A GENERALIZATION OF THE LANDEN CONNECTION FORMULA AND GENERALIZED POLY-BERNOULLI NUMBERS

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Abstract

We introduce a generalization of multiple polylogarithms and give their Landen-type connection formulas. Also, we define generalized poly-Bernoulli numbers by using these polylogarithms and prove some relations.

1. Introduction

For an index $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}_{>0}^r$, we set $d(\boldsymbol{k}) := r$ and $|\boldsymbol{k}| = k := k_1 + \cdots + k_r$. They are called the *depth* and the *weight* of $\boldsymbol{k}$, respectively. For any index $\boldsymbol{k} = (k_1, \ldots, k_r)$, define *multiple polylogarithms* $\mathrm{Li}_{\boldsymbol{k}}(z)$ as

$$\mathrm{Li}_{\boldsymbol{k}}(z) := \sum_{0<m_1<\cdots<m_r} \frac{z^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}} \quad \in \mathbb{Q}[[z]]$$

(for more general settings, see, e.g., [3] and [16]). When $\boldsymbol{k} = (k)$ $(k > 0)$, the function $\mathrm{Li}_k(z)$ is the classical polylogarithm.

For indices $\boldsymbol{k}$ and $\boldsymbol{k}'$, the notation $\boldsymbol{k}' \preceq \boldsymbol{k}$ means that $\boldsymbol{k}'$ is obtained from $\boldsymbol{k}$ by combining some consecutive entries, e.g., $(3,5) \preceq (1,2,1,3,1)$. Okuda and Ueno [14] proved the following beautiful relation called the Landen connection formula:

Theorem 1 ([14, Prop. 9])**.** *For any index $\boldsymbol{k} \in \mathbb{Z}_{>0}^r$, we have that*

$$\mathrm{Li}_{\boldsymbol{k}}\left(\frac{z}{z-1}\right) = (-1)^{d(\boldsymbol{k})} \sum_{\boldsymbol{k} \preceq \boldsymbol{k}'} \mathrm{Li}_{\boldsymbol{k}'}(z). \tag{1}$$

Let $\mathrm{Li}^{\star}_{\boldsymbol{k}}(z)$ be the non-strict multiple polylogarithm defined by

$$\mathrm{Li}^{\star}_{\boldsymbol{k}}(z) := \sum_{0<m_1\leq\cdots\leq m_r} \frac{z^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}}.$$

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We note that $\mathrm{Li}^{\star}_{\boldsymbol{k}}(z)=\sum_{\boldsymbol{k}'\preceq\boldsymbol{k}}\mathrm{Li}_{\boldsymbol{k}'}(z)$. If an index $\boldsymbol{k}=(k_1,\ldots,k_r)$ satisfies $k_r\geq 2$, then the values $\mathrm{Li}_{\boldsymbol{k}}(1)$ and $\mathrm{Li}^{\star}_{\boldsymbol{k}}(1)$ converge. These limit values, denoted by $\zeta(\boldsymbol{k})$ and $\zeta^{\star}(\boldsymbol{k})$, are called *multiple zeta values* and *multiple zeta star values*, respectively.

For an index $\boldsymbol{k}=(k_1,\ldots,k_r)$ with $k_r\geq 2$, Yamamoto [15] introduced the following interpolated multiple zeta values:

$$\zeta^t(\boldsymbol{k}):=\sum_{\boldsymbol{k}'\preceq\boldsymbol{k}}t^{d(\boldsymbol{k})-d(\boldsymbol{k}')}\zeta(\boldsymbol{k}')\in\mathbb{R}[t].$$

This polynomial interpolates multiple zeta values and multiple zeta star values because $\zeta^0(\boldsymbol{k})=\zeta(\boldsymbol{k})$ and $\zeta^1(\boldsymbol{k})=\zeta^{\star}(\boldsymbol{k})$. Yamamoto studied the function $\zeta^t(\boldsymbol{k})$ in connection with the so-called "two-one formula" [13], and proved a sum formula and a cyclic sum formula for $\zeta^t(\boldsymbol{k})$.

There are some studies on a polylogarithm version of the interpolated multiple zeta values. Li and Qin [10] introduced interpolated multiple polylogarithms as

$$\mathrm{Li}_{\boldsymbol{k}}(t,z):=\sum_{\boldsymbol{k}'\preceq\boldsymbol{k}}t^{\mathrm{dep}(\boldsymbol{k})-\mathrm{dep}(\boldsymbol{k}')}\mathrm{Li}_{\boldsymbol{k}'}(z) \tag{2}$$

and gave a formula relating to the so-called "Ohno-Zagier relation" [12]. When $k_r\geq 2$, it clearly follows that $\mathrm{Li}_{\boldsymbol{k}}(t,1)=\zeta^t(\boldsymbol{k})$. Ohno and Wayama [11] investigated the interpolated Arakawa-Kaneko zeta functions and considered another type of interpolated multiple polylogarithm as follows:

$$\mathrm{Li}^t_{\boldsymbol{k}}(z):=\sum_{\boldsymbol{k}\preceq\boldsymbol{k}'}t^{\mathrm{dep}(\boldsymbol{k}')-\mathrm{dep}(\boldsymbol{k})}\mathrm{Li}_{\boldsymbol{k}'}(z). \tag{3}$$

This function interpolates $\mathrm{Li}^0_{\boldsymbol{k}}(z)=\mathrm{Li}_{\boldsymbol{k}}(z)$ and $\mathrm{Li}^1_{\boldsymbol{k}}(z)=\sum_{\boldsymbol{k}\preceq\boldsymbol{k}'}\mathrm{Li}_{\boldsymbol{k}'}(z)$. The latter value $\mathrm{Li}^1_{\boldsymbol{k}}(z)$ appears in the Landen connection formula (1).

For an index $\boldsymbol{k}=(k_1,\ldots,k_r)\in\mathbb{Z}^r_{>0}$, two kinds of poly-Bernoulli numbers $C_n^{(\boldsymbol{k})}$ and $B_n^{(\boldsymbol{k})}$ $(n\geq 0)$ are defined as follows:

$$\frac{\mathrm{Li}_{\boldsymbol{k}}(1-e^{-t})}{e^t-1}=\sum_{n=0}^{\infty}C_n^{(\boldsymbol{k})}\frac{t^n}{n!}\quad\text{and}\quad\frac{\mathrm{Li}_{\boldsymbol{k}}(1-e^{-t})}{1-e^{-t}}=\sum_{n=0}^{\infty}B_n^{(\boldsymbol{k})}\frac{t^n}{n!}$$

(see, e.g., [9]). We remark that, in general, these numbers can be defined even if k_i's are non-positive. When $r=1$, the numbers $C_n^{(k)}$ and $B_n^{(k)}$ are introduced by Arakawa-Kaneko [1] and Kaneko [6]. Since $\mathrm{Li}_1(z)=-\log(1-z)$, we have $C_n^{(1)}=B_n$ and $B_n^{(1)}=(-1)^nB_n$ for $n\geq 0$. Here B_n $(n\geq 0)$ are ordinary Bernoulli numbers defined by

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}t^n.$$

In the present paper, we introduce generalized multiple polylogarithms including both Equations (2) and (3) and give a generalization of the Landen connection formula (1). Moreover, we define the corresponding poly-Bernoulli numbers, which are generalizations of the ordinary poly-Bernoulli numbers $C_n^{(\boldsymbol{k})}$ and $B_n^{(\boldsymbol{k})}$, and prove some identities.

2. Generalized Multiple Polylogarithms

For a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$, let us consider $z = \dfrac{ae^t+b}{ce^t+d}$ which is a linear fractional transformation of e^t. We remark that $z = \dfrac{ae^t+b}{ce^t+d}$ can be written *formally* as

$$t = \log\frac{-b+dz}{a-cz} = \int_{\frac{a+b}{c+d}}^{z} \left(\frac{c}{a-cu} - \frac{d}{b-du}\right) du. \tag{4}$$

In particular, when $a = 1$ and $b = -1$, we have

$$t = \log\frac{1+dz}{1-cz} = \int_0^z \left(\frac{c}{1-cu} + \frac{d}{1+du}\right) du. \tag{5}$$

For an integer $s \geq 1$, set $g_0 = \begin{pmatrix} 1 & -1 \\ c_0 & d_0 \end{pmatrix} \in GL_2(\mathbb{R})$ and $g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in GL_2(\mathbb{R})$ $(1 \leq i \leq s)$. The symbol $\boldsymbol{g}$ stands for a sequence $\boldsymbol{g} = (g_0, g_1, \ldots, g_s)$. Let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_s)$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_s) \in \mathbb{R}^s$.

For an index $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}_{>0}^r$, we use the arrow notation $\boldsymbol{k}_\uparrow := (k_1, \ldots, k_r + 1)$ and $\boldsymbol{k}_\rightarrow := (k_1, \ldots, k_r, 1)$. Then we define generalized multiple polylogarithms $\mathrm{Li}_{\boldsymbol{k}}(\boldsymbol{g}, (\boldsymbol{\beta}, \boldsymbol{\gamma}); z) \in \mathbb{R}[[z]]$ inductively as

$$\mathrm{Li}_{\boldsymbol{k}_\uparrow}(\boldsymbol{g}, (\boldsymbol{\beta}, \boldsymbol{\gamma}); z) = \int_0^z \sum_{i=1}^{s} \beta_i \left(\frac{c_i}{a_i - c_i u} - \frac{d_i}{b_i - d_i u}\right) \mathrm{Li}_{\boldsymbol{k}}(\boldsymbol{g}, (\boldsymbol{\beta}, \boldsymbol{\gamma}); u)\, du,$$
$$\mathrm{Li}_{\boldsymbol{k}_\rightarrow}(\boldsymbol{g}, (\boldsymbol{\beta}, \boldsymbol{\gamma}); z) = \int_0^z \sum_{i=1}^{s} \gamma_i \left(\frac{c_i}{a_i - c_i u} - \frac{d_i}{b_i - d_i u}\right) \mathrm{Li}_{\boldsymbol{k}}(\boldsymbol{g}, (\boldsymbol{\beta}, \boldsymbol{\gamma}); u)\, du$$

with an initial condition

$$\begin{aligned} \mathrm{Li}_{(1)}(\boldsymbol{g}, (\boldsymbol{\beta}, \boldsymbol{\gamma}); z) &= \int_0^z \left(\frac{c_0}{1-c_0 u} + \frac{d_0}{1+d_0 u}\right) du \\ &= \sum_{n=1}^{\infty} \frac{c_0^n - (-d_0)^n}{n} z^n. \end{aligned} \tag{6}$$

Here we give some examples of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $G_g(t) := \frac{c}{a-ct} - \frac{d}{b-dt}$ in Table 1.

g	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&-1\\1&1\end{pmatrix}$	$\begin{pmatrix}1&-1\\1&-w\end{pmatrix}$ $(w\neq 1)$
$G_g(t)$	$\frac{1}{t}$	$\frac{1}{1-t}$	$\frac{2}{1-t^2}$	$\frac{1}{1-t}-\frac{w}{1-wt}$

Table 1: Examples of $G_g(t)$

If $\boldsymbol{g}=(g_0,g_1,g_2)$ with $g_0=g_2=\begin{pmatrix}1&-1\\1&0\end{pmatrix}$ and $g_1=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, and $\boldsymbol{\beta}=(1,0)$ and $\boldsymbol{\gamma}=(0,1)$, then the function $\mathrm{Li}_{\boldsymbol{k}}\left(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma});z\right)$ is expressed by the well-known iterated integral representation of the multiple polylogarithm $\mathrm{Li}_{\boldsymbol{k}}(z)$. By definition, the function $\mathrm{Li}_{\boldsymbol{k}}\left(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma});z\right)$ has no constant term, i.e., $\mathrm{Li}_{\boldsymbol{k}}\left(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma});z\right)\in z\mathbb{R}[[z]]$ for any index $\boldsymbol{k}$. Also it follows that $\mathrm{Li}_{(1)}\left(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma});\dfrac{e^t-1}{c_0e^t+d_0}\right)=t$ because of Equation (4).

For an index $\boldsymbol{k}\in\mathbb{Z}_{>0}^r$, Hoffman's dual index $\boldsymbol{k}^\vee$ of $\boldsymbol{k}$ is defined inductively by the identities $(\boldsymbol{k}_\uparrow)^\vee=(\boldsymbol{k}^\vee)_\to$ and $(\boldsymbol{k}_\to)^\vee=(\boldsymbol{k}^\vee)_\uparrow$ with $(1)^\vee=(1)$. This index $\boldsymbol{k}^\vee$ appears in Hoffman's duality formula for finite multiple zeta values (see [4]). Interchanging $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ corresponds to taking Hoffman's dual index $\boldsymbol{k}^\vee$, i.e., the following identity holds:

$$\mathrm{Li}_{\boldsymbol{k}}\left(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma});z\right)=\mathrm{Li}_{\boldsymbol{k}^\vee}\left(\boldsymbol{g},(\boldsymbol{\gamma},\boldsymbol{\beta});z\right).$$

The following is a kind of sum formula for $\mathrm{Li}_{\boldsymbol{k}}(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma});z)$.

Proposition 1. *For an integer $k\geq 1$ and an indeterminate μ, we have*

$$\sum_{|\boldsymbol{k}|=k}\mu^{d(\boldsymbol{k})-1}\mathrm{Li}_{\boldsymbol{k}}(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma});z)=\mathrm{Li}_{k}(\boldsymbol{g},(\boldsymbol{\beta}+\mu\boldsymbol{\gamma},\boldsymbol{\delta});z). \tag{7}$$

Here $\boldsymbol{\delta}\in\mathbb{R}^s$ is an arbitrary vector and $\boldsymbol{\beta}+\mu\boldsymbol{\gamma}:=(\beta_i+\mu\gamma_i)_{1\leq i\leq s}$.

Proof. We prove Identity (7) by induction on k. When $k=1$, Identity (7) is trivial because $\mathrm{Li}_{(1)}(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma});z)$ defined by Identity (6) does not depend on $(\boldsymbol{\beta},\boldsymbol{\gamma})$.

We assume that Identity (7) holds for some $k\geq 1$. Then

$$\begin{aligned}
&\frac{d}{dz}\sum_{|\boldsymbol{k}|=k+1}\mu^{d(\boldsymbol{k})-1}\mathrm{Li}_{\boldsymbol{k}}(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma});z)\\
&\qquad=\frac{d}{dz}\left(\sum_{|\boldsymbol{k}|=k}\mu^{d(\boldsymbol{k})-1}\mathrm{Li}_{\boldsymbol{k}_\uparrow}(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma});z)+\sum_{|\boldsymbol{k}|=k}\mu^{d(\boldsymbol{k})}\mathrm{Li}_{\boldsymbol{k}_\to}(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma});z)\right)\\
&\qquad=\sum_{i=1}^{s}\beta_i\left(\frac{c_i}{a_i-c_iz}-\frac{d_i}{b_i-d_iz}\right)\sum_{|\boldsymbol{k}|=k}\mu^{d(\boldsymbol{k})-1}\mathrm{Li}_{\boldsymbol{k}}(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma});z)
\end{aligned}$$

$$+\sum_{i=1}^{s}\mu\gamma_i\left(\frac{c_i}{a_i-c_iz}-\frac{d_i}{b_i-d_iz}\right)\sum_{|\boldsymbol{k}|=k}\mu^{d(\boldsymbol{k})-1}\mathrm{Li}_{\boldsymbol{k}}(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma});z)$$
$$=\sum_{i=1}^{s}(\beta_i+\mu\gamma_i)\left(\frac{c_i}{a_i-c_iz}-\frac{d_i}{b_i-d_iz}\right)\mathrm{Li}_k(\boldsymbol{g},(\boldsymbol{\beta}+\mu\boldsymbol{\gamma},\boldsymbol{\delta});z).$$

Hence, by inductive assumption, we have

$$\frac{d}{dz}\sum_{|\boldsymbol{k}|=k+1}\mu^{d(\boldsymbol{k})-1}\mathrm{Li}_{\boldsymbol{k}}(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma});z)=\frac{d}{dz}\mathrm{Li}_{k+1}(\boldsymbol{g},(\boldsymbol{\beta}+\mu\boldsymbol{\gamma},\boldsymbol{\delta});z).$$

Because every multiple polylogarithm has no constant term, Identity (7) also holds for $k+1$. $\square$

For $\boldsymbol{g}=(g_0,g_1,\dots,g_s)$, we define

$$\tilde{g}_i=\begin{pmatrix}\tilde{a}_i & \tilde{b}_i\\ \tilde{c}_i & \tilde{d}_i\end{pmatrix}:=\begin{pmatrix}1 & 0\\ c_0-d_0 & -1\end{pmatrix}g_i=\begin{pmatrix}a_i & b_i\\ (c_0-d_0)a_i-c_i & (c_0-d_0)b_i-d_i\end{pmatrix}$$

for $0\le i\le s$ and set $\tilde{\boldsymbol{g}}=(\tilde{g}_0,\tilde{g}_1,\dots,\tilde{g}_s)$. We remark that each $\tilde{g}_i$ $(1\le i\le s)$ depends on g_0 and

$$\tilde{g}_0=\begin{pmatrix}1 & 0\\ c_0-d_0 & -1\end{pmatrix}\begin{pmatrix}1 & -1\\ c_0 & d_0\end{pmatrix}=\begin{pmatrix}1 & -1\\ -d_0 & -c_0\end{pmatrix}.$$

Also we can see that this operation is an involution, i.e., $\widetilde{(\tilde{\boldsymbol{g}})}=\boldsymbol{g}$. Then we obtain the following theorem which is a generalization of the Landen connection formula (1).

Theorem 2. *For any index $\boldsymbol{k}\in\mathbb{Z}_{>0}^r$, we have*

$$\mathrm{Li}_{\boldsymbol{k}}\left(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma});\frac{z}{(c_0-d_0)z-1}\right)=\mathrm{Li}_{\boldsymbol{k}}\left(\tilde{\boldsymbol{g}},(\boldsymbol{\beta},\boldsymbol{\gamma});z\right). \tag{8}$$

Proof. We will prove the following: if $u=\dfrac{v}{(c_0-d_0)v-1}$, then

$$\left(\frac{c_i}{a_i-c_iu}-\frac{d_i}{b_i-d_iu}\right)du=\left(\frac{\tilde{c}_i}{\tilde{a}_i-\tilde{c}_iv}-\frac{\tilde{d}_i}{\tilde{b}_i-\tilde{d}_iv}\right)dv\quad(1\le i\le s).$$

Then, by considering the transformation of variables, we obtain Equation (8).

When $u=\dfrac{v}{(c_0-d_0)v-1}$, we have $du=\frac{-1}{((c_0-d_0)v-1)^2}dv$ and

$$\frac{c_i}{a_i-c_iu}du=\frac{c_i}{a_i-c_i\frac{v}{(c_0-d_0)v-1}}\cdot\frac{-1}{((c_0-d_0)v-1)^2}dv$$

$$
\begin{aligned}
&= \frac{c_i}{(a_i(c_0-d_0)-c_i)v-a_i} \cdot \frac{-1}{(c_0-d_0)v-1} dv \\
&= \left(\frac{a_i(c_0-d_0)-c_i}{a_i-(a_i(c_0-d_0)-c_i)v} - \frac{c_0-d_0}{1-(c_0-d_0)v} \right) dv.
\end{aligned}
$$

Therefore we obtain that

$$
\begin{aligned}
\left(\frac{c_i}{a_i-c_iu} - \frac{d_i}{b_i-d_iu} \right) du &= \left(\frac{a_i(c_0-d_0)-c_i}{a_i-(a_i(c_0-d_0)-c_i)v} - \frac{b_i(c_0-d_0)-d_i}{b_i-(b_i(c_0-d_0)-d_i)v} \right) dv \\
&= \left(\frac{\tilde{c}_i}{\tilde{a}_i-\tilde{c}_iv} - \frac{\tilde{d}_i}{\tilde{b}_i-\tilde{d}_iv} \right) dv.
\end{aligned}
$$

□

3. Examples

3.1. The Classical Case

Throughout this subsection we assume that $s = 2$ and

$$
\boldsymbol{g} = (g_0, g_1, g_2) \text{ with } g_0 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } g_2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}. \tag{9}
$$

The function $\mathrm{Li}_{\boldsymbol{k}}(\boldsymbol{g}, ((\beta_1, \beta_2), (\gamma_1, \gamma_2)); z)$ is a generalization of both $\mathrm{Li}_{\boldsymbol{k}}(t, z)$ and $\mathrm{Li}_{\boldsymbol{k}}^t(z)$ defined by Equations (2) and (3). In fact, we have

$$
\mathrm{Li}_{\boldsymbol{k}}(\boldsymbol{g}, ((1,0),(t,1)); z) = \mathrm{Li}_{\boldsymbol{k}}(t, z) \quad \text{and} \quad \mathrm{Li}_{\boldsymbol{k}}(\boldsymbol{g}, ((1,t),(0,1)); z) = \mathrm{Li}_{\boldsymbol{k}}^t(z).
$$

We give some examples of the correspondence of g and $G_g(t)$ in Table 2.

$g_0 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$	$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$g_2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$
$G_{g_0}(t) = \frac{1}{1-t}$	$G_{g_1}(t) = \frac{1}{t}$	$G_{g_2}(t) = \frac{1}{1-t}$
$\tilde{g}_0 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$	$\tilde{g}_1 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$	$\tilde{g}_2 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$
$G_{\tilde{g}_0}(t) = -\frac{1}{1-t}$	$G_{\tilde{g}_1}(t) = \frac{1}{1-t} + \frac{1}{t}$	$G_{\tilde{g}_2}(t) = -\frac{1}{1-t}$

Table 2: Examples of $G_{g_i}(t)$ and $G_{\tilde{g}_i}(t)$

Under the assumption (9), Theorem 2 can be written in the following form.

Theorem 3. *Assume* $\boldsymbol{g}$ *satisfies the condition* (9)*. Then*

$$\mathrm{Li}_{\boldsymbol{k}}\left(\boldsymbol{g},((\beta_1,\beta_2),(\gamma_1,\gamma_2));\frac{z}{z-1}\right)=-\mathrm{Li}_{\boldsymbol{k}}\left(\boldsymbol{g},\left((\beta_1,\beta_1-\beta_2),(\gamma_1,\gamma_1-\gamma_2)\right);z\right). \tag{10}$$

Proof. By Theorem 2, we have

$$\mathrm{Li}_{\boldsymbol{k}}\left(\boldsymbol{g},((\beta_1,\beta_2),(\gamma_1,\gamma_2));\frac{z}{z-1}\right)=\mathrm{Li}_{\boldsymbol{k}}\left(\tilde{\boldsymbol{g}},\left((\beta_1,\beta_2),(\gamma_1,\gamma_2)\right);z\right). \tag{11}$$

The differential form $\beta_1\frac{1}{t}dt+\beta_2\frac{1}{1-t}dt$ is transformed as

$$\left(\beta_1\left(\frac{1}{t}+\frac{1}{1-t}\right)+\beta_2\left(-\frac{1}{1-t}\right)\right)dt=\left(\beta_1\frac{1}{t}+(\beta_1-\beta_2)\frac{1}{1-t}\right)dt$$

under the transformation $t\mapsto\frac{t}{t-1}$. Hence we have

$$\mathrm{Li}_{\boldsymbol{k}}\left(\tilde{\boldsymbol{g}},\left((\beta_1,\beta_2),(\gamma_1,\gamma_2)\right);z\right)=-\mathrm{Li}_{\boldsymbol{k}}\left(\boldsymbol{g},\left((\beta_1,\beta_1-\beta_2),(\gamma_1,\gamma_1-\gamma_2)\right);z\right)$$

and this completes the proof. □

Example 1. By applying $(\beta_1,\beta_2)=(1,t)$ and $(\gamma_1,\gamma_2)=(0,1)$ in Equation (10), we obtain that

$$\mathrm{Li}_{\boldsymbol{k}}^{t}\left(\frac{z}{z-1}\right)=(-1)^{d(\boldsymbol{k})}\mathrm{Li}_{\boldsymbol{k}}^{1-t}(z),$$

where $\mathrm{Li}_{\boldsymbol{k}}^{t}(z)$ is the function defined by Equation (3). When $t=0$, this equation gives the original Landen connection formula (1).

Example 2. For $\beta\in\mathbb{R}$, set $\mathcal{L}_{\boldsymbol{k}}^{(\beta)}(z):=\mathrm{Li}_{\boldsymbol{k}}\left(\boldsymbol{g},((1,\beta),(1,1-\beta));z\right)$. By Theorem 3, we can get

$$\mathcal{L}_{\boldsymbol{k}}^{(\beta)}\left(\frac{z}{z-1}\right)=-\mathcal{L}_{\boldsymbol{k}^\vee}^{(\beta)}(z). \tag{12}$$

In particular, when $\beta=0$, this equation gives

$$\mathrm{Li}_{\boldsymbol{k}}^{\star}\left(\frac{z}{z-1}\right)=-\mathrm{Li}_{\boldsymbol{k}^\vee}^{\star}(z).$$

3.2. A Case with One Parameter

Throughout this subsection we assume that $s=2$ and

$$\boldsymbol{g}=(g_0,g_1,g_2)\text{ with }g_0=\begin{pmatrix}1&-1\\1&-w\end{pmatrix},\ g_1=\begin{pmatrix}1&0\\0&1\end{pmatrix}\text{ and }g_2=\begin{pmatrix}1&-1\\1&-w\end{pmatrix},$$

$g_0 = \begin{pmatrix} 1 & -1 \\ 1 & -w \end{pmatrix}$	$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$g_2 = \begin{pmatrix} 1 & -1 \\ 1 & -w \end{pmatrix}$
$G_{g_0}(t) = \frac{1}{1-t} - \frac{w}{1-wt}$	$G_{g_1}(t) = \frac{1}{t}$	$G_{g_2}(t) = \frac{1}{1-t} - \frac{w}{1-wt}$
$\tilde{g}_0 = \begin{pmatrix} 1 & -1 \\ w & -1 \end{pmatrix}$	$\tilde{g}_1 = \begin{pmatrix} 1 & 0 \\ 1+w & -1 \end{pmatrix}$	$\tilde{g}_2 = \begin{pmatrix} 1 & -1 \\ w & -1 \end{pmatrix}$
$G_{\tilde{g}_0}(t) = -\frac{1}{1-t} + \frac{w}{1-wt}$	$G_{\tilde{g}_1}(t) = \frac{1+w}{1-(1+w)t} + \frac{1}{t}$	$G_{\tilde{g}_2}(t) = -\frac{1}{1-t} + \frac{w}{1-wt}$

Table 3: Examples of $G_{g_i}(t)$ and $G_{\tilde{g}_i}(t)$ with a parameter w

where w is a real parameter with $-1 \leq w < 1$. The case $w = 0$ is the one treated in the previous section.

We also give a correspondence between g and $G_g(t)$ in Table 3.

If $\boldsymbol{\beta} = (1, 0)$ and $\boldsymbol{\gamma} = (0, 1)$, then the function $\mathrm{Li}_{\boldsymbol{k}}(\boldsymbol{g}, (\boldsymbol{\beta}, \boldsymbol{\gamma}); z)$ coincides with the function $\mathrm{Li}_{\boldsymbol{k}}^{(w)}(z)$ defined by the author [5] (in [5] the parameter c is used instead of w and the function is denoted by $\mathrm{Li}_{\boldsymbol{k}}^{c}(z)$). The value $\mathrm{Li}_{\boldsymbol{k}}^{(w)}(1)$ is the *multiple T-value with one parameter* defined by Chapoton [2].

When $w = 0$, we have $\mathrm{Li}_{\boldsymbol{k}}^{(0)}(z) = \mathrm{Li}_{\boldsymbol{k}}(z)$ and when $w = -1$, we have $\mathrm{Li}_{\boldsymbol{k}}^{(-1)}(z) = 2^{d(\boldsymbol{k})}\mathrm{Ath}(\boldsymbol{k}, z)$. Here $\mathrm{Ath}(\boldsymbol{k}, z)$ is a kind of multiple polylogarithm of level two defined by

$$\mathrm{Ath}(\boldsymbol{k}, z) := \sum_{\substack{m_i \equiv 1(2) \\ m_i > 0}} \frac{z^{m_1+\cdots+m_r}}{m_1^{k_1}(m_1+m_2)^{k_2}\cdots(m_1+\cdots+m_r)^{k_r}}$$

(see [7, Eq. (5.1)]). By Theorem 2, we get the following Landen-type connection formula:

$$\mathrm{Li}_{\boldsymbol{k}}^{(w)}\left(\frac{z}{(w+1)z-1}\right) = (-1)^{d(\boldsymbol{k})}\widetilde{\mathrm{Li}}_{\boldsymbol{k}}^{(w)}(z).$$

Here

$$\widetilde{\mathrm{Li}}_{\boldsymbol{k}}^{(w)}(z) = \sum_{m_1 \geq 1, m_2 \geq 0, \ldots, m_k \geq 0} \frac{R(1)\cdots R(k)}{m_1(m_1+m_2)\cdots(m_1+\cdots+m_k)} z^{m_1+\cdots+m_k},$$

where

$$R(i) := \begin{cases} 1 - w^{m_i} & (i \in \{1, k_1+1, \ldots, k_1+\cdots+k_{r-1}+1\}), \\ (1+w)^{m_i} & (\text{otherwise}). \end{cases}$$

4. Generalized Poly-Bernoulli Numbers

Imatomi [8] introduced two kinds of multi-poly-Bernoulli-star numbers $C_{n,\star}^{(\boldsymbol{k})}$ and $B_{n,\star}^{(\boldsymbol{k})}$ $(n \geq 0)$ by the following generating series:

$$\frac{\mathrm{Li}_{\boldsymbol{k}}^{\star}(1-e^{-t})}{e^t-1} = \sum_{n=0}^{\infty} C_{n,\star}^{(\boldsymbol{k})} \frac{t^n}{n!}, \tag{13}$$

$$\frac{\mathrm{Li}_{\boldsymbol{k}}^{\star}(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_{n,\star}^{(\boldsymbol{k})} \frac{t^n}{n!}. \tag{14}$$

The following is a duality formula for multi-poly-Bernoulli-star numbers.

Theorem 4 ([8, Theorem 3.2])**.** *For an index $\boldsymbol{k} \in \mathbb{Z}_{>0}^r$ and an integer $n \geq 0$, we have that*

$$C_{n,\star}^{(\boldsymbol{k})} = (-1)^n B_{n,\star}^{(\boldsymbol{k}^\vee)}.$$

It can be easily checked that the left-hand sides of Equations (13) and (14) can be written as

$$\frac{\mathrm{Li}_{\boldsymbol{k}}^{\star}(1-e^{-t})}{e^t-1} = \frac{d}{dt}\mathrm{Li}_{\boldsymbol{k}_\uparrow}^{\star}(1-e^{-t}),$$
$$\frac{\mathrm{Li}_{\boldsymbol{k}}^{\star}(1-e^{-t})}{1-e^{-t}} = \frac{d}{dt}\mathrm{Li}_{\boldsymbol{k}_\rightarrow}^{\star}(1-e^{-t}).$$

As an analogy, we define two types of Bernoulli numbers $C_n^{(\boldsymbol{k},\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma}))}$ and $B_n^{(\boldsymbol{k},\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma}))}$ as

$$\frac{d}{dt}\mathrm{Li}_{\boldsymbol{k}_\uparrow}\left(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma}); \frac{e^t-1}{c_0e^t+d_0}\right) = \sum_{n=0}^{\infty} C_n^{(\boldsymbol{k},\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma}))} \frac{t^n}{n!},$$
$$\frac{d}{dt}\mathrm{Li}_{\boldsymbol{k}_\rightarrow}\left(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma}); \frac{e^t-1}{c_0e^t+d_0}\right) = \sum_{n=0}^{\infty} B_n^{(\boldsymbol{k},\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma}))} \frac{t^n}{n!}.$$

These numbers are generalizations of multi-poly-Bernoulli-star numbers defined by Equations (13) and (14). In fact, if $\boldsymbol{g}$ satisfies the condition (9) and $\boldsymbol{\beta} = (1,1)$ and $\boldsymbol{\gamma} = (0,1)$, then $\mathrm{Li}_{\boldsymbol{k}}(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma}); z) = \mathrm{Li}_{\boldsymbol{k}}^{\star}(z)$ and we have $C_n^{(\boldsymbol{k},\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma}))} = C_{n,\star}^{(\boldsymbol{k})}$ and $B_n^{(\boldsymbol{k},\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma}))} = B_{n,\star}^{(\boldsymbol{k})}$. By definition, it is clear that $B_n^{(\boldsymbol{k},\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma}))} = C_n^{(\boldsymbol{k}^\vee,\boldsymbol{g},(\boldsymbol{\gamma},\boldsymbol{\beta}))}$ and these values are essentially the same objects.

By straightforward calculation, an explicit expression of the generating function of $C_n^{(\boldsymbol{k},\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma}))}$ can be given as

$$\begin{aligned}\frac{d}{dt}\mathrm{Li}_{\boldsymbol{k}_\uparrow}&\left(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma});\frac{e^t-1}{c_0e^t+d_0}\right)\\&=\sum_{i=1}^{s}\beta_i\left(\frac{a_id_0+c_i}{(a_ic_0-c_i)e^t+a_id_0+c_i}-\frac{b_id_0+d_i}{(b_ic_0-d_i)e^t+b_id_0+d_i}\right)\\&\quad\times\mathrm{Li}_{\boldsymbol{k}}\left(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma});\frac{e^t-1}{c_0e^t+d_0}\right).\end{aligned}\tag{15}$$

The generating function of $B_n^{(\boldsymbol{k},\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma}))}$ is obtained by replacing β_i with γ_i in the right-hand side of the above equation.

By the sum formula (Proposition 1), we get the following proposition.

Proposition 2. *For an integer $k\geq 1$, we have*

$$\sum_{|\boldsymbol{k}|=k}\mu^{d(\boldsymbol{k})-1}C_n^{(\boldsymbol{k},\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma}))}=B_n^{(k,\boldsymbol{g},(\boldsymbol{\beta}+\mu\boldsymbol{\gamma},\boldsymbol{\beta}))}=C_n^{(\{1\}^k,\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\beta}+\mu\boldsymbol{\gamma}))},\tag{16}$$

$$\sum_{|\boldsymbol{k}|=k}\mu^{d(\boldsymbol{k})-1}B_n^{(\boldsymbol{k},\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma}))}=B_n^{(k,\boldsymbol{g},(\boldsymbol{\beta}+\mu\boldsymbol{\gamma},\boldsymbol{\gamma}))}=C_n^{(\{1\}^k,\boldsymbol{g},(\boldsymbol{\gamma},\boldsymbol{\beta}+\mu\boldsymbol{\gamma}))}.\tag{17}$$

Here $\{1\}^k$ stands for the index $(\overbrace{1,\dots,1}^{k})$ for $k\geq 1$.

Proof. We prove only Identity (16), and Identity (17) can be proved similarly. By Proposition 1, we have

$$\sum_{|\boldsymbol{k}|=k}\mu^{d(\boldsymbol{k})-1}\mathrm{Li}_{\boldsymbol{k}}\left(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma});\frac{e^t-1}{c_0e^t+d_0}\right)=\mathrm{Li}_k\left(\boldsymbol{g},(\boldsymbol{\beta}+\mu\boldsymbol{\gamma},\boldsymbol{\beta});\frac{e^t-1}{c_0e^t+d_0}\right).\tag{18}$$

By multiplying

$$\sum_{i=1}^{s}\beta_i\left(\frac{a_id_0+c_i}{(a_ic_0-c_i)e^t+a_id_0+c_i}-\frac{b_id_0+d_i}{(b_ic_0-d_i)e^t+b_id_0+d_i}\right)$$

both sides of Equation (18) and by considering the generating function (15) of $C_n^{(\boldsymbol{k},\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma}))}$, we have

$$\sum_{|\boldsymbol{k}|=k}\mu^{d(\boldsymbol{k})-1}\sum_{n=0}^{\infty}C_n^{(\boldsymbol{k},\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma}))}\frac{t^n}{n!}=\sum_{n=0}^{\infty}B_n^{(k,\boldsymbol{g},(\boldsymbol{\beta}+\mu\boldsymbol{\gamma},\boldsymbol{\beta}))}\frac{t^n}{n!}.\tag{19}$$

By comparing the coefficients, we obtain the first equation of (16). The second equation is obtained immediately because of $(k)^\vee=\{1\}^k$. □

By using Theorem 2, we can get the following theorem.

Theorem 5. *For any index $\boldsymbol{k}\in\mathbb{Z}_{>0}^r$, we have*

$$C_n^{(\boldsymbol{k},\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma}))}=(-1)^{n-1}C_n^{(\boldsymbol{k},\tilde{\boldsymbol{g}},(\boldsymbol{\beta},\boldsymbol{\gamma}))}\quad(n\geq 0).$$

Proof. Set $u(z):=\dfrac{z}{(c_0-d_0)z-1}$ and $z(t):=\dfrac{e^t-1}{c_0e^t+d_0}$. Then we can easily show that $u(z(t))=z(-t)$. By this identity and Theorem 2, we have

$$\mathrm{Li}_{\boldsymbol{k}_\uparrow}\left(\boldsymbol{g},(\boldsymbol{\beta},\boldsymbol{\gamma});z(t)\right)=\mathrm{Li}_{\boldsymbol{k}_\uparrow}\left(\tilde{\boldsymbol{g}},(\boldsymbol{\beta},\boldsymbol{\gamma});z(-t)\right).$$

By differentiating both sides in t and comparing the coefficients, the desired identity is obtained. □

In the last of the paper, we give a formula which generalizes Theorem 4. Assume that $\boldsymbol{g}$ satisfies the condition (9). Substituting $z=1-e^t$ in Identity (12), we have

$$\mathcal{L}_{\boldsymbol{k}}^{(\beta)}\left(1-e^{-t}\right)=-\mathcal{L}_{\boldsymbol{k}^\vee}^{(\beta)}\left(1-e^{t}\right).$$

By the relation $(\boldsymbol{k}_\uparrow)^\vee=(\boldsymbol{k}^\vee)_\rightarrow$, we have

$$\mathcal{L}_{\boldsymbol{k}_\uparrow}^{(\beta)}\left(1-e^{-t}\right)=-\mathcal{L}_{(\boldsymbol{k}^\vee)_\rightarrow}^{(\beta)}\left(1-e^{t}\right).$$

By differentiating both sides of this equation with respect to t and by comparing the coefficients, we obtain that

$$C_n^{(\boldsymbol{k},\boldsymbol{g},((1,\beta),(1,1-\beta)))}=(-1)^nB_n^{(\boldsymbol{k}^\vee,\boldsymbol{g},((1,\beta),(1,1-\beta)))}\quad(n\geq 0).$$

This formula gives Theorem 4 when $\beta=0$.

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