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NEW CONGRUENCES FOR PARTITIONS WHERE THE EVEN PARTS ARE DISTINCT

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Abstract

We denote the number of partitions of n wherein the even parts are distinct (and the odd parts are unrestricted) by $ped(n)$. In this paper, we will use generating function manipulations to obtain new congruences for $ped(n)$ modulo 24.

1. Introduction and Main Result

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is equal to n. If $p(n)$ denotes the number of partitions of a positive integer n and we adopt the convention $p(0) = 1$, then the generating function for $p(n)$ satisfies the identity

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n), \quad |q| < 1.$$

Throughout this paper, we write

$$f_k := (q^k;q^k)_\infty, \quad \text{for any integer} \quad k \geq 1.$$

The number of partitions of n wherein the even parts are distinct (and the odd parts are unrestricted) is denoted by $ped(n)$. The generating function for $ped(n)$ [6] is

$$\sum_{n=0}^{\infty} ped(n)q^n = \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{(q^4;q^4)_\infty}{(q;q)_\infty}. \tag{1.1}$$

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Note that by (1.1), the number of partitions of n wherein the even parts are distinct (and the odd parts are unrestricted) equals the number of partitions of n with no parts divisible by 4, i.e., the 4-regular partitions (see [6] and references therein). In recent years many congruences for the number of 4-regular partitions have been discovered (see [2–4, 11, 14–17] and references therein).

Numerous congruence properties are known for the function $ped(n)$. For example, Andrews, Hirschhorn and Sellers [6] proved that for $\alpha \geq 1$ and $n \geq 0$,

$$\begin{aligned}
ped(3n+2) &\equiv 0 \pmod{2},\\
ped(9n+4) &\equiv 0 \pmod{4},\\
ped(9n+7) &\equiv 0 \pmod{12},\\
ped\left(3^{2\alpha+2}n + \frac{11 \cdot 3^{2\alpha+1} - 1}{8}\right) &\equiv 0 \pmod{2},\\
ped\left(3^{2\alpha+1}n + \frac{17 \cdot 3^{2\alpha} - 1}{8}\right) &\equiv 0 \pmod{6},\\
ped\left(3^{2\alpha+2}n + \frac{19 \cdot 3^{2\alpha+1} - 1}{8}\right) &\equiv 0 \pmod{6}.
\end{aligned}$$

Recently, Xia [5] obtained many interesting infinite families of congruences modulo 8 for $ped(n)$.

The aim of this paper is to establish new congruences modulo 24 for $ped(n)$. In the next theorem, we state our main results.

Theorem 1.1. *For every $n \geq 0$, we have*

$$\begin{aligned}
ped(225n+43) &\equiv 0 \pmod{24},\\
ped(225n+88) &\equiv 0 \pmod{24},\\
ped(225n+133) &\equiv 0 \pmod{24},\\
ped(225n+223) &\equiv 0 \pmod{24}.
\end{aligned}$$

Furthermore, for every $k \geq 1$ and $n \geq 0$, we have

$$ped(9n+7) \equiv ped\left(9 \cdot 5^{2k} n + \frac{57 \cdot 5^{2k} - 1}{8}\right) \pmod{24}.$$

The paper is organised as follows: In Section 2, we present some preliminaries required for our proofs. In Sections 3, we present the proof of Theorem 1.1.

2. Preliminaries

In this section, we collect the q-series identities that are used in our proofs. Recall that Ramanujan's general theta function $f(a,b)$ [1] is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$

Important special cases of $f(a,b)$ are the theta functions $\varphi(q)$, $\psi(q)$ and $f(-q)$, which satisfies the identities

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2},$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1},$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = (q;q)_\infty = f_1.$$

In terms of $f(a,b)$, Jacobi's triple product identity [1] is given by

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty.$$

Lemma 2.1 (Hirschhorn [10])**.** *We have that*

$$f_1 = f_{25}\left(R(q^5) - q - q^2 R(q^5)^{-1}\right), \tag{2.1}$$

where

$$R(q) = \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

3. Proof of Theorem 1.1

Andrews, Hirschhorn and Sellers [6] proved that

$$\sum_{n=0}^{\infty} ped(9n+7) q^n = 12 \frac{f_2^4 f_3^6 f_4}{f_1^{11}}. \tag{3.1}$$

Therefore,

$$ped(9n+7) \equiv 0 \pmod{12}. \tag{3.2}$$

It follows from (3.1) that

$$\sum_{n=0}^{\infty} ped(9n+7)q^n \equiv 12\frac{f_2^4 f_3^6 f_4}{f_1^{11}} \pmod{24}. \tag{3.3}$$

But, by the binomial theorem, $f_t^{2m} \equiv f_{2t}^m \pmod{2}$, for all positive integers t and m.

Therefore, it follows from (3.3) that

$$\sum_{n=0}^{\infty} ped(9n+7)q^n \equiv 12 f_1 f_6 f_{12} \pmod{24}. \tag{3.4}$$

Employing (2.1) in (3.4), we arrive at

$$\begin{aligned}\sum_{n=0}^{\infty} ped(9n+7)q^n \equiv 12 f_{25} f_{150} f_{300} \Bigg(& R_{30}R_5R_{60} - R_{30}R_{60}q^2 - \frac{R_{30}R_{60}q^2}{R_5} + R_5R_{60}q^6 \\ & + R_{60}q^7 + \frac{R_{60}q^8}{R_5} - R_{30}R_5q^{12} - \frac{R_5R_{60}q^{12}}{R_{30}} + R_{30}q^{13} + \frac{R_{60}q^{13}}{R_{30}} \\ & + \frac{R_{30}q^{14}}{R_5} + \frac{R_{60}q^{14}}{R_{30}R_5} + R_5q^{18} - q^{19} - \frac{q^{20}}{R_5} + \frac{R_5q^{24}}{R_{30}} - \frac{R_{30}R_5q^{24}}{R_{60}} \\ & - \frac{q^{25}}{R_{30}} + \frac{R_{30}q^{25}}{R_{60}} - \frac{q^{26}}{R_{30}R_5} + \frac{R_{30}q^{26}}{R_5R_{60}} + \frac{R_5q^{30}}{R_{60}} - \frac{q^{31}}{R_{60}} - \frac{q^{32}}{R_5R_{60}} \\ & + \frac{R_5q^{36}}{R_{30}R_{60}} - \frac{q^{37}}{R_{30}R_{60}} - \frac{q^{38}}{R_{30}R_5R_{60}} \Bigg) \pmod{24}. \end{aligned} \tag{3.5}$$

Extracting the terms involving q^{5n+4} from both sides of (3.5), dividing both sides by q^4 and then replacing q^5 by q, yields

$$\sum_{n=0}^{\infty} ped(9(5n+4)+7)q^n \equiv 12 f_5 f_{30} f_{60} \left(2q^2\frac{R_6}{R_1} - q^3\right) \pmod{24},$$

from which it follows that

$$\sum_{n=0}^{\infty} ped(45n+43)q^n \equiv 12 q^3 f_5 f_{30} f_{60} \pmod{24}. \tag{3.6}$$

Next, equating the coefficients of q^{5n+j} on both sides of this congruence, where $j = 0, 1, 2, 4$, gives the congruences in Theorem 1.1.

Further, extracting the terms involving q^{5n+3} from both sides of (3.6), dividing both sides by q^3 and then replacing q^5 by q, yields

$$ped(225n+178) \equiv 12 f_1 f_6 f_{12} \pmod{24}.$$

which is equivalent to

$$ped(9n + 7) \equiv ped(225n + 178) \pmod{24}. \tag{3.7}$$

Successive iterations of (3.7) give

$$\begin{aligned} ped(9n + 7) &\equiv ped(9(25n + 19) + 7) \\ &\equiv ped(225(25n + 19) + 178) \\ &\equiv ped(9 \cdot 5^4 n + 9 \cdot 5^2 \cdot 19 + 9 \cdot 19 + 7) \\ &\ \ \vdots \\ &\equiv ped(9 \cdot 5^{2k} n + 9 \cdot 19 \cdot 5^{2k-2} + \ldots + 9 \cdot 19 + 7) \\ &\equiv ped\left(9 \cdot 5^{2k} n + \frac{57 \cdot 5^{2k} - 1}{8}\right) \pmod{24}. \end{aligned}$$

This completes the proof.

The author would like to end this section with the following conjecture:

Conjecture 3.1. For each nonnegative integer n,

$$\begin{aligned} ped(225n + 43) &\equiv 0 \pmod{192}, \\ ped(225n + 88) &\equiv 0 \pmod{192}, \\ ped(225n + 133) &\equiv 0 \pmod{192}, \\ ped(225n + 223) &\equiv 0 \pmod{192}. \end{aligned}$$

4. Concluding Remarks

Recently, Chen [14] proved some vanishing results on the coefficients of $\theta_\chi(z)$ and the product of two theta functions. Using these results and some generating function manipulations we can find many more congruences for $ped(n)$ modulo 24.

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References

[1] B. C. Berndt, *Number Theory in the Spirit of Ramanujan*, AMS, Providence, 2006.

[2] B. Dandurand and D. Penniston, ℓ-divisibility of ℓ-regular partition functions, *Ramanujan J.* **19** (2009), 63–70.

[3] B. Gordon and K. Ono, Divisibility of certain partition functions by powers of primes, *Ramanujan J.* **1** (1997), 25–34.

[4] D. Furcy and D. Penniston, Congruences for ℓ-regular partition functions modulo 3, *Ramanujan J.* **27** (2012), 101–108.

[5] E. X. W. Xia, New infinite families of congruences modulo 8 for partitions with even parts distinct, *The Electronic Journal of Combinatorics* **21** (2014), 4–8.

[6] G.E. Andrews, M.D. Hirschhorn and J.A. Sellers, Arithmetic properties of partitions with even parts distinct, *Ramanujan J.* **22** (2010), 273–284.

[7] H. Dai, Congruences for the number of partitions and bipartitions with distinct even parts, *Discrete Math.* **338** (2015), 133–138.

[8] H. Nath, The pod function and its connection with other partition functions, *Rad HAZU, Matematičke Znanosti*, to appear.

[9] H. Nath, Parity results of PEND partition, preprint `arXiv:2407.10428`.

[10] M. D. Hirschhorn, *The power of q, a personal journey*, Developments in Mathematics **49**, Springer, 2017.

[11] O. Y. M. Yao, New congruences modulo powers of 2 and 3 for 9-regular partitions, *J. Number Theory* **142** (2014), 89–101.

[12] S. C. Chen, On the number of partitions with distinct even parts, *Discrete Math.* **313** (2013), 1565–1568.

[13] S. C. Chen, The Number of Partitions with Distinct Even Parts Revisited, *Discrete Math.* **346** (2023), 133–138.

[14] S. C. Chen, Partition congruences and vanishing coefficients of products of theta functions, *Ramanujan J.* **62** (2023), 1125–1144.

[15] S. P. Cui and N. S. S. Gu, Arithmetic properties of ℓ-regular partitions, *Adv. Appl. Math.* **51** (2013), 507–523.

[16] S. P. Cui and N. S. S. Gu, Congruences for 9-regular partitions modulo 3, *Ramanujan J.* **38** (2015), 503–512.

[17] W. J. Keith, Congruences for 9-regular partitions modulo 3, *Ramanujan J.* **35** (2014), 157–164.