#A80 INTEGERS 24 (2024)

DIVISIBILITY PROPERTIES FOR OVERCUBIC PARTITION TRIPLES

S. Shivaprasada Nayaka[1]
JSS Banashankari Arts, Commerce and S. K. Gubbi Science College, Vidyagiri, Dharwad, Karnataka, India.
shivprasadnayaks@gmail.com

B. N. Dharmendra
Department of Mathematics, Government College of Women, Kolar, Karnataka, India.
bndharma@gmail.com

M. C. Mahesh Kumar
Department of Mathematics, Government First Grade College, K R Pura, Bengaluru, Karnataka, India.
softmahe15@gmail.com

Received: 4/19/24, Accepted: 8/14/24, Published: 9/16/24

Abstract

Let $\overline{bt}(n)$ counts all of the overlined version of the cubic partition triples of a positive integer n. In this paper, we obtain several infinite families of congruences modulo small powers of 2 for $\overline{bt}(n)$. For example, we obtain $\overline{bt}(8n+7) \equiv 0 \pmod{32}$ and $\overline{bt}(8 \cdot 9^{\alpha+2}n + 33 \cdot 9^{\alpha+1}) \equiv 0 \pmod{8}$, for all nonnegative integers α and n.

1. Introduction

A *partition* of a positive integer n is a finite non-increasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of a positive integer n, whose generating function is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \frac{1}{E_1},$$

where

$$E_k := (q^k;q^k)_\infty = \prod_{n=1}^{\infty}(1-q^{nk}).$$

DOI: 10.5281/zenodo.13768824

[1]Corresponding author

Chan [2, 3, 4] studied the congruence properties of the *cubic partition function* $a(n)$, the function that counts the number of partitions of n in which the even parts can appear in two colors, whose generating function for $a(n)$ is given by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty} = \frac{1}{E_1 E_2}.$$

He obtained the identity

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\frac{E_3^3 E_6^3}{E_1^4 E_2^4},$$

which implies

$$a(3n+2) \equiv 0 \pmod 3.$$

In [8] Kim studied the number of *overcubic partition function* $\overline{a}(n)$, the function that counts all of the overlined version of the cubic partitions counted by $a(n)$. In this case, the first instance of each part is allowed to be overlined (although such overlining is not required). whose generating function for $\overline{a}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{a}(n)q^n = \frac{(-q;q)_\infty (-q^2;q^2)_\infty}{(q;q)_\infty (q^2;q^2)_\infty} = \frac{E_4}{E_1^2 E_2}.$$

Kim obtained the following identity by using the theory of modular forms:

$$\sum_{n=0}^{\infty} \overline{a}(3n+2)q^n = 3\frac{E_3^6 E_4^3}{E_1^8 E_2^3}.$$

Hirschhorn [6] gave an elementary proof of the result satisfied by $\overline{a}(n)$, which is appeared in Kim's paper [8]. Sellers [16] has proved a number of arithmetic properties of $\overline{a}(n)$. Zhao and Zhong [17] studied the number of cubic partition pairs, denoted by $b(n)$, whose generating function is

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{(q;q)_\infty^2 (q^2;q^2)_\infty^2} = \frac{1}{E_1^2 E_2^2}.$$

Recently, Kim [9] studied congruence properties of $\overline{b}(n)$, which denotes the number of overcubic partition pairs of n, whose generating function is given by

$$\sum_{n=0}^{\infty} \overline{b}(n)q^n = \frac{(-q;q)_\infty^2 (-q^2;q^2)_\infty^2}{(q;q)_\infty^2 (q^2;q^2)_\infty^2} = \frac{E_4^2}{E_1^4 E_2^2}.$$

More recently, Many authors have obtained families of congruences satisfied by $\overline{b}(n)$. One can see [10, 11, 12, 13, 14, 15].

By the motivation of the above works, we will continue to study the divisibility properties of the function $\overline{bt}(n)$, the number of overcubic partition triples of a positive integer n, whose generating function is given by

$$\sum_{n=0}^{\infty} \overline{bt}(n)q^n = \frac{(-q;q)_\infty^3(-q^2;q^2)_\infty^3}{(q;q)_\infty^3(q^2;q^2)_\infty^3} = \frac{E_4^3}{E_1^6E_2^3}. \tag{1}$$

The main purpose of this paper is to prove the following results.

Theorem 1. *For any integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$\overline{bt}(8n+7) \equiv 0 \pmod{32}, \tag{2}$$
$$\overline{bt}(8n+5) \equiv 0 \pmod{8}, \tag{3}$$
$$\overline{bt}(72n+33) \equiv 0 \pmod{8}, \tag{4}$$
$$\overline{bt}(72n+57) \equiv 0 \pmod{8}, \tag{5}$$
$$\overline{bt}\left(8\cdot 9^{\alpha+2}n + 33\cdot 9^{\alpha+1}\right) \equiv 0 \pmod{8}. \tag{6}$$

Theorem 2. *For any prime $p \geq 5$, $\alpha \geq 0$ and $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{bt}\left(24p^{2\alpha}n + p^{3\alpha}\right)q^n \equiv 2E_1 \pmod{4}. \tag{7}$$

Theorem 3. *For any prime $p \geq 5$, $\alpha \geq 0$, $n \geq 0$ and $l = 1, 2, ...p-1$, we have*

$$\overline{bt}\left(24p^{2\alpha}(pn+l) + p^{3\alpha}\right) \equiv 0 \pmod{4}. \tag{8}$$

Theorem 4. *For any integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$\overline{bt}(16n+14) \equiv 0 \pmod{16}, \tag{9}$$
$$\overline{bt}(16n+10) \equiv 0 \pmod{16}, \tag{10}$$
$$\overline{bt}(144n+66) \equiv 0 \pmod{8}, \tag{11}$$
$$\overline{bt}(144n+114) \equiv 0 \pmod{8}, \tag{12}$$
$$\overline{bt}(1296n+594) \equiv 0 \pmod{8}, \tag{13}$$
$$\overline{bt}(1296n+1026) \equiv 0 \pmod{8}, \tag{14}$$
$$\overline{bt}\left(16\cdot 9^{\alpha+3}n + 66\cdot 9^{\alpha+2}\right) \equiv 0 \pmod{8}, \tag{15}$$
$$\overline{bt}(432n+18) \equiv \overline{bt}(48n+2) \pmod{8}, \tag{16}$$
$$\overline{bt}(432n+306) \equiv \overline{bt}(48n+34) \pmod{8}. \tag{17}$$

Theorem 5. *For any integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$\overline{bt}(32n+28) \equiv 0 \pmod{16}, \tag{18}$$
$$\overline{bt}(32n+20) \equiv 0 \pmod{16}, \tag{19}$$
$$\overline{bt}(64n+56) \equiv 0 \pmod{8}, \tag{20}$$
$$\overline{bt}(576n+408) \equiv 0 \pmod{8}, \tag{21}$$
$$\overline{bt}(5184n+216) \equiv 0 \pmod{8}, \tag{22}$$
$$\overline{bt}(5184n+3672) \equiv 0 \pmod{8}, \tag{23}$$
$$\overline{bt}\left(64 \cdot 81^{\alpha+2}n + 216 \cdot 81^{\alpha+1}\right) \equiv 0 \pmod{8}. \tag{24}$$

Theorem 6. *For any prime $p \geq 5$, $\alpha \geq 0$ and $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{bt}\left(576p^{2\alpha}n + 24p^{3\alpha}\right) q^n \equiv 4E_1 \pmod{8}. \tag{25}$$

Theorem 7. *For any prime $p \geq 5$, $\alpha \geq 0$, $n \geq 0$ and $l = 1, 2, ...p-1$, we have*

$$\overline{bt}\left(576p^{2\alpha}(pn+l) + 24p^{3\alpha}\right) \equiv 0 \pmod{8}. \tag{26}$$

Theorem 8. *For any integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$\overline{bt}(64n+40) \equiv 0 \pmod{8}, \tag{27}$$
$$\overline{bt}(576n+264) \equiv 0 \pmod{8}, \tag{28}$$
$$\overline{bt}(576n+456) \equiv 0 \pmod{8}, \tag{29}$$
$$\overline{bt}\left(64 \cdot 9^{\alpha+2}n + 264 \cdot 9^{\alpha+1}\right) \equiv 0 \pmod{8}. \tag{30}$$

Theorem 9. *For any prime $p \geq 5$, $\alpha \geq 0$ and $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{bt}\left(192p^{2\alpha}n + 8p^{3\alpha}\right) q^n \equiv 2E_1 \pmod{4}. \tag{31}$$

Theorem 10. *For any prime $p \geq 5$, $\alpha \geq 0$, $n \geq 0$ and $l = 1, 2, ...p-1$, we have*

$$\overline{bt}\left(192p^{2\alpha}(pn+l) + 8p^{3\alpha}\right) \equiv 0 \pmod{4}. \tag{32}$$

Theorem 11. *For any integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$\overline{bt}(64n+48) \equiv 0 \pmod{4}, \tag{33}$$
$$\overline{bt}(128n+80) \equiv 0 \pmod{4}, \tag{34}$$
$$\overline{bt}(128n+96) \equiv 0 \pmod{4}, \tag{35}$$
$$\overline{bt}(256n+160) \equiv 0 \pmod{4}, \tag{36}$$
$$\overline{bt}(256n+32) \equiv \overline{bt}(128n+16) \pmod{4}, \tag{37}$$
$$\overline{bt}\left(256 \cdot 2^{\alpha}n + 192\right) \equiv 0 \pmod{4}. \tag{38}$$

2. Preliminaries

Ramanujan's general theta function $f(a,b)$ is defined as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$

The product representation of $f(a,b)$ arises from Jacobi's triple product identity [1, p. 35, Entry 19] as

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty.$$

The most important special cases of $f(a,b)$ are as follows:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty = \frac{E_2^5}{E_1^2 E_4^2},$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{E_2^2}{E_1}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty = E_1.$$

Lemma 1 ([1, page 40, Entry 25]). We have

$$E_1^2 = \frac{E_2 E_8^5}{E_4^2 E_{16}^2} - 2q \frac{E_2 E_{16}^2}{E_8} \tag{39}$$

and

$$\frac{1}{E_1^2} = \frac{E_8^5}{E_2^5 E_{16}^2} + 2q \frac{E_4^2 E_{16}^2}{E_2^5 E_8}. \tag{40}$$

Lemma 2 ([1, page 345, Entry 1 (iv)]). We have

$$E_1^3 = \frac{E_6 E_9^6}{E_3 E_{18}^3} + 4q^3 \frac{E_3^2 E_{18}^6}{E_6^2 E_9^3} - 3q E_9^3. \tag{41}$$

Lemma 3 ([7]). We have

$$E_1 E_2 = \frac{E_6 E_9^4}{E_3 E_{18}^2} - q E_9 E_{18} - 2q^2 \frac{E_3 E_{18}^4}{E_6 E_9^2}. \tag{42}$$

Lemma 4 ([5, Theorem 2.2]). For any prime $p \geq 5$, then

$$E_1 = \sum_{\substack{k=\frac{1-p}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} E\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} E_{p^2}, \tag{43}$$

where

$$\frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod 6, \\ \frac{-p-1}{6}, & \text{if } p \equiv -1 \pmod 6. \end{cases}$$

Lemma 5. For any prime p and positive integer n, then

$$E_1^{p^n} \equiv E_p^{p^{n-1}} \pmod{p^n}. \tag{44}$$

3. Proofs of Main Results

In this section, we provide the proofs of Theorems 1 to 3. For brevity, we omit the proofs of Theorems 4, 5, 8 and 11 as they closely resemble the proof of Theorem 1. Similarly, we omit the proofs of Theorems 6 and 9 as well as Theorems 7 and 10 since they are similar to the proofs of Theorems 2 and 3, respectively.

Proof of Theorem 1. Employing (40) in (1), we arrive at

$$\sum_{n=0}^{\infty} \overline{bt}(n)q^n = \frac{E_4^3 E_8^{15}}{E_2^{18} E_{16}^6} + 6q\frac{E_4^5 E_8^9}{E_2^{18} E_{16}^2} + 12q^2\frac{E_4^7 E_8^3 E_{16}^2}{E_2^{18}} + 8q^3\frac{E_4^9 E_{16}^6}{E_2^{18} E_8^3}. \tag{45}$$

Extracting the terms involving odd powers of q from (45), we deduce that

$$\sum_{n=0}^{\infty} \overline{bt}(2n+1)q^n = 6\frac{E_2^5 E_4^9}{(E_1^2)^9 E_8^2} + 8q\frac{E_2^9 E_8^6}{(E_1^2)^9 E_4^3}. \tag{46}$$

Employing (40) in (46), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{bt}(2n+1)q^n \equiv{}& 6\frac{E_4^9 E_8^{43}}{E_2^{40} E_{16}^{18}} + 108q\frac{E_4^{11} E_8^{37}}{E_2^{40} E_{16}^{14}} + 96q^2\frac{E_4^{13} E_8^{31}}{E_2^{40} E_{16}^{10}} \\ &+ 64q^3\frac{E_4^{15} E_8^{25}}{E_2^{40} E_{16}^6} + 64q^4\frac{E_4^{17} E_8^{19}}{E_2^{40} E_{16}^2} \\ &+ 8q\frac{E_8^{51}}{E_2^{36} E_4^3 E_{16}^{18}} + 16q^2\frac{E_8^{45}}{E_2^{36} E_4 E_{16}^{14}} \pmod{128}. \end{aligned} \tag{47}$$

Extracting the terms involving odd powers of q from (47), we see that

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{bt}(4n+3)q^n \equiv{}& 108\frac{E_2^{11} E_4^{37}}{(E_1^2)^{20} E_8^{14}} + 64q\frac{E_2^{15} E_4^{25}}{(E_1^2)^{20} E_8^{16}} \\ &+ 8\frac{E_4^{51}}{(E_1^2)^{18} E_2^3 E_8^{18}} \pmod{128}. \end{aligned} \tag{48}$$

Employing (40) in (48), we arrive at

$$\sum_{n=0}^{\infty} \overline{bt}(4n+3)q^n \equiv 12\frac{E_4^{37}E_8^{86}}{E_2^{89}E_{16}^{40}} + 8\frac{E_4^{51}E_8^{108}}{E_2^{93}E_{16}^{36}} \pmod{34}. \tag{49}$$

Congruence (2) follows from (49).

Extracting the terms involving even powers of q from (47), we have

$$\sum_{n=0}^{\infty} \overline{bt}(4n+1)q^n \equiv 6\frac{E_2^{9}E_4^{43}}{E_1^{40}E_8^{18}} \pmod{8}. \tag{50}$$

Using (44) in (50), we obtain

$$\sum_{n=0}^{\infty} \overline{bt}(4n+1)q^n \equiv 6\frac{E_4^{7}}{E_2^{11}} \pmod{8}. \tag{51}$$

Congruence (3) follows from (51).

Extracting the terms involving even powers of q from (51), we see that

$$\sum_{n=0}^{\infty} \overline{bt}(8n+1)q^n \equiv 6\frac{E_2^{7}}{E_1^{11}} \pmod{8}. \tag{52}$$

Using (44) in (52), we get

$$\sum_{n=0}^{\infty} \overline{bt}(8n+1)q^n \equiv 6E_1E_2 \pmod{8}. \tag{53}$$

Employing (42) in (53), we obtain

$$\sum_{n=0}^{\infty} \overline{bt}(8n+1)q^n \equiv 6\frac{E_6E_9^{4}}{E_3E_{18}^{2}} + 2qE_9E_{18} + 4q^2\frac{E_3E_{18}^{4}}{E_6E_9^{2}} \pmod{8}. \tag{54}$$

Extracting the terms involving q^{3n+1} from (54), we arrive at

$$\sum_{n=0}^{\infty} \overline{bt}(24n+9)q^n \equiv 2E_3E_6 \pmod{8}. \tag{55}$$

Congruences (4) and (5) follow from (55).

Extracting the terms involving q^{3n} from (55), we deduce that

$$\sum_{n=0}^{\infty} \overline{bt}(72n+9)q^n \equiv 2E_1E_2 \pmod{8}. \tag{56}$$

In view of congruences (56) and (53), we establish that

$$\overline{bt}(72n+9) \equiv \overline{bt}(8n+1) \pmod{8}. \tag{57}$$

Using (57) and by the principle of mathematical induction, we have

$$\overline{bt}\left(8\cdot 9^{\alpha+1}n+9^{\alpha+1}\right)\equiv \overline{bt}(8n+1) \pmod 8. \tag{58}$$

Using (4) in (58), we get (6).

Proof of Theorem 2. Extracting the terms involving q^{3n} from (54), we arrive at

$$\sum_{n=0}^{\infty}\overline{bt}(24n+1)q^n\equiv 6\frac{E_2E_3^4}{E_1E_6^2} \pmod 8. \tag{59}$$

Using (44) in (59), we obtain

$$\sum_{n=0}^{\infty}\overline{bt}(24n+1)q^n\equiv 2E_1 \pmod 4. \tag{60}$$

Employing (43) in (60), we deduce that

$$\sum_{n=0}^{\infty}\overline{bt}\left(24\left(pn+\frac{p^2-1}{24}\right)+1\right)q^n\equiv 2E_p \pmod 4, \tag{61}$$

which implies

$$\sum_{n=0}^{\infty}\overline{bt}\left(24p^2n+p^3\right)q^n\equiv 2E_1 \pmod 4. \tag{62}$$

Therefore, it follows that

$$\overline{bt}\left(24p^2n+p^3\right)\equiv \overline{bt}(24n+1) \pmod 4.$$

Using the above relation and by the principle of mathematical induction on α, we arrive at (7).

Proof of Theorem 3. Combining Equation (61) with Equation (7), we derive that for $\alpha\geq 0$,

$$\sum_{n=0}^{\infty}\overline{bt}\left(24p^{2\alpha+1}n+p^{3\alpha}\right)q^n\equiv 2E_p \pmod 4.$$

Therefore, it follows that

$$\overline{bt}\left(24p^{2\alpha+1}(pn+l)+p^{3\alpha}\right)\equiv 0 \pmod 4.$$

where $l=1,2,...,p-1$, we obtain (8).

Acknowledgement. We are extremely grateful to the anonymous referees for their careful reading and suggestions.

References

[1] B. C. Berndt, *Ramanujan's Notebooks Part III*, Springer-Verlag, New York, 1991.

[2] H. C. Chan, Ramanujan's cubic continued fraction and a generalization of his "most beautiful identity", *Int. J. Number Theory* **6** (2010), 673–680.

[3] H. C. Chan, Ramanujan's cubic continued fraction and Ramanujan type congruences for a certain partition function, *Int. J. Number Theory* **6** (4) (2010), 819–834.

[4] H. C. Chan, Distribution of a certain partition function modulo powers of primes, *Acta Math. Sin.* **27** (2011), 625–634.

[5] S. P. Cui and N. S. S. Gu, Arithmetic properties of ℓ-regular partitions, *Adv. Appl. Math.* **51** (2013), 507–523.

[6] M. D. Hirschhorn, A note on overcubic partitions, *New Zealand J. Math.* **42** (2012), 229–234.

[7] M. D. Hirschhorn and J. A. Sellers, A congruence modulo 3 for partitions into distinct non-multiples of four, *J. Integer Seq.* **17** (2014), Article 14.9.6.

[8] B. Kim, The overcubic partition function mod 3, Ramanujan Rediscovered, *Ramanujan Math. Soc. Lect. Notes Ser.* **14** (2010), 157–163.

[9] B. Kim, On partition congruences for overcubic partition pairs, *Commun. Korean Math. Soc.* **27** (2012), 477–482.

[10] M. S. Mahadeva Naika and C. Shivashankar, New Congruence for overcubic partition pairs, *Adv. Stud. Euro-Tbil. Math. J.* **10** (4) (2017), 117—128.

[11] M. S. Mahadeva Naika and S. Shivaprasada Nayaka, Congruences for ℓ- regular cubic partition pairs, *Rend. Circ. Mat. Palermo (2)* **67** (2018), 465—476.

[12] M. S. Mahadeva Naika and S. Shivaprasada Nayaka, Congruences modulo large powers of 3 for 3^ℓ- regular cubic bipartition, *Proc. Jangjeon Math. Soc.* **23** (3) (2020), 399–411.

[13] M. S. Mahadeva Naika and T. Harishkumar, On ℓ-regular cubic partitions with odd parts overlined, *Bull. Braz. Math. Soc. (N.S.)* **53** (2022), 647—663.

[14] M. S. Mahadeva Naika, T. Harishkumar and Y. Veeranna, Some congruences for (2, 5)-regular cubic partition pairs, *Integers* **20** (2020), #A14.

[15] S. Shivaprasada Nayaka, Arithmetic properties of $(2, 3)$–regular overcubic bipartitions, *Arab. J. Math. Sci.* **29** (1) (2023), 119–129.

[16] J. A. Sellers, Elementary proofs of congruences for the cubic and overcubic partition functions, *Australas. J. Combin.* **60** (2) (2014), 191–197.

[17] H. Zhao and Z. Zhong, Ramanujan type congruences for a partition function, *Electron. J. Combin.* **18** (2011), #P58.