments [6]). □

Proof of Corollary 3. Let $char(\xi)$ be the group of Dirichlet characters modulo ξ. We have [7, Proposition 4.2.5]

$$\begin{aligned} \tilde{D}_{B_\kappa}(x,a,\xi) &= \frac{1}{\varphi(\xi)} \sum_{\chi\, \in\, char(\xi)} \overline{\chi(a)} \sum_{n \le x} \chi(n) q_{B_\kappa}(n) \sum_{d|n} 1 \\ &= \frac{1}{\varphi(\xi)} \sum_{\chi\, \in\, char(\xi)} \overline{\chi(a)} \sum_{n \le x} q_{B_\kappa}(n)(\chi * \chi)(n), \end{aligned} \tag{32}$$

where φ is the Euler totient function. For a non-principal character χ, we apply Theorem 1 for $u = v = \chi$ and $w = q_{B_\kappa}$. By (22),

$$\sum_{n \le x} q_{B_\kappa}(n)(\chi * \chi)(n) \;=\; \sum_{r^\kappa \le x} \mu(r)\chi(r^\kappa) \sum_{d|r^\kappa} \sum_{m|d} \mu(m)\chi(m) \sum_{j \le \frac{x}{r^\kappa m}} (\chi * \chi)(j). \tag{33}$$

Combining Theorem 8.18 of [2] with Theorem 1.1 of [18], we get

$$\sum_{j \le \frac{x}{r^\kappa m}} (\chi * \chi)(j) \;=\; O_\xi \left(\left(\frac{x}{r^\kappa m} \right)^{1/3} \right), \tag{34}$$

so the overall contribution for a non-principal character χ in (32) is $O_{a,\xi}\left(x^{1/2+\epsilon}\right)$, $\kappa_{\min} = 2$, and $O_{a,\xi}\left(x^{1/3+\epsilon}\right)$, $\kappa_{\min} \ge 3$, in this case.

For the principal character χ_ξ, we apply Theorem 1 for $w = \chi_\xi . q_{B_\kappa}$ and $u = v \equiv 1$ ("." means pointwise product). By (8) and (23), we get

$$\sum_{n \le x} w(n) \sum_{d|n} 1 \;=\; \sum_{\ell|\xi} \mu(\ell) \sum_{r^\kappa \le x/\ell} \mu(r)\chi_\xi(r) \sum_{d|\ell r^\kappa} \sum_{m|d} \mu(m) D\left(\frac{x}{m\ell r^\kappa}\right).$$

Note that $\gcd(\ell, r^\kappa) = 1$, so we write each divisor d of ℓr^κ as $\ell' d'$, with $\ell' \mid \ell$ and $d' \mid r^\kappa$. Similarly, we write $m = \ell'' d''$, with $\ell'' \mid \ell'$ and $d'' \mid d'$ and we use that $D(y) = 0$ for $y < 1$ to replace the condition $r^\kappa \le x/\ell$ by $r^\kappa \le x/(\ell\ell'')$:

$$\begin{aligned} \sum_{n \le x} w(n) \sum_{d|n} 1 &= \sum_{\ell|\xi} \mu(\ell) \sum_{\ell'|\ell} \sum_{\ell''|\ell'} \mu(\ell'') \sum_{r^\kappa \le x/(\ell\ell'')} \mu(r)\chi_\xi(r) \sum_{d'|r^\kappa} \sum_{d''|d'} \mu(d'') D\left(\frac{x}{\ell'' d'' \ell r^\kappa}\right) \\ &= \sum_{\ell|\xi} \mu(\ell) \sum_{\ell'|\ell} \sum_{\ell''|\ell'} \mu(\ell'') L\left(\frac{x}{\ell\ell''}\right), \end{aligned} \tag{35}$$

where

$$L(x) \;=\; \sum_{r^\kappa \le x} \mu(r)\chi_\xi(r) \sum_{d'|r^\kappa} \sum_{d''|d'} \mu(d'') D\left(\frac{x}{d'' r^\kappa}\right). \tag{36}$$

The expression in the right-hand side of (36) is quite similar to the one for $\tilde{D}_{B_\kappa}$ on (24). Therefore, proceeding as in the proof of Corollary 2, we conclude that

$$\begin{aligned} L(x) &= \left(\prod_{\chi_\xi(p_i)=1} \left[1 - \frac{\kappa_i + 1}{p_i^{\kappa_i}} + \frac{\kappa_i}{p_i^{\kappa_i+1}} \right] \right) x \log(x) + c_{\kappa,\xi,a}\, x + \Delta_\kappa^*(x), \\ \Delta_\kappa^*(x) &= O\left(x^{\eta_\kappa + \epsilon}\right), \quad \text{with } \eta_\kappa = \begin{cases} 1/2, & \kappa_{min} = 2, \\ 1/3 & \kappa_{min} \geq 4, \end{cases} \end{aligned} \tag{37}$$

for some $c_{\kappa,\xi,a}$. It remains to prove that

$$\sum_{\ell|\xi} \frac{\mu(\ell)}{\ell} \sum_{\ell'|\ell} \sum_{\ell''|\ell'} \frac{\mu(\ell'')}{\ell''} = \left(\frac{\varphi(\xi)}{\xi} \right)^2. \tag{38}$$

In this regard, note that the sum on the left-hand side of (38) is made only over square-free divisors of ξ, so it is sufficient to prove (38) for square–free integers ξ. This gives

$$\begin{aligned} \sum_{\ell|\xi} \frac{\mu(\ell)}{\ell} \sum_{\ell'|\ell} \sum_{\ell''|\ell'} \frac{\mu(\ell'')}{\ell''} &= \sum_{\ell|\xi} \frac{\mu(\ell)}{\ell} \sum_{\ell'|\ell} \frac{\varphi(\ell')}{\ell'} = \sum_{\ell'|\xi} \frac{\varphi(\ell')}{\ell'} \sum_{\eta|\xi/\ell'} \frac{\mu(\ell'\eta)}{\ell'\eta} \\ &= \sum_{\ell'|\xi} \frac{\mu(\ell')}{\ell'} \frac{\varphi(\ell')}{\ell'} \sum_{\eta|\xi/\ell'} \frac{\mu(\eta)}{\eta} = \sum_{\ell'|\xi} \frac{\mu(\ell')}{\ell'} \frac{\varphi(\ell')}{\ell'} \frac{\varphi(\xi/\ell')}{\xi/\ell'} \\ &= \sum_{\ell'|\xi} \frac{\mu(\ell')}{\ell'} \frac{\varphi(\xi)}{\xi} = \left(\frac{\varphi(\xi)}{\xi} \right)^2. \end{aligned}$$

The hypothesis "ξ is square-free" was used above to write $\mu(\ell'\eta) = \mu(\ell')\mu(\eta)$ and $\varphi(\xi) = \varphi(\ell')\varphi(\xi/\ell')$. □

4. Some Remarks about the Jakimczuk-Lalín Method

Jakimczuk and Lalín proved Equation (3) in [8] by combining Perron's formula with an Euler-type product formula for the Dirichlet series with coefficients $a(j) = q_k(j) \sum_{d|j} 1$. As an intermediate step, they showed that

$$\sum_{j=1}^{\infty} \frac{a(j)}{j^s} = \zeta^2(s) F_{0,k}(s), \tag{39}$$

with

$$F_{0,k}(s) = \sum_{j=1}^{\infty} \frac{\tau(j)}{j^s}$$

absolutely convergent for $\Re e(s) > \frac{1}{k}$. It passed unnoticed that (39) and (10) are already enough to provide an upper bound $O_{k,\epsilon}(x^{\eta_k+\epsilon})$ for the error term $\tilde{\Delta}_k(x)$ as in Corollary 1. In fact,

$$\tilde{D}_k(x) \ = \ \sum_{j\leq x} \tau(j) D(x/j),$$

with D defined in (10), and the absolute convergence of $F_{0,k}$ and Abel's formula tell us that

$$\sum_{j=1}^{x} \frac{\tau(j)}{j^{1/k+\epsilon}} \frac{1}{j^{1-1/k-\epsilon}} \ = \ \sum_{j=1}^{\infty} \frac{\tau(j)}{j} \ + \ O_{k,\epsilon}(x^{-1+\frac{1}{k}+\epsilon}),$$

for every $\epsilon > 0$.

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