SOME DIRECT AND INVERSE PROBLEMS FOR THE RESTRICTED SIGNED SUMSET IN THE SET OF INTEGERS

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Abstract

Given a positive integer h and a nonempty finite set of integers $A = \{a_1, a_2, \ldots, a_k\}$, the *restricted h-fold signed sumset of A*, denoted by $h^{\wedge}_{\pm}A$, is defined as

$$h^{\wedge}_{\pm}A := \left\{ \sum_{i=1}^{k} \lambda_i a_i : \lambda_i \in \{-1, 0, 1\} \text{ for } i = 1, 2, \ldots, k \text{ and } \sum_{i=1}^{k} |\lambda_i| = h \right\}.$$

A direct problem associated with this sumset is to find the optimal lower bound for $|h^{\wedge}_{\pm}A|$, and an inverse problem associated with this sumset is to determine the structure of the underlying set A when $|h^{\wedge}_{\pm}A|$ attains the optimal lower bound. Bhanja, Komatsu and Pandey studied these problems for the restricted h-fold signed sumset for $h = 2$, 3, and k, and conjectured some direct and inverse results for $h \geq 4$. In this paper, we prove these conjectures for $h = 4$. We also prove some direct and inverse theorems for arbitrary h under certain restrictions on the set A, which are particular cases of the conjectures. Moreover, we prove the conjectures for arithmetic progressions.

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1. Introduction

Let $\mathbb{Z}$ denote the set of integers. For integers a and b with $a \leq b$, we denote the set $\{n \in \mathbb{Z} : a \leq n \leq b\}$ by $[a, b]$. For a nonempty finite set A of integers, let $\max(A)$, $\min(A)$, $\max_-(A)$, and $\min_+(A)$ denote the largest, the smallest, the second largest and the second smallest elements of A, respectively. For an integer c, we denote the set $\{ca : a \in A\}$ by $c * A$, and write $-A$ for $(-1) * A$. By a *k-term arithmetic progression of integers with common difference d*, we mean a set A of the form $\{a + id : i = 0, 1, \ldots, k-1\}$. Let $a, b, u, v, u_1, u_2, \ldots, u_n$ be integers. Then we write $a < \{u_1, u_2, \ldots, u_n\} < b$ to mean $a < u_i < b$ for all $i = 1, 2, \ldots, n$. We also write $a < \{u \text{ or } v\} < b$ to mean either $a < u < b$ or $a < v < b$. A set S is said to be *symmetric* if whenever $x \in S$, it is also true that $-x \in S$.

Let $A = \{a_1, a_2, \ldots, a_k\}$ be a nonempty finite subset of an additive abelian group G. For a positive integer h, the usual *h-fold sumset hA* and the *restricted h-fold sumset $h^\wedge A$* are defined as follows:

$$hA := \left\{ \sum_{i=1}^{k} \lambda_i a_i : \lambda_i \in [0, h] \text{ for } i = 1, 2, \ldots, k \text{ and } \sum_{i=1}^{k} \lambda_i = h \right\},$$

and

$$h^\wedge A := \left\{ \sum_{i=1}^{k} \lambda_i a_i : \lambda_i \in [0, 1] \text{ for } i = 1, 2, \ldots, k \text{ and } \sum_{i=1}^{k} \lambda_i = h \right\}.$$

Other related sumsets of two subsets A and B of G are the *Minkowski sumset* $A+B := \{a+b : a \in A, b \in B\}$ and the *restricted sumset* $A \dot{+} B := \{a+b : a \in A, b \in B, a \neq b\}$. The study of sumsets dates back to Cauchy [7] who obtained a lower bound for the cardinality of the sumset $A+B$, where A and B are nonempty subsets of the group of residue classes modulo a prime p. The result is known as the Cauchy-Davenport Theorem after Davenport rediscovered this result in 1935 [8, 9]. These types of sumsets have been studied extensively in the literature. A classical book by Nathanson [19] on additive number theory contains a detailed study of these sumsets and other kind of sumsets, and has a comprehensive bibliography (see [22] and [4] also). For some old and recent articles in the context of h-fold sumsets and restricted h-fold sumsets and their generalizations, see [10, 11, 14, 15, 18, 23, 16, 20, 21, 17].

Two other variants of these sumsets that appeared recently in the literature [1, 4, 6, 12, 13] are the *h-fold signed sumset $h_\pm A$* and the *restricted h-fold signed sumset $h^\wedge_\pm A$* of the set A. These are defined as follows:

$$h_\pm A := \left\{ \sum_{i=1}^{k} \lambda_i a_i : \lambda_i \in [-h, h] \text{ for } i = 1, 2, \ldots, k \text{ and } \sum_{i=1}^{k} |\lambda_i| = h \right\}$$

and

$$h^{\wedge}_{\pm}A := \left\{ \sum_{i=1}^{k} \lambda_i a_i : \lambda_i \in [-1,1] \text{ for } i = 1,2,\ldots,k \text{ and } \sum_{i=1}^{k} |\lambda_i| = h \right\}.$$

The study of the optimal lower bound for the cardinality of a sumset of a given set A is called a direct problem and the study of the structure of the underlying set A when the optimal lower bound of the cardinality of the sumset is known, is called an inverse problem. The problems associated with the sumsets $h_{\pm}A$ and $h^{\wedge}_{\pm}A$ are these direct and inverse problems.

The signed sumset first appeared in the work of Bajnok and Ruzsa [1] in the context of the independence number of a subset of an abelian group. Later, it also appeared in the work of Klopsch and Lev [12, 13] in a different context. Not much is known about the signed sumset. For the results in this contex, one may refer Bajnok and Matzke [2, 3]. Recently, Bhanja and Pandey [5] have studied some direct and inverse problems in the group of integers. They obtained the optimal lower bound for the cardinality of the sumset $h_{\pm}A$. They also proved that if the optimal lower bound is achieved by the cardinality of $h_{\pm}A$, then A must be an arithmetic progression.

Much less is known about the restricted signed sumset $h^{\wedge}_{\pm}A$. Recently, Bhanja et al. [6] solved some cases of the above mentioned direct and inverse problems for $h^{\wedge}_{\pm}A$ in $\mathbb{Z}$ and made conjectures in the rest of the cases. More precisely, they proved the following result.

Theorem 1 ([6, Theorem 2.1, Theorem 3.1])**.** *Let h and k be positive integers with $h \leq k$. Let A be a set of k nonnegative integers. If $0 \notin A$, then*

$$\left|h^{\wedge}_{\pm}A\right| \geq 2(hk - h^2) + \frac{h(h+1)}{2} + 1. \tag{1}$$

If $0 \in A$, then

$$\left|h^{\wedge}_{\pm}A\right| \geq 2(hk - h^2) + \frac{h(h-1)}{2} + 1. \tag{2}$$

These lower bounds are best possible for $h = 1, 2$, and k.

In the same paper, they also proved some inverse theorems for $h = 2$ and $h = k$ (see [6, Theorem 2.2, Theorem 2.3, Theorem 3.2, and Theorem 3.3]). It can be verified that the lower bounds in (1) and (2) are not optimal for $3 \leq h \leq k-1$. In this case, they made conjectures and proved them for the case $h = 3$ (see [6, Theorem 2.5 and Theorem 3.5])). The precise statements of the conjectures are the following.

Conjecture 1 ([6, Conjecture 2.4, Conjecture 2.6])**.** Let A be a set of $k \geq 4$ positive integers and let h be an integer with $3 \leq h \leq k-1$. Then

$$\left|h^{\wedge}_{\pm}A\right| \geq 2hk - h^2 + 1. \tag{3}$$

This lower bound is best possible. Moreover, if $|h^{\wedge}_{\pm}A| = 2hk - h^2 + 1$, then $A = d * \{1, 3, \ldots, 2k-1\}$ for some positive integer d.

Conjecture 2 ([6, Conjecture 3.4, Conjecture 3.7])**.** Let A be a set of $k \geq 5$ nonnegative integers with $0 \in A$ and let h be an integer with $3 \leq h \leq k-1$. Then

$$\left|h^{\wedge}_{\pm}A\right| \geq 2hk - h(h+1) + 1.$$

This lower bound is best possible. Moreover, if $\left|h^{\wedge}_{\pm}A\right| = 2hk - h(h+1) + 1$, then $A = d * [0, k-1]$ for some positive integer d.

In Section 3, We prove these conjectures for the case $h = 4$. In Section 2, we prove some auxiliary lemmas which are crucial for the proofs of the conjectures for $h = 4$. Using these lemmas, we also prove the conjectures for certain special type of sets including arithmetic progression (see the results in Subsection 2.2). We remark that Bhanja et al. [6] proved the lower bound in (3) for super increasing sequences.

2. Two Auxiliary Lemmas and Some Special Cases

2.1. Auxiliary Lemmas

We need the following result for the proofs of Lemma 1 and Lemma 2.

Theorem 2 ([19, Theorem 1.9, Theorem 1.10])**.** *Let A be a nonempty finite set of integers and let $1 \leq h \leq |A|$. Then*

$$|h^{\wedge}A| \geq h|A| - h^2 + 1.$$

Furthermore, if $|A| \geq 5$, $2 \leq h \leq |A| - 2$, and $|h^{\wedge}A| = h|A| - h^2 + 1$, then A is an arithmetic progression.

Lemma 1. *Let h and k be integers such that $3 \leq h \leq k-1$. Let $A = \{a_1, a_2, \ldots, a_k\}$ be a finite set of k positive integers with $a_1 < a_2 < \cdots < a_k$. Let $A_{h+1} = \{a_1, a_2, \ldots, a_{h+1}\} \subseteq A$. If $\left|h^{\wedge}_{\pm}A_{h+1}\right| \geq (h+1)^2 + t$, where $t \geq 0$, then*

$$\left|h^{\wedge}_{\pm}A\right| \geq 2hk - h^2 + 1 + t.$$

Proof. Let $A' = \{a_2, a_3, \ldots, a_k\}$. Then $(-h^{\wedge}A') \cup h^{\wedge}_{\pm}A_{h+1} \cup h^{\wedge}A' \subseteq h^{\wedge}_{\pm}A$. Since

$$h^{\wedge}_{\pm}A_{h+1} \cap h^{\wedge}A' = \{a_2 + a_3 + \cdots + a_{h+1}\}$$

and

$$h^{\wedge}_{\pm}A_{h+1} \cap (-h^{\wedge}A') = \{-(a_2 + a_3 + \cdots + a_{h+1})\},$$

it follows from Theorem 2 that

$$\left|h^{\wedge}_{\pm}A\right| \geq \left|h^{\wedge}_{\pm}A_{h+1}\right| + 2\left|h^{\wedge}A'\right| - 2 \geq (h+1)^2 + t + 2(h(k-1) - h^2 + 1) - 2 = 2hk - h^2 + 1 + t.$$

This proves the lemma. $\square$

A similar argument also proves the following lemma.

Lemma 2. *Let h and k be integers such that $k \geq 5$ and $3 \leq h \leq k-1$. Let $A = \{a_0, a_1, a_2, \ldots, a_{k-1}\}$ be a finite set of k nonnegative integers with $0 = a_0 < a_1 < a_2 < \cdots < a_{k-1}$. Let $A_h = \{a_0, a_1, a_2, \ldots, a_h\} \subseteq A$. If $\left|h^{\wedge}_{\pm} A_h\right| \geq h(h+1)+1+t$, where $t \geq 0$, then*

$$\left|h^{\wedge}_{\pm} A\right| \geq 2hk - h(h+1) + 1 + t.$$

2.2. Some Special Cases

First we verify the Conjectures 1 and 2 for finite arithmetic progression.

Theorem 3. *Let $h \geq 3$ be an integer and let A be a $(h+1)$-term arithmetic progression of positive integers with common difference d. Then*

$$\left|h^{\wedge}_{\pm} A\right| \geq \begin{cases} (h+1)^2, & \text{if } d = 2\min(A); \\ (h+1)^2+1, & \text{otherwise.} \end{cases}$$

Furthermore, $\left|h^{\wedge}_{\pm} A\right| = (h+1)^2$ if and only if $d = 2\min(A)$.

Proof. Let $A = \{a_1, a_2, \ldots, a_{h+1}\}$, where $a_i = a_1 + (i-1)d$ for $i = 1, 2, \ldots, h+1$. Set $A_h = A \setminus \{a_{h+1}\}$. We use induction on h to prove the result. If $h = 3$, then $A = \{a_1, a_2, a_3, a_4\}$ with $a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = d$. Consider the following increasing sequence of the elements of $3^{\wedge}_{\pm} A$:

$$\begin{aligned} -a_4-a_3-a_2 &< -a_4-a_3-a_1 < -a_4-a_2-a_1 < \{-a_3-a_2-a_1 \text{ or } -a_4-a_2+a_1\} \\ &< -a_3-a_2+a_1 < -a_4+a_2-a_1 < -a_3+a_2-a_1 < -a_2+a_4-a_3 < -a_1+a_3-a_2 \\ &< a_1-a_2+a_3 < a_1-a_2+a_4 < -a_1+a_2+a_3 < a_1+a_2+a_3 < a_1+a_2+a_4 \\ &< a_1+a_3+a_4 < a_2+a_3+a_4. \end{aligned}$$

Since $d = 2a_1$ if and only if $-a_3-a_2-a_1 = -a_4-a_2+a_1$, it follows that

$$\left|3^{\wedge}_{\pm} A\right| \geq \begin{cases} 16, & \text{if } d = 2a_1; \\ 17, & \text{if } d \neq 2a_1. \end{cases}$$

Now assume that the result holds for $h-1 \geq 3$. Then

$$\left|(h-1)^{\wedge}_{\pm} A_h\right| \geq \begin{cases} h^2, & \text{if } d = 2a_1; \\ h^2+1, & \text{if } d \neq 2a_1. \end{cases}$$

Since $(h-1)^{\wedge}_{\pm} A_h + a_{h+1} \subseteq h^{\wedge}_{\pm} A$ and $|(h-1)^{\wedge}_{\pm} A_h + a_{h+1}| = |(h-1)^{\wedge}_{\pm} A_h|$, we have at least h^2 elements in $h^{\wedge}_{\pm} A$ if $d = 2a_1$ and at least h^2+1 elements in $h^{\wedge}_{\pm} A$ if $d \neq 2a_1$.

Now we construct some more elements of $h^{\wedge}_{\pm}A$ which are distinct from the elements of $(h-1)^{\wedge}_{\pm}A_h + a_{h+1}$. For each $i \in [3, h+1]$, let

$$S_i = -a_2 + a_i - \left(\sum_{j=3, j\neq i}^{h+1} a_j \right).$$

For each $i \in [3, h]$,

$$T_i = -a_1 + a_i - \left(\sum_{j=3, j\neq i}^{h+1} a_j \right).$$

Let

$$U_0 = -\sum_{j=2}^{h+1} a_j, \quad U_1 = -\sum_{j=1, j\neq 2}^{h+1} a_j,$$

and for each $i \in [2, 5]$,

$$U_i = a_1 - \left(\sum_{j=2, j\neq i}^{h+1} a_j \right).$$

It is easy to see that

$$\begin{aligned} \min((h-1)^{\wedge}_{\pm}A_h + a_{h+1}) = S_{h+1} > T_h > S_h > T_{h-1} > S_{h-1} > \cdots > T_3 \\ > S_3 = U_5 > U_4 > U_3 > U_2 > U_1 > U_0 = \min(h^{\wedge}_{\pm}A). \end{aligned}$$

Hence

$$\left| h^{\wedge}_{\pm}A \right| \geq \begin{cases} (h+1)^2, & \text{if } d = 2a_1; \\ (h+1)^2 + 1, & \text{if } d \neq 2a_1. \end{cases}$$

If $d = 2a_1$, then $A = a_1 * \{1, 3, 5, \ldots, 2h+1\}$. Therefore, if h is a positive odd integer, then $h^{\wedge}_{\pm}A$ contains only odd multiples of a_1 and if h is a positive even integer, then $h^{\wedge}_{\pm}A$ contains only even multiples of a_1. Since $h^{\wedge}_{\pm}A \subseteq a_1 * [-(h+1)^2+1, (h+1)^2-1]$, we get $\left| h^{\wedge}_{\pm}A \right| \leq (h+1)^2$. This completes the proof of the theorem. □

Corollary 1. *Let $h \geq 3$ be an integer. Let $A = \{a_1, a_2, \ldots, a_k\}$ be an arithmetic progression of integers of length $k \geq h+1$ with common difference d with $0 < a_1 < a_2 < \cdots < a_k$. Then*

$$\left| h^{\wedge}_{\pm}A \right| \geq \begin{cases} 2hk - h^2 + 1, & \textit{if } d = 2a_1; \\ 2hk - h^2 + 2, & \textit{otherwise}. \end{cases}$$

Furthermore, $\left| h^{\wedge}_{\pm}A \right| = 2hk - h^2 + 1$ if and only if $d = 2\min(A)$.

Proof. Let $A_{h+1} = \{a_1, a_2, \ldots, a_{h+1}\}$ and $A' = \{a_2, a_3, \ldots, a_k\}$. Then $(-h^\wedge A') \cup h^\wedge_\pm A_{h+1} \cup h^\wedge A' \subseteq h^\wedge_\pm A$. Note that

$$h^\wedge_\pm A_{h+1} \cap h^\wedge A' = \{a_2 + a_3 + \cdots + a_{h+1}\}$$

and

$$h^\wedge_\pm A_{h+1} \cap (-h^\wedge A') = \{-(a_2 + a_3 + \cdots + a_{h+1})\}.$$

Using Theorem 2 and Theorem 3, we have

$$\begin{aligned} \left|h^\wedge_\pm A\right| &\geq \left|h^\wedge_\pm A_{h+1}\right| + 2\left|h^\wedge A'\right| - 2 \qquad (4)\\ &\geq \begin{cases} (h+1)^2 + 2(h(k-1) - h^2 + 1) - 2, & \text{if } d = 2a_1;\\ (h+1)^2 + 1 + 2(h(k-1) - h^2 + 1) - 2, & \text{otherwise.} \end{cases}\\ &= \begin{cases} 2hk - h^2 + 1, & \text{if } d = 2a_1;\\ 2hk - h^2 + 2, & \text{otherwise.} \end{cases} \end{aligned}$$

Now if $d = 2a_1$, then $A = a_1 * \{1, 3, 5, \ldots, 2k-1\}$ and so $\left|h^\wedge_\pm A\right| = 2hk - h^2 + 1$. If $\left|h^\wedge_\pm A\right| = 2hk - h^2 + 1$, then by (4), we get $\left|h^\wedge_\pm A_{h+1}\right| = (h+1)^2$ and so $d = 2a_1$. □

Theorem 4. *Let h and k be integers such that $4 \leq h \leq k-1$. If A is an arithmetic progression of k nonnegative integers with $0 \in A$, then $|h^\wedge_\pm A| = 2hk - h(h+1) + 1$.*

Proof. If A is an arithmetic progression of k nonnegative integers with $0 \in A$, then $A = d * [0, k-1]$, where d is the common difference of the arithmetic progression. Since the cardinality of the restricted h-fold signed sumset is dilation invariant, we may assume $A = [0, k-1]$. Clearly, $h^\wedge_\pm A$ contains disjoint sets $h^\wedge A$ and $h^\wedge(-A)$. It is easy to see that $h^\wedge A = \left[\frac{h(h-1)}{2}, hk - \frac{h(h+1)}{2}\right]$ and $h^\wedge(-A) = -(h^\wedge A)$. Now we construct some more elements of $h^\wedge_\pm A$ that are different from the elements of $h^\wedge A$ and $h^\wedge(-A)$. It is easy to see that

$$\begin{aligned} \max(h^\wedge(-A)) &= 0 - 1 - 2 - \cdots - (h-2) - (h-1)\\ &< 0 + 1 - 2 - \cdots - (h-2) - h\\ &< 0 + 1 - 2 - \cdots - (h-2) - (h-1)\\ &< 0 - 1 + 2 - \cdots - (h-2) - h\\ &< 0 - 1 + 2 - \cdots - (h-2) - (h-1)\\ &\ \vdots\\ &< 0 - 1 - \cdots - (h-3) + (h-2) - (h-1)\\ &< 0 + 1 - \cdots - (h-3) + (h-2) - h\\ &< 0 + 1 - \cdots - (h-3) + (h-2) - (h-1)\\ &< 0 - 1 + 2 \cdots - (h-3) + (h-2) - h \end{aligned}$$

$$
\begin{aligned}
&< 0 - 1 + 2 \cdots - (h-3) + (h-2) - (h-1) \\
&\vdots \\
&< 0 - 1 \cdots + (h-3) + (h-2) - h \\
&\vdots \\
&< 0 + 1 + 2 + \cdots + (h-2) - (h-1) \\
&= 0 - 1 + 2 + \cdots + (h-3) - (h-2) + (h-1) \\
&< 0 - 1 + 2 + \cdots + (h-3) - (h-2) + h \\
&< 0 - 1 + 2 + \cdots - (h-3) + (h-2) + (h-1) \\
&\vdots \\
&< 0 - 1 - 2 + \cdots + (h-2) + h \\
&< 0 + 1 - 2 + \cdots + (h-2) + (h-1) \\
&< 0 + 1 - 2 + \cdots + (h-2) + h \\
&< 0 - 1 + 2 + \cdots + (h-2) + (h-1) \\
&< 0 - 1 + 2 + \cdots + (h-2) + h < 0 + 1 + 2 + \cdots + (h-2) + (h-1) = \min(h^\wedge A).
\end{aligned}
$$

Hence $h^\wedge_\pm A = \left[-hk + \dfrac{h(h+1)}{2}, hk - \dfrac{h(h+1)}{2}\right]$ and $|h^\wedge_\pm A| = 2hk - h(h+1) + 1$. $\square$

Next theorem is a partial inverse theorem.

Theorem 5. *Let h and k be integers with $4 \le h \le k-1$. Let $A = \{a_1, a_2, \ldots, a_k\}$ be a set of k positive integers with $a_1 < a_2 < \cdots < a_k$. Let $A_{h+1} = \{a_1, a_2, \ldots, a_{h+1}\}$ and $A' = A \setminus \{a_1\}$. If*

$$|h^\wedge_\pm A| = 2hk - h^2 + 1 \tag{5}$$

and at least one of the following holds:

(a) *A is an arithmetic progression,*

(b) *A_{h+1} is an arithmetic progression,*

(c) *$\left|h^\wedge_\pm A_{h+1}\right| \ge (h+1)^2$ and $4 \le h \le k-3$,*

(d) *$h^\wedge_\pm A = h^\wedge(-A') \cup h^\wedge_\pm A_{h+1} \cup h^\wedge A'$ and A' is an arithmetic progression,*

(e) *$\left|h^\wedge_\pm A_{h+1}\right| \ge (h+1)^2$ and A' is an arithmetic progression,*

then $A = a_1 * \{1, 3, 5, \ldots, 2k-1\}$.

Proof. We have

$$(-h^\wedge A') \cup h^\wedge_\pm A_{h+1} \cup h^\wedge A' \subseteq h^\wedge_\pm A.$$

(a) If A is an arithmetic progression, then Corollary 1 implies that $A = a_1 * \{1, 3, 5, \ldots, 2k-1\}$.

(b) If A_{h+1} is an arithmetic progression, then by Theorem 3, $\left|h^{\wedge}_{\pm}A_{h+1}\right| \geq (h+1)^2$. If $k = h+1$, then $|h^{\wedge}_{\pm}A_{h+1}| = |h^{\wedge}_{\pm}A| = (h+1)^2$. So, by Theorem 3, we have $A_{h+1} = a_1 * \{1, 3, 5, \ldots, 2h+1\}$. Assume $k \geq h+2$. Note that $(-h^{\wedge}A') \cup h^{\wedge}_{\pm}A_{h+1} \cup h^{\wedge}A' \subseteq h^{\wedge}_{\pm}A$ and there is exactly one element in each of $h^{\wedge}_{\pm}A_h \cap h^{\wedge}A'$ and $h^{\wedge}_{\pm}A_h \cap (-h^{\wedge}A')$. Therefore, by Theorem 2, we have

$$\begin{aligned} 2hk - h^2 + 1 &= \left|h^{\wedge}_{\pm}A\right| \\ &\geq \left|h^{\wedge}_{\pm}A_{h+1}\right| + |h^{\wedge}A'| + |h^{\wedge}(-A')| - 2 \\ &\geq (h+1)^2 + 2h(k-1) - 2h^2 + 2 - 2 = 2hk - h^2 + 1. \end{aligned}$$

This gives

$$\left|h^{\wedge}_{\pm}A_{h+1}\right| = (h+1)^2. \tag{6}$$

So, by Theorem 3, $A_{h+1} = a_1 * \{1, 3, 5, \ldots, 2h+1\}$. Also, we have

$$|h^{\wedge}A'| + |h^{\wedge}(-A')| = 2h(k-1) - 2h^2 + 2 \tag{7}$$

and

$$h^{\wedge}_{\pm}A = h^{\wedge}(-A') \cup h^{\wedge}_{\pm}A_{h+1} \cup h^{\wedge}A'. \tag{8}$$

Let $x = a_1 + a_3 + \cdots + a_h + a_{h+2} \in h^{\wedge}_{\pm}A$. Note that

$$\begin{aligned} \max\nolimits_{-}(h^{\wedge}_{\pm}A_{h+1}) = a_1 + a_3 + \cdots + a_h + a_{h+1} < x \\ < a_2 + a_3 + \cdots + a_h + a_{h+2} = \min\nolimits_{+}(h^{\wedge}A') \end{aligned}$$

and

$$\max\nolimits_{-}(h^{\wedge}_{\pm}A_{h+1}) < \max(h^{\wedge}_{\pm}A_{h+1}) = \min(h^{\wedge}A') < \min\nolimits_{+}(h^{\wedge}A').$$

From (6), (7), and (8), we have

$$x = a_1 + a_3 + \cdots + a_h + a_{h+2} = a_2 + a_3 + \cdots + a_h + a_{h+1} = \max(h^{\wedge}_{\pm}A_{h+1}).$$

This gives $a_{h+2} - a_{h+1} = a_2 - a_1$. Therefore, if $k = h+2$, then $A = a_1 * \{1, 3, 5, \ldots, 2h+3\}$. If $4 \leq h \leq k-3$, then from (7) and Theorem 2, A' is an arithmetic progression. Hence $A = a_1 * \{1, 3, 5, \ldots, 2k-1\}$.

(c) If $\left|h^{\wedge}_{\pm}A_{h+1}\right| \geq (h+1)^2$ and $4 \leq h \leq k-3$, then by the similar argument as in (b), we get $a_{h+2} - a_{h+1} = a_2 - a_1$ and $|h^{\wedge}A'| = h(k-1) - h^2 + 1$. So, by Theorem 2, A' is an arithmetic progression. Consequently, A is an arithmetic progression. Since $|h^{\wedge}_{\pm}A| = 2hk - h^2 + 1$, we have by Corollary 1 that $A = a_1 * \{1, 3, 5, \ldots, 2k-1\}$.

(d) If $h^{\wedge}_{\pm}A = h^{\wedge}(-A') \cup h^{\wedge}_{\pm}A_{h+1} \cup h^{\wedge}A'$ and A' is an arithmetic progression, then $|h^{\wedge}A'| = h(k-1) - h^2 + 1$ and

$$\begin{aligned} 2hk - h^2 + 1 &= \left|h^{\wedge}_{\pm}A\right| \\ &= \left|h^{\wedge}_{\pm}A_{h+1}\right| + |h^{\wedge}A'| + |h^{\wedge}(-A')| - 2 \\ &= \left|h^{\wedge}_{\pm}A_{h+1}\right| + 2h(k-1) - 2h^2 + 2 - 2. \end{aligned}$$

Therefore, $\left|h^{\wedge}_{\pm}A_{h+1}\right| = (h+1)^2$. By the similar argument as in (b), we get $a_{h+2} - a_{h+1} = a_2 - a_1$. So, A is an arithmetic progression. Since $|h^{\wedge}_{\pm}A| = 2hk - h^2 + 1$, we have by Corollary 1 that $A = a_1 * \{1, 3, 5, \ldots, 2k-1\}$.

(e) This case is similar to the case (d).

□

Using a similar argument as in Theorem 5, we can also prove the following theorem.

Theorem 6. *Let h and k be integers with $4 \le h \le k-1$. Let $A = \{a_0, a_1, a_2, \ldots, a_{k-1}\}$ be a set of k nonnegative integers with $0 = a_0 < a_1 < a_2 < \cdots < a_{k-1}$. Let $A_h = \{a_0, a_1, a_2, \ldots, a_h\}$ and $A' = A \setminus \{a_0\}$. If*

$$|h^{\wedge}_{\pm}A| = 2hk - h(h+1) + 1 \tag{9}$$

and at least one of the following holds:

(a) *A is an arithmetic progression,*

(b) *A_h is an arithmetic progression,*

(c) *$\left|h^{\wedge}_{\pm}A_h\right| \ge h(h+1) + 1$ and $4 \le h \le k-3$,*

(d) *$h^{\wedge}_{\pm}A = h^{\wedge}(-A') \cup h^{\wedge}_{\pm}A_h \cup h^{\wedge}A'$ and A' is an arithmetic progression,*

(e) *$\left|h^{\wedge}_{\pm}A_h\right| \ge h(h+1) + 1$ and A' is an arithmetic progression,*

then $A = a_1 * [0, k-1]$.

Theorem 7. *Let $h \ge 3$ be an integer. Let $A = \{a_1, a_2, \ldots, a_{h+1}\}$ be a set of positive integers with $a_1 < a_2 < \cdots < a_{h+1}$ and $a_i \ge a_{i-1} + a_{i-2}$ for $i = 4, \ldots, h+1$. Then*

$$\left|h^{\wedge}_{\pm}A\right| \ge (h+1)^2 + 1. \tag{10}$$

Proof. We use induction on h to prove the lower bound in (10). The base case $h = 3$ was proved by Bhanja et al. (see the proof of Theorem 2.5 in [6]). So $h \ge 4$ and assume that the result holds for $h-1$. Let $A_h = \{a_1, a_2, \ldots, a_h\}$. Then by the induction hypothesis,

$$\left|(h-1)^{\wedge}_{\pm}A_h\right| \ge h^2 + 1.$$

Since $(h-1)^{\wedge}_{\pm}A_h + a_{h+1} \subseteq h^{\wedge}_{\pm}A$, it is sufficient to construct $2h+1$ more elements to complete the proof. Since $-a_{h-1} - a_h + a_{h+1} \geq a_{h-1} + a_h - a_{h+1}$, we have

$$\begin{aligned}\min((h-1)^{\wedge}_{\pm}A_h + a_{h+1}) &= -a_2 - a_3 - \ldots - a_{h-1} - a_h + a_{h+1} \\ &\geq -a_2 \ldots - a_{h-2} + a_{h-1} + a_h - a_{h+1}.\end{aligned}$$

Consider the following elements of $h^{\wedge}_{\pm}A$:

$$S_i = -\left(\sum_{j=2, j\neq i}^{h-1} a_j\right) + a_i + a_h - a_{h+1} \text{ for } i = 2, 3, \ldots, h-1;$$

$$T_i = -\left(\sum_{j=2, j\neq i}^{h+1} a_j\right) + a_i \text{ for } i = 2, \ldots, h-1;$$

$$X_1 = a_1 - \left(\sum_{i=3, i\neq h}^{h+1} a_i\right) + a_h,\ X = a_1 - \left(\sum_{i=3}^{h+1} a_i\right);$$

$$Y_1 = -a_1 - \left(\sum_{i=3, i\neq h}^{h+1} a_i\right) + a_h,\ Y = -a_1 - \left(\sum_{i=3}^{h+1} a_i\right);$$

$$Z_1 = -\left(\sum_{i=2, i\neq h}^{h+1} a_i\right) + a_h, \text{ and } Z = -\left(\sum_{i=2}^{h+1} a_i\right).$$

It is easy to see that

$$\begin{aligned}\min((h-1)^{\wedge}_{\pm}A_h + a_{h+1}) &\geq S_{h-1} > S_{h-2} > \cdots > S_2 > X_1 > Y_1 > Z_1 \\ &> T_{h-1} > T_{h-2} > \cdots > T_2 > X > Y > Z = \min(h^{\wedge}_{\pm}A).\end{aligned}$$

Hence

$$|h^{\wedge}_{\pm}A| \geq (h+1)^2 + 1.$$

□

Theorem 8. *Let $h \geq 3$ be an integer. Let $A = \{a_1, a_2, \ldots, a_{h+1}\}$ be a set of $h+1$ positive integers with $a_1 < a_2 < \cdots < a_{h+1}$, $a_3 - a_2 < 2a_1$, and $a_i - a_{i-1} > \dfrac{a_2 - a_1}{2}$ for $i = 4, \ldots, h+1$. Then*

$$|h^{\wedge}_{\pm}A| \geq (h+1)^2 + 1.$$

Proof. We use induction on h to prove the lower bound. The base case $h = 3$ is proved by Bhanja et al. (see the proof of Theorem 2.5 in [6]). Let $h \geq 4$ and assume the result holds for $h-1$. Let $A_h = \{a_1, a_2, \ldots, a_h\} \subseteq A$. Then the induction hypothesis implies that

$$\left|(h-1)^{\wedge}_{\pm}A_h\right| \geq h^2+1.$$

Since $(h-1)^{\wedge}_{\pm}A_h + a_{h+1} \subseteq h^{\wedge}_{\pm}A$, it is sufficient to construct $2h+1$ more elements to complete the proof. Let

$$S_1 = a_1 - \left(\sum_{j=3}^{h+1} a_j\right).$$

For each $i \in [2, h+1]$, let

$$S_i = a_i - \left(\sum_{j=2, j\neq i}^{h+1} a_i\right).$$

For $i \in [3, h]$, let

$$T_i = -a_1 + a_i - \left(\sum_{j=3, j\neq i}^{h+1} a_i\right).$$

For each $i \in [1, 3]$, let

$$U_i = -(\sum_{j=1, j\neq i}^{h+1} a_j).$$

It is easy to see that

$$\begin{aligned}\min((h-1)^{\wedge}_{\pm}A + a_{h+1}) = S_{h+1} > T_h > S_h > T_{h-1} > S_{h-1} > \cdots > T_3 \\ > S_3 > S_2 > S_1 > U_3 > U_2 > U_1 = \min(h^{\wedge}_{\pm}A).\end{aligned}$$

Hence

$$|h^{\wedge}_{\pm}A| \geq (h+1)^2+1.$$

□

3. Proofs of Conjecture 1 and Conjecture 2 for $h = 4$

In this section, we prove the following theorem which is a special case of Conjecture 1 in the case $h = 4$.

Theorem 9. *Let A be a set of $k \geq 5$ positive integers. Then*

$$\left|4^{\wedge}_{\pm}A\right| \geq 8k-15.$$

Furthermore, if $\left|4^{\wedge}_{\pm}A\right| = 8k-15$, then $A = \min(A) * \{1, 3, \ldots, 2k-1\}$.

In view of Lemma 1 and Theorem 5, it suffices to prove the following theorem.

Theorem 10. *Let A be a set of positive integers with $|A| = 5$. Then*

$$\left|4^{\wedge}_{\pm}A\right| \geq 25.$$

Furthermore, if $\left|4^{\wedge}_{\pm}A\right| = 25$, then $A = \min(A) * \{1,3,5,7,9\}$.

For the proof, we consider various cases as lemmas. Throughout this section, the following list of (not necessarily distinct) elements of $4^{\wedge}_{\pm}A$ is used in the proofs:

$x_1 = -a_1 + a_2 - a_3 + a_4,$

$x_2 = a_1 + a_2 - a_3 + a_4,$

$x_3 = a_1 - a_2 + a_3 + a_4,$

$x_4 = -a_1 + a_2 + a_3 + a_4,$

$x_5 = a_1 + a_2 + a_3 + a_4,$

$x_6 = a_1 + a_2 + a_3 + a_5,$

$x_7 = a_1 + a_2 + a_4 + a_5,$

$x_8 = a_1 + a_3 + a_4 + a_5,$

$x_9 = a_2 + a_3 + a_4 + a_5,$

$y_1 = -a_1 + a_2 + a_4 - a_5,$

$y_2 = -a_1 + a_2 - a_4 + a_5,$

$y_3 = -a_1 - a_3 + a_4 + a_5,$

$y_4 = -a_1 - a_2 + a_4 + a_5,$

$y_5 = a_1 - a_2 + a_4 + a_5,$

$y_6 = -a_1 + a_2 + a_4 + a_5,$

$y_7 = -a_1 + a_3 + a_4 + a_5,$

$z_1 = a_1 + a_2 + a_4 - a_5,$

$z_2 = a_1 + a_2 - a_4 + a_5,$

$z_3 = a_1 + a_2 - a_3 + a_5,$

$z_4 = a_1 - a_2 + a_3 + a_5,$

$z_5 = -a_1 + a_2 + a_3 + a_5,$

$\alpha_1 = -a_2 + a_3 - a_4 + a_5,$

$\alpha_2 = -a_2 - a_3 + a_4 + a_5,$

$\alpha_3 = -a_2 + a_3 + a_4 + a_5,$

$\beta_1 = -a_1 - a_2 + a_3 + a_4,$

$\beta_2 = -a_1 - a_2 + a_3 + a_5,$

$\gamma_1 = -a_1 + a_2 + a_3 - a_4,$

$\gamma_2 = a_1 + a_2 + a_3 - a_4,$

$\delta_1 = a_2 - a_3 - a_4 + a_5,$

$\delta_2 = -a_1 + a_2 - a_3 + a_5,$

$\epsilon_1 = -a_1 + a_3 + a_4 - a_5,$

$\epsilon_2 = a_1 + a_3 + a_4 - a_5.$

It is easy to see that

$$\left.\begin{array}{ll} 0 < x_i < x_{i+1} \text{ for } i = 1, 2, \ldots, 8, & y_1 < z_1, \\ y_i < y_{i+1} \text{ for } i = 1, 2, \ldots, 6, & \{-\gamma_1, \gamma_1\} < x_1 < \beta_1 < x_3, \\ z_1 < z_2 < z_3 < z_4 < z_5, & -\delta_1 < \alpha_1 < \alpha_2 < \alpha_3, \\ -\gamma_1 < \beta_1 < \beta_2, & \gamma_1 < \gamma_2, \\ \delta_1 < \delta_2, & \epsilon_1 < \epsilon_2, \\ x_4 < y_7 < x_8, & x_4 < y_6 < x_7, \\ x_4 < z_5 < x_6, & y_1 < y_2 < x_6, \\ \alpha_2 < y_3, & z_5 < y_6, \\ x_2 < z_3, & x_3 < z_4 < y_5, \\ x_3 < y_5 < y_6 < x_7, & \beta_2 < z_4, \\ \delta_2 < z_3, & x_1 < \delta_2. \end{array}\right\} \quad (11)$$

In (11), $\{-\gamma_1, \gamma_1\} = \{0\}$ if $\gamma_1 = 0$ and $\{-\gamma_1, \gamma_1\}$ is a symmetric set of cardinality two if $\gamma_1 \neq 0$.

Lemma 3. *Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ be a set of positive integers with $a_1 < a_2 < a_3 < a_4 < a_5$. Let $a_i - a_{i-1} \neq 2a_1$ for exactly one $i \in [2, 5]$ and $a_i - a_{i-1} = 2a_1$ for the remaining $i \in [2, 5]$. Then $\left|4_{\pm}^{\wedge} A\right| \geq 26$.*

Proof. Since signed sumset is symmetric, it suffices to prove that there are at least 13 positive integers in $4_{\pm}^{\wedge} A$.

`Case A`: $a_2 - a_1 \neq 2a_1$ and $a_5 - a_4 = a_4 - a_3 = a_3 - a_2 = 2a_1$. In this case,

$$\gamma_1 = -a_1 + a_2 + a_3 - a_4 = -3a_1 + a_2,$$

$$\gamma_2 = a_1 + a_2 + a_3 - a_4 = a_2 - a_1,$$

$$x_1 = -a_1 + a_2 - a_3 + a_4 = a_2 + a_1,$$

$$x_2 = a_1 + a_2 - a_3 + a_4 = 3a_1 + a_2,$$

$$\beta_1 = -a_1 - a_2 + a_3 + a_4 = 3a_1 + a_3,$$

$$x_3 = a_1 - a_2 + a_3 + a_4 = 3a_1 + a_4,$$

$$\alpha_1 = -a_2 + a_3 - a_4 + a_5 = 4a_1.$$

Clearly,

$$0 \neq \gamma_1 < \gamma_2 < x_1, \ -\gamma_1 < \alpha_1 < x_2 < \beta_1 < x_3, \text{ and } \alpha_1 \neq x_1. \quad (12)$$

If $a_2 - a_1 = a_1$, then $A = a_1 * \{1, 2, 4, 6, 8\}$, and so $\left|4_{\pm}^{\wedge} A\right| \geq 26$. If $a_2 - a_1 \neq a_1$, then $\gamma_2 \neq -\gamma_1$. Note that, if $\gamma_1 > 0$, then we have the following 12 distinct elements of $4_{\pm}^{\wedge} A$:

$$0 < \gamma_1 < \gamma_2 < x_1 < x_2 < \beta_1 < x_3 < x_4 < x_5 < x_6 < x_7 < x_8 < x_9, \quad (13)$$

and if $-\gamma_1 > 0$, then we have the following 12 distinct elements of $4^{\wedge}_{\pm}A$:

$$0 < \{-\gamma_1, \gamma_2\} < x_1 < x_2 < \beta_1 < x_3 < x_4 < x_5 < x_6 < x_7 < x_8 < x_9. \tag{14}$$

Therefore, in each case, we have at least 12 positive integers in $4^{\wedge}_{\pm}A$. Next we show the existence of at least one more positive integer in $4^{\wedge}_{\pm}A$ which is different from the elements listed in (13) and (14). Consider the positive integer α_1. Since $-\gamma_1 < \alpha_1 < x_2$ and $\alpha_1 \neq x_1$, we have the following subcases.

(*i*) If $\gamma_1 > 0$, $\alpha_1 \neq \gamma_1$, and $\alpha_1 \neq \gamma_2$, then we can include α_1 in the list (13) to get the required number of elements as follows:

$$0 < \{\gamma_1, \gamma_2, \alpha_1, x_1\} < x_2 < \beta_1 < x_3 < x_4 < x_5 < x_6 < x_7 < x_8 < x_9.$$

(*ii*) If $-\gamma_1 > 0$ and $\alpha_1 \neq \gamma_2$, then we can add α_1 in (14) to get required number of elements:

$$0 < \{-\gamma_1, \gamma_2, \alpha_1, x_1\} < x_2 < \beta_1 < x_3 < x_4 < x_5 < x_6 < x_7 < x_8 < x_9.$$

(*iii*) If $\alpha_1 = \gamma_1$, then $a_2 = 7a_1$. This gives $A = a_1 * \{1, 7, 9, 11, 13\}$, and so $|4^{\wedge}_{\pm}A| \geq 26$.

(*iv*) If $\alpha_1 = \gamma_2$, then $a_2 = 5a_1$. This gives $A = a_1 * \{1, 5, 7, 9, 11\}$, and so $|4^{\wedge}_{\pm}A| \geq 26$.

In each of the cases, either $|4^{\wedge}_{\pm}A| \geq 26$ or there are at least 13 distinct positive elements in $4^{\wedge}_{\pm}A$.

`Case B`: $a_3 - a_2 \neq 2a_1$ and $a_5 - a_4 = a_4 - a_3 = a_2 - a_1 = 2a_1$. In this case,

$$a_2 = 3a_1,\ a_3 \neq 5a_1,$$

$$\gamma_2 = a_1 + a_2 + a_3 - a_4 = 2a_1,$$

$$x_1 = -a_1 + a_2 - a_3 + a_4 = 4a_1,$$

$$\alpha_1 = -a_2 + a_3 - a_4 + a_5 = -a_1 + a_3,$$

$$\alpha_2 = -a_2 - a_3 + a_4 + a_5 = -a_1 + a_5,$$

$$y_3 = -a_1 - a_3 + a_4 + a_5,$$

$$x_7 = a_1 + a_2 + a_4 + a_5,$$

$$y_7 = -a_1 + a_3 + a_4 + a_5,$$

$$x_8 = a_1 + a_3 + a_4 + a_5,$$

$$y_6 = -a_1 + a_2 + a_4 + a_5,$$

$$z_2 = a_1 + a_2 - a_4 + a_5 = 6a_1.$$

Clearly,

$$0 < \gamma_2 < \{x_1,\ \alpha_1\}, x_1 \neq \alpha_1,\ y_7 \neq x_7,\ x_1 < z_2, \tag{15}$$

and

$$x_1 < z_2 = 6a_1 = 3a_1 + a_2 < 3a_1 + a_3 = a_1 + a_4 = -a_1 + a_5 = \alpha_2. \tag{16}$$

It follows from (15), (16), and (11) that

$$0 < \gamma_2 < \{x_1, \alpha_1\} < \alpha_2 < y_3 < y_4 < y_5 < y_6 < \{y_7, x_7\} < x_8 < x_9. \tag{17}$$

Therefore, we have at least 12 positive integers in $4^{\wedge}_{\pm}A$. Now we show the existence of at least one more positive integer in $4^{\wedge}_{\pm}A$ which is different from the integers in (17). Since $x_1 < z_2 < \alpha_2$, we have the following cases.
(i) If $z_2 \neq \alpha_1$, then we have at least 13 positive integers in $4^{\wedge}_{\pm}A$ as follows:

$$0 < \gamma_2 < \{x_1, \alpha_1, z_2\} < \alpha_2 < y_3 < y_4 < y_5 < y_6 < \{y_7, x_7\} < x_8 < x_9.$$

(ii) If $z_2 = \alpha_1$, then $a_3 = 7a_1$. This implies that $A = a_1 * \{1, 3, 7, 9, 11\}$. So $|4^{\wedge}_{\pm}A| \geq 26$.

`Case C`: $a_4 - a_3 \neq 2a_1$ and $a_5 - a_4 = a_3 - a_2 = a_2 - a_1 = 2a_1$. In this case,

$$a_2 = 3a_1, a_3 = 5a_1 < a_4, a_4 \neq 7a_1,$$

$$\gamma_1 = -a_1 + a_2 + a_3 - a_4 = 7a_1 - a_4,$$

$$x_1 = -a_1 + a_2 - a_3 + a_4 = -3a_1 + a_4,$$

$$x_2 = a_1 + a_2 - a_3 + a_4 = -a_1 + a_4,$$

$$\beta_1 = -a_1 - a_2 + a_3 + a_4 = a_1 + a_4,$$

$$x_3 = a_1 - a_2 + a_3 + a_4 = 3a_1 + a_4,$$

$$z_1 = a_1 + a_2 + a_4 - a_5 = 2a_1,$$

$$x_6 = a_1 + a_2 + a_3 + a_5 = 9a_1 + a_5,$$

$$y_6 = -a_1 + a_2 + a_4 + a_5 = 2a_1 + a_4 + a_5.$$

Clearly,

$$-\gamma_1 < x_1 < x_2 < \beta_1 < x_3 \text{ and } x_6 \neq y_6. \tag{18}$$

Also

$$\gamma_1 = -a_1 + a_2 + a_3 - a_4 = 2a_1 + a_3 - a_4 < 2a_1 = z_1 < -3a_1 + a_4 = x_1, \tag{19}$$

and

$$x_5 = a_1 + a_2 + a_3 + a_4 < a_1 + a_2 + 2a_4 = 2a_1 + a_4 + a_5 = y_6. \tag{20}$$

Since $\gamma_1 \neq 0$, it follows from (18), (19), (20), and (11) that

$$0 < \{-\gamma_1 \text{ or } \gamma_1\} < x_1 < x_2 < \beta_1 < x_3 < x_4 < x_5 < \{x_6, y_6\} < x_7 < x_8 < x_9,$$

and

$$\gamma_1 < z_1 < x_1.$$

Therefore, we have at least 13 positive integers in $4^\wedge_\pm A$ except for the case $z_1 = -\gamma_1$. If $z_1 = -\gamma_1$, then $a_4 = 9a_1$, which implies that $A = a_1 * \{1, 3, 5, 9, 11\}$. So $|4^\wedge_\pm A| \geq 26$.

`Case D`: $a_5 - a_4 \neq 2a_1$ and $a_4 - a_3 = a_3 - a_2 = a_2 - a_1 = 2a_1$. In this case,

$$a_2 = 3a_1, a_3 = 5a_1, a_4 = 7a_1,$$

$$a_5 \neq 9a_1, a_5 > 7a_1,$$

$$\gamma_2 = a_1 + a_2 + a_3 - a_4 = 2a_1,$$

$$x_1 = -a_1 + a_2 - a_3 + a_4 = 4a_1,$$

$$x_2 = a_1 + a_2 - a_3 + a_4 = 6a_1,$$

$$\beta_1 = -a_1 - a_2 + a_3 + a_4 = 8a_1,$$

$$x_3 = a_1 - a_2 + a_3 + a_4 = 10a_1,$$

$$x_4 = -a_1 + a_2 + a_3 + a_4 = 14a_1,$$

$$z_5 = -a_1 + a_2 + a_3 + a_5 = 7a_1 + a_5,$$

$$x_5 = a_1 + a_2 + a_3 + a_4 = 16a_1,$$

$$x_6 = a_1 + a_2 + a_3 + a_5 = 9a_1 + a_5,$$

$$y_2 = -a_1 + a_2 - a_4 + a_5 = 2a_1 - a_4 + a_5.$$

Clearly, we have

$$0 < \gamma_2 < x_1 < x_2 < \beta_1 < x_3 < x_4 < \{z_5, x_5\} < x_6 < x_7 < x_8 < x_9. \tag{21}$$

Therefore, we have at least 12 positive integers in $4^\wedge_\pm A$. Next, we show the existence of at least one more positive integer in $4^\wedge_\pm A$ which is different from the elements listed in (21). Since $y_2 \neq x_1$ and $\gamma_2 < y_2 < z_5 < x_6$, we have the following subcases. (i) If $y_2 \notin \{x_2, \beta_1, x_3, x_4, x_5\}$, then we have at least 13 positive integers in $4^\wedge_\pm A$ as follows:

$$0 < \gamma_2 < \{x_1, x_2, \beta_1, x_3, x_4, z_5, x_5, y_2\} < x_6 < x_7 < x_8 < x_9.$$

(ii) If $y_2 = x_2$, then $a_5 = 11a_1$. This gives $A = a_1 * \{1, 3, 5, 7, 11\}$. So $|4^{\wedge}_{\pm}A| \geq 26$.
(iii) If $y_2 = \beta_1$, then $a_5 = 13a_1$. This gives $A = a_1 * \{1, 3, 5, 7, 13\}$. So $|4^{\wedge}_{\pm}A| \geq 26$.
(iv) If $y_2 = x_3$, then $a_5 = 15a_1$. This gives $A = a_1 * \{1, 3, 5, 7, 15\}$. So $|4^{\wedge}_{\pm}A| \geq 26$.
(v) If $y_2 = x_4$, then $a_5 = 19a_1$. This gives $A = a_1 * \{1, 3, 5, 7, 19\}$. So $|4^{\wedge}_{\pm}A| \geq 26$.
(vi) If $y_2 = x_5$, then $a_5 = 21a_1$. This gives $A = a_1 * \{1, 3, 5, 7, 21\}$. So $|4^{\wedge}_{\pm}A| \geq 26$. □

Lemma 4. *Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ be a set of positive integers with $a_1 < a_2 < a_3 < a_4 < a_5$. Let $a_i - a_{i-1} \neq 2a_1$ for exactly two $i \in [2, 5]$ and $a_i - a_{i-1} = 2a_1$ for the remaining two $i \in [2, 5]$. Then $\left|4^{\wedge}_{\pm}A\right| \geq 26$.*

Proof. To prove $\left|4^{\wedge}_{\pm}A\right| \geq 26$, it is sufficient to prove that there are at least 13 positive integers in $4^{\wedge}_{\pm}A$.

`Case A`: $a_5 - a_4 = a_4 - a_3 = 2a_1$, $a_3 - a_2 \neq 2a_1$, and $a_2 - a_1 \neq 2a_1$. In this case,

$$z_1 = a_1 + a_2 + a_4 - a_5 = -a_1 + a_2,$$

$$\epsilon_2 = a_1 + a_3 + a_4 - a_5 = -a_1 + a_3,$$

$$x_1 = -a_1 + a_2 - a_3 + a_4 = a_1 + a_2,$$

$$\beta_1 = -a_1 - a_2 + a_3 + a_4 = a_1 - a_2 + 2a_3,$$

$$\beta_2 = -a_1 - a_2 + a_3 + a_5 = a_1 - a_2 + a_3 + a_4,$$

$$y_4 = -a_1 - a_2 + a_4 + a_5 = a_1 - a_2 + 2a_4,$$

$$x_5 = a_1 + a_2 + a_3 + a_4 = 3a_1 + a_2 + 2a_3,$$

$$x_6 = a_1 + a_2 + a_3 + a_5,$$

$$x_7 = a_1 + a_2 + a_4 + a_5,$$

$$y_7 = -a_1 + a_3 + a_4 + a_5,$$

$$x_8 = a_1 + a_3 + a_4 + a_5,$$

$$x_9 = a_2 + a_3 + a_4 + a_5,$$

$$x_4 = -a_1 + a_2 + a_3 + a_4,$$

$$z_3 = a_1 + a_2 - a_3 + a_5 = a_1 + a_2 + 4a_1 = 5a_1 + a_2.$$

Clearly, $\epsilon_2 \neq x_1$, $x_7 \neq y_7$, $x_6 < y_7$, and $x_5 > y_4$. Therefore, we have

$$0 < z_1 < \{\epsilon_2, x_1\} < \beta_1 < \beta_2 < y_4 < x_5 < x_6 < \{x_7, y_7\} < x_8 < x_9, \qquad (22)$$

$$\beta_2 < x_4 < x_5,$$

and

$$x_1 < z_3 < y_4.$$

In (22), we have 12 distinct positive integers of $4^{\wedge}_{\pm}A$. Now, we show that there exists at least one more positive integer in $4^{\wedge}_{\pm}A$ which is distinct from the integers in (22). Consider the positive integers x_4 and z_3. If $x_4 \neq y_4$, then we have at least 13 positive integers in $4^{\wedge}_{\pm}A$ as follows:

$$0 < z_1 < \{\epsilon_2, x_1\} < \beta_1 < \beta_2 < \{y_4, x_4\} < x_5 < x_6 < \{x_7, y_7\} < x_8 < x_9.$$

Assume $x_4 = y_4$. Then $a_2 = 2a_1$. Now consider the following subcases.
(i) If $z_3 \notin \{\epsilon_2, \beta_1, \beta_2\}$, then we have at least 13 positive integers in $4^{\wedge}_{\pm}A$ as follows:

$$0 < z_1 < \{\epsilon_2, x_1, \beta_1, \beta_2, z_3\} < y_4 < x_5 < x_6 < \{x_7, y_7\} < x_8 < x_9.$$

(ii) If $z_3 = \epsilon_2$, then $a_3 = 8a_1$. This gives $A = a_1 * \{1, 2, 8, 10, 12\}$. So $\left|4^{\wedge}_{\pm}A\right| \geq 26$.
(iii) If $z_3 = \beta_1$, then $a_3 = 4a_1$. This gives $A = a_1 * \{1, 2, 4, 6, 8\}$. So $\left|4^{\wedge}_{\pm}A\right| \geq 26$.
(iv) If $z_3 = \beta_2$, then $a_3 = 3a_1$. This gives $A = a_1 * \{1, 2, 3, 5, 7\}$. So $\left|4^{\wedge}_{\pm}A\right| \geq 26$.

`Case B`: $a_5 - a_4 = a_3 - a_2 = 2a_1$, $a_4 - a_3 \neq 2a_1$, and $a_2 - a_1 \neq 2a_1$. In this case,

$$a_5 - a_3 > a_5 - a_4 = 2a_1,$$

$$a_4 - a_2 > a_3 - a_2 = 2a_1,$$

$$z_1 = a_1 + a_2 + a_4 - a_5 = a_2 - a_1,$$

$$x_1 = -a_1 + a_2 - a_3 + a_4 = -3a_1 + a_4,$$

$$x_2 = a_1 + a_2 - a_3 + a_4 = -a_1 + a_4,$$

$$\beta_1 = -a_1 - a_2 + a_3 + a_4 = a_1 + a_4,$$

$$x_3 = a_1 - a_2 + a_3 + a_4 = 3a_1 + a_4,$$

$$x_4 = -a_1 + a_2 + a_3 + a_4 = a_1 + 2a_2 + a_4,$$

$$x_5 = a_1 + a_2 + a_3 + a_4 = 3a_1 + 2a_2 + a_4,$$

$$x_6 = a_1 + a_2 + a_3 + a_5 = 3a_1 + 2a_2 + a_5,$$

$$y_6 = -a_1 + a_2 + a_4 + a_5,$$

$$x_7 = a_1 + a_2 + a_4 + a_5,$$

$$x_8 = a_1 + a_3 + a_4 + a_5,$$

$$x_9 = a_2 + a_3 + a_4 + a_5,$$

$$z_2 = a_1 + a_2 - a_4 + a_5 = 3a_1 + a_2,$$

$$z_4 = a_1 - a_2 + a_3 + a_5 = 3a_1 + a_5.$$

Note also that $x_6 = a_1 + a_2 + a_3 + a_5 \neq -a_1 + a_2 + a_4 + a_5 = y_6$. So, we have

$$0 < z_1 < x_1 < x_2 < \beta_1 < x_3 < x_4 < x_5 < \{x_6, y_6\} < x_7 < x_8 < x_9, \qquad (23)$$

and

$$z_1 < z_2 < x_3 < z_4 < x_5.$$

In (23), we have 12 distinct positive integers of $4^{\wedge}_{\pm}A$. Next, we show that there is at least one more positive integer in $4^{\wedge}_{\pm}A$ which is distinct from the elements listed in (23). Consider the following subcases.

(i) If $z_4 \neq x_4$, then we have at least 13 positive integers in $4^{\wedge}_{\pm}A$ as follows:

$$0 < z_1 < x_1 < x_2 < \beta_1 < x_3 < \{x_4, z_4\} < x_5 < \{x_6, y_6\} < x_7 < x_8 < x_9.$$

(ii) If $z_4 = x_4$ and $z_2 \neq x_1$, then $a_2 = 2a_1$, $a_3 = 4a_1$, $z_2 < \beta_1$, and $z_2 \neq x_2$. Therefore, we have at least 13 positive integers in $4^{\wedge}_{\pm}A$ as follows:

$$0 < z_1 < \{x_1, x_2, z_2\} < \beta_1 < x_3 < x_4 < x_5 < \{x_6, y_6\} < x_7 < x_8 < x_9.$$

(iii) If $z_4 = x_4$ and $z_2 = x_1$, then $a_2 = 2a_1$, $a_3 = 4a_1$, and $a_4 = 8a_1$. This gives $A = a_1 * \{1, 2, 4, 8, 10\}$. So $\left|4^{\wedge}_{\pm}A\right| \geq 26$.

`Case C`: $a_4 - a_3 = a_3 - a_2 = 2a_1$, $a_5 - a_4 \neq 2a_1$, and $a_2 - a_1 \neq 2a_1$. In this case,

$$a_2 \neq 3a_1,$$

$$\gamma_1 = -a_1 + a_2 + a_3 - a_4 \neq 0,$$

$$\gamma_2 = a_1 + a_2 + a_3 - a_4 = -a_1 + a_2,$$

$$x_1 = -a_1 + a_2 - a_3 + a_4 = -3a_1 + a_4,$$

$$x_2 = a_1 + a_2 - a_3 + a_4 = -a_1 + a_4,$$

$$\beta_1 = -a_1 - a_2 + a_3 + a_4 = a_1 + a_4,$$

$$x_3 = a_1 - a_2 + a_3 + a_4 = 3a_1 + a_4,$$

$$x_4 = -a_1 + a_2 + a_3 + a_4 = a_1 + 2a_2 + a_4,$$

$$x_5 = a_1 + a_2 + a_3 + a_4 = 3a_1 + 2a_2 + a_4,$$

$$x_6 = a_1 + a_2 + a_3 + a_5 = 3a_1 + 2a_2 + a_5,$$

$$z_5 = -a_1 + a_2 + a_3 + a_5 = a_1 + 2a_2 + a_5,$$

$$y_4 = -a_1 - a_2 + a_4 + a_5.$$

Clearly,

$$0 < \gamma_2 < x_1 < x_2 < \beta_1 < x_3 < x_4 < \{x_5, z_5\} < x_6 < x_7 < x_8 < x_9. \qquad (24)$$

In (24), we have 12 distinct positive integers of $4^{\wedge}_{\pm}A$. Next, we show the existence of at least one more positive integer in $4^{\wedge}_{\pm}A$ which is different from the elements listed in (24). Now consider the following subcases.
(i) If $\gamma_1 > 0$, then we have at least 13 positive integers in $4^{\wedge}_{\pm}A$ as follows:

$$0 < \gamma_1 < \gamma_2 < x_1 < x_2 < \beta_1 < x_3 < x_4 < \{x_5, z_5\} < x_6 < x_7 < x_8 < x_9.$$

(ii) If $-\gamma_1 > 0$ and $\gamma_2 \neq -\gamma_1$, then we have at least 13 positive integers in $4^{\wedge}_{\pm}A$ as follows:

$$0 < \{-\gamma_1, \gamma_2\} < x_1 < x_2 < \beta_1 < x_3 < x_4 < \{x_5, z_5\} < x_6 < x_7 < x_8 < x_9.$$

(iii) If $-\gamma_1 > 0$, $\gamma_2 = -\gamma_1$, and $y_4 \neq x_5$, then $a_2 = 2a_1$, $a_3 = 4a_1, a_4 = 6a_1 < a_5$, and $y_4 = -a_1 - a_2 + a_4 + a_5 = 3a_1 + a_5$. It is easy to see that $x_3 < y_4 < z_5 < x_6$ and $y_4 \neq x_4$. Therefore, we have at least 13 positive integers in $4^{\wedge}_{\pm}A$ as follows:

$$0 < \gamma_2 < x_1 < x_2 < \beta_1 < x_3 < \{x_4, x_5, z_5, y_4\} < x_6 < x_7 < x_8 < x_9.$$

(iv) If $-\gamma_1 > 0$, $\gamma_2 = -\gamma_1$, and $y_4 = x_5$, then $a_2 = 2a_1$, $a_3 = 4a_1, a_4 = 6a_1$, and $a_5 = 10a_1$. So $A = a_1 * \{1, 2, 4, 6, 10\}$.

In each of these subcases, we have $\left|4^{\wedge}_{\pm}A\right| \geq 26$.

`Case D`: $a_5 - a_4 = a_2 - a_1 = 2a_1$, $a_4 - a_3 \neq 2a_1$, and $a_3 - a_2 \neq 2a_1$. In this case,

$$a_2 = 3a_1, a_3 \neq 5a_1, a_3 > 3a_1,$$

$$z_1 = a_1 + a_2 + a_4 - a_5 = 2a_1,$$

$$y_2 = -a_1 + a_2 - a_4 + a_5 = 4a_1,$$

$$x_2 = a_1 + a_2 - a_3 + a_4 = 4a_1 - a_3 + a_4,$$

$$z_2 = a_1 + a_2 - a_4 + a_5 = 6a_1,$$

$$z_3 = a_1 + a_2 - a_3 + a_5 = 6a_1 - a_3 + a_4,$$

$$z_4 = a_1 - a_2 + a_3 + a_5 = -2a_1 + a_3 + a_5,$$

$$y_5 = a_1 - a_2 + a_4 + a_5 = -2a_1 + a_4 + a_5,$$

$$y_6 = -a_1 + a_2 + a_4 + a_5 = 2a_1 + a_4 + a_5,$$

$$x_7 = a_1 + a_2 + a_4 + a_5 = 4a_1 + a_4 + a_5,$$

$$y_7 = -a_1 + a_3 + a_4 + a_5,$$

$$x_1 = -a_1 + a_2 - a_3 + a_4 = 2a_1 - a_3 + a_4,$$

$$\beta_2 = -a_1 - a_2 + a_3 + a_5 = -2a_1 + a_3 + a_4.$$

Clearly,

$$0 < z_1 < y_2 < \{x_2, z_2\} < z_3 < z_4 < y_5 < y_6 < \{x_7, y_7\} < x_8 < x_9, \qquad (25)$$

$$x_1 \neq y_2, z_1 < x_1 < x_2 < z_3, \text{ and } x_2 < \beta_2 < z_4. \qquad (26)$$

We have 12 distinct positive integers in the list (25). Now we show the existence of at least one more positive integer which is different from the integers in (25). Consider x_1 and β_2, and the following subcases.
(i) If $x_1 \neq z_2$, then we have at least 13 positive integers in $4\hat{}_{\pm}A$ as follows:

$$0 < z_1 < \{y_2, x_1, x_2, z_2\} < z_3 < z_4 < y_5 < y_6 < \{x_7, y_7\} < x_8 < x_9.$$

(ii) If $x_1 = z_2$ and $\beta_2 \neq z_3$, then we have at least 13 positive integers in $4\hat{}_{\pm}A$ as follows:

$$0 < z_1 < y_2 < x_1 < x_2 < \{z_3, \beta_2\} < z_4 < y_5 < y_6 < \{x_7, y_7\} < x_8 < x_9.$$

(iii) If $x_1 = z_2$ and $\beta_2 = z_3$, then $a_3 = 4a_1, a_4 = 8a_1$, and $a_5 = 10a_1$. This implies that $A = a_1 * \{1, 3, 4, 8, 10\}$. So $\left|4\hat{}_{\pm}A\right| \geq 26$.

`Case E`: $a_4 - a_3 = a_2 - a_1 = 2a_1, a_3 - a_2 \neq 2a_1$, and $a_5 - a_4 \neq 2a_1$. In this case,

$$a_2 = 3a_1,$$

$$a_4 = 2a_1 + a_3 > 2a_1 + a_2 = 5a_1,$$

$$\gamma_2 = a_1 + a_2 + a_3 - a_4 = 2a_1,$$

$$x_1 = -a_1 + a_2 - a_3 + a_4 = 4a_1,$$

$$x_2 = a_1 + a_2 - a_3 + a_4 = 6a_1,$$

$$x_3 = a_1 - a_2 + a_3 + a_4 = -2a_1 + a_3 + a_4,$$

$$x_4 = -a_1 + a_2 + a_3 + a_4 = 2a_1 + a_3 + a_4,$$

$$x_5 = a_1 + a_2 + a_3 + a_4 = 4a_1 + a_3 + a_4,$$

$$z_5 = -a_1 + a_2 + a_3 + a_5 = 2a_1 + a_3 + a_5,$$

$$x_6 = a_1 + a_2 + a_3 + a_5 = 4a_1 + a_3 + a_5,$$

$$x_7 = a_1 + a_2 + a_4 + a_5 = 4a_1 + a_4 + a_5,$$

$$\beta_1 = -a_1 - a_2 + a_3 + a_4 = -2a_1 + 2a_3,$$

$$z_4 = a_1 - a_2 + a_3 + a_5 = -2a_1 + a_3 + a_5,$$

$$y_7 = -a_1 + a_3 + a_4 + a_5 = a_1 + 2a_3 + a_5.$$

Clearly,

$$0 < \gamma_2 < x_1,\ x_5 \neq z_5,\ x_7 \neq y_7,$$

and

$$y_7 = a_1 + 2a_3 + a_5 > a_1 + a_2 + a_3 + a_5 = x_6.$$

Therefore, we have

$$0 < \gamma_2 < x_1 < x_2 < x_3 < x_4 < \{x_5, z_5\} < x_6 < \{x_7, y_7\} < x_8 < x_9 \tag{27}$$

and

$$x_1 < \beta_1 < x_3 < z_4 < z_5.$$

We have 12 distinct positive integers in (27). Next, we show the existence of at least one more positive integer which is different from the elements listed in (27). Consider the following subcases.
(i) If $\beta_1 \neq x_2$, then we have at least 13 positive integers in $4\hat{\underset{\pm}{}}A$ as follows:

$$0 < \gamma_2 < x_1 < \{x_2, \beta_1\} < x_3 < x_4 < \{x_5, z_5\} < x_6 < \{x_7, y_7\} < x_8 < x_9.$$

(ii) If $z_4 \neq x_4$ and $z_4 \neq x_5$, then we have at least 13 positive integers in $4\hat{\underset{\pm}{}}A$ as follows:

$$0 < \gamma_2 < x_1 < x_2 < x_3 < \{x_4, x_5, z_4, z_5\} < x_6 < \{x_7, y_7\} < x_8 < x_9.$$

(iii) If $\beta_1 = x_2$ and $z_4 = x_4$, then $a_3 = 4a_1, a_4 = 6a_1$, and $a_5 = 10a_1$. This implies that $A = a_1 * \{1, 3, 4, 6, 10\}$. So $\left|4\hat{\underset{\pm}{}}A\right| \geq 26$.
(iv) If $\beta_1 = x_2$ and $z_4 = x_5$, then $a_3 = 4a_1, a_4 = 6a_1$, and $a_5 = 12a_1$. So $A = a_1 * \{1, 3, 4, 6, 12\}$, hence $\left|4\hat{\underset{\pm}{}}A\right| \geq 26$.

`Case F`: $a_3 - a_2 = a_2 - a_1 = 2a_1$, $a_5 - a_4 \neq 2a_1$, and $a_4 - a_3 \neq 2a_1$. In this case,

$$a_2 = 3a_1,\ a_3 = 5a_1 < a_4 \neq 7a_1,$$

$$\gamma_1 = -a_1 + a_2 + a_3 - a_4 = 7a_1 - a_4,$$

$$x_1 = -a_1 + a_2 - a_3 + a_4 = -3a_1 + a_4,$$

$$x_2 = a_1 + a_2 - a_3 + a_4 = -a_1 + a_4,$$

$$\beta_1 = -a_1 - a_2 + a_3 + a_4 = a_1 + a_4,$$

$$x_3 = a_1 - a_2 + a_3 + a_4 = 3a_1 + a_4,$$

$$x_4 = -a_1 + a_2 + a_3 + a_4 = 7a_1 + a_4,$$
$$x_5 = a_1 + a_2 + a_3 + a_4 = 9a_1 + a_4,$$
$$z_5 = -a_1 + a_2 + a_3 + a_5 = 7a_1 + a_5,$$
$$x_6 = a_1 + a_2 + a_3 + a_5 = 9a_1 + a_5,$$
$$y_6 = -a_1 + a_2 + a_4 + a_5 > 7a_1 + a_5,$$
$$x_7 = a_1 + a_2 + a_4 + a_5,$$
$$\delta_1 = a_2 - a_3 - a_4 + a_5.$$

Clearly,

$$0 < \{\gamma_1 \text{ or } -\gamma_1\} < x_1 < x_2 < \beta_1 < x_3 < x_4 < \{x_5, z_5\} < x_6 < x_7 < x_8 < x_9, \quad (28)$$

$$x_4 < z_5 < y_6 < x_7, \text{ and } y_6 \neq x_6.$$

In (28), we have 12 distinct positive integers of $4\hat{}_{\pm}A$. Next, we show the existence of at least one more positive integer in $4\hat{}_{\pm}A$ which is different from the elements listed in (28). Consider the following subcases.

(*i*) If $y_6 \neq x_5$, then we have the following list of 13 positive integers:

$$0 < \{\gamma_1 \text{ or } -\gamma_1\} < x_1 < x_2 < \beta_1 < x_3 < x_4 < \{x_5, z_5, y_6, x_6\} < x_7 < x_8 < x_9.$$

(*ii*) If $y_6 = x_5$ and $-\delta_1 \neq \gamma_1$, then $a_5 = 7a_1, \gamma_1 > 0$, and $0 < -\delta_1 = -a_2 + a_3 + a_4 - a_5 = -5a_1 + a_4 < x_1$. Thus we have the following list of 13 positive integers:

$$0 < \{\gamma_1, -\delta_1\} < x_1 < x_2 < \beta_1 < x_3 < x_4 < \{x_5, z_5\} < x_6 < x_7 < x_8 < x_9.$$

(*iii*) If $y_6 = x_5$ and $-\delta_1 = \gamma_1$, then $a_5 = 7a_1$ and $a_4 = 6a_1$, which implies that $A = a_1 * \{1, 3, 5, 6, 7\}$. So $\left|4\hat{}_{\pm}A\right| \geq 26$. □

Lemma 5. *Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ be a set of positive integers with $a_1 < a_2 < a_3 < a_4 < a_5$. Let $a_i - a_{i-1} = 2a_1$ for exactly one $i \in [2, 5]$ and $a_i - a_{i-1} \neq 2a_1$ for the remaining $i \in [2, 5]$. Then $\left|4\hat{}_{\pm}A\right| \geq 26$.*

Proof. It is sufficient to prove that there are 13 positive integers in $4\hat{}_{\pm}A$. Consider the following cases.

`Case A`: $a_5 - a_4 \neq 2a_1$, $a_4 - a_3 \neq 2a_1$, $a_3 - a_2 \neq 2a_1$, and $a_2 - a_1 = 2a_1$. Consider the following list of integers which are elements of $4\hat{}_{\pm}A$:

$$\gamma_1 = -a_1 + a_2 + a_3 - a_4,$$
$$x_1 = -a_1 + a_2 - a_3 + a_4,$$
$$x_2 = a_1 + a_2 - a_3 + a_4,$$
$$\delta_2 = -a_1 + a_2 - a_3 + a_5,$$
$$z_3 = a_1 + a_2 - a_3 + a_5,$$
$$z_4 = a_1 - a_2 + a_3 + a_5,$$
$$z_5 = -a_1 + a_2 + a_3 + a_5,$$
$$x_6 = a_1 + a_2 + a_3 + a_5,$$
$$y_6 = -a_1 + a_2 + a_4 + a_5,$$
$$x_7 = a_1 + a_2 + a_4 + a_5,$$
$$x_8 = a_1 + a_3 + a_4 + a_5,$$
$$x_9 = a_2 + a_3 + a_4 + a_5,$$
$$y_7 = -a_1 + a_3 + a_4 + a_5,$$
$$\gamma_2 = a_1 + a_2 + a_3 - a_4,$$
$$x_5 = a_1 + a_2 + a_3 + a_4,$$
$$y_5 = a_1 - a_2 + a_4 + a_5,$$
$$y_2 = -a_1 + a_2 - a_4 + a_5.$$

Note that

$$0 < \{\gamma_1 \text{ or } -\gamma_1\} < x_1 < \{x_2, \delta_2\} < z_3 < z_4 < z_5 < \{x_6, y_6\} < x_7 < x_8 < x_9. \quad (29)$$

Also

$$z_5 < y_6 < y_7 < x_8, \ y_7 \neq x_7, \text{ and } z_4 < y_5 < y_6 < x_7.$$

Therefore, we have at least 12 distinct positive integers in $4^{\wedge}_{\pm}A$ that are listed in (29). Next, we show the existence of at least one more positive integer in $4^{\wedge}_{\pm}A$ which is different from the elements listed in (29). Consider the following subcases.

(i) If $y_7 \neq x_6$, then we have a list of 13 distinct positive integers of $4^{\wedge}_{\pm}A$ as follows:

$$0 < \{\gamma_1 \text{ or } -\gamma_1\} < x_1 < \{x_2, \delta_2\} < z_3 < z_4 < z_5 < \{x_6, y_6, x_7, y_7\} < x_8 < x_9.$$

(ii) If $y_7 = x_6$, then $a_4 = 5a_1$ and $y_5 < z_5$. Therefore, we have

$$0 < \{\gamma_1 \text{ or } -\gamma_1\} < x_1 < \{x_2, \delta_2\} < z_3 < z_4 < y_5 < z_5 < \{x_6, y_6\} < x_7 < x_8 < x_9.$$

Thus, we have at least 26 elements in $4^{\wedge}_{\pm}A$ in this case.

`Case B`: $a_5 - a_4 \neq 2a_1$, $a_4 - a_3 \neq 2a_1$, $a_3 - a_2 = 2a_1$, and $a_2 - a_1 \neq 2a_1$. Consider the following list of integers which are elements of $4^{\wedge}_{\pm}A$:

$$x_1 = -a_1 + a_2 - a_3 + a_4,$$
$$x_2 = a_1 + a_2 - a_3 + a_4,$$
$$\beta_1 = -a_1 - a_2 + a_3 + a_4,$$
$$z_3 = a_1 + a_2 - a_3 + a_5,$$
$$\beta_2 = -a_1 - a_2 + a_3 + a_5,$$
$$y_4 = -a_1 - a_2 + a_4 + a_5,$$
$$z_4 = a_1 - a_2 + a_3 + a_5,$$
$$y_5 = a_1 - a_2 + a_4 + a_5,$$

$$y_6 = -a_1 + a_2 + a_4 + a_5, \qquad \delta_2 = -a_1 + a_2 - a_3 + a_5,$$
$$x_7 = a_1 + a_2 + a_4 + a_5,$$
$$\delta_1 = a_2 - a_3 - a_4 + a_5,$$
$$x_8 = a_1 + a_3 + a_4 + a_5,$$
$$x_9 = a_2 + a_3 + a_4 + a_5, \qquad y_2 = -a_1 + a_2 - a_4 + a_5.$$

Note that

$$0 < x_1 < x_2 < \{\beta_1, z_3\} < \beta_2 < \{y_4, z_4\} < y_5 < y_6 < x_7 < x_8 < x_9. \tag{30}$$

Also

$$x_1 < \delta_2 < z_3 < \beta_2, \ \delta_1 < y_2 < \delta_2 < z_3 < \beta_2, \ \text{and } \delta_2 \neq x_2.$$

In (30), we have 12 distinct positive integers of $4^\wedge_\pm A$. Next, we show the existence of at least one more positive integer in $4^\wedge_\pm A$ which is different from the elements listed in (30). Consider the following subcases.
(i) If $\delta_2 \neq \beta_1$, then we have at least 13 positive integers in $4^\wedge_\pm A$ as follows:

$$0 < x_1 < \{x_2, \beta_1, z_3, \delta_2\} < \beta_2 < \{y_4, z_4\} < y_5 < y_6 < x_7 < x_8 < x_9.$$

(ii) If $\delta_2 = \beta_1$ and $\delta_1 \neq x_1$, then $a_5 - a_4 = 2a_3 - 2a_2 = 4a_1$ and $\delta_1 = 2a_1 < x_2$. Therefore, we have at least 13 positive integers in $4^\wedge_\pm A$ as follows:

$$0 < \{x_1, \delta_1\} < x_2 < \delta_2 < z_3 < \beta_2 < \{y_4, z_4\} < y_5 < y_6 < x_7 < x_8 < x_9.$$

(iii) If $\delta_2 = \beta_1$ and $\delta_1 = x_1$, then $a_4 = 5a_1$ and $a_5 = 9a_1$. Consequently, $x_2 < y_2 < z_3$. So $\delta_1 = x_1 < x_2 < y_2 < \delta_2 = \beta_1 < z_3 < \beta_2 < \{y_4, z_4\} < y_5 < y_6 < x_7 < x_8 < x_9$. Thus, we have at least 26 elements in $4^\wedge_\pm A$.

`Case C`: $a_5 - a_4 \neq 2a_1$, $a_4 - a_3 = 2a_1$, $a_3 - a_2 \neq 2a_1$, and $a_2 - a_1 \neq 2a_1$. Consider the following list of positive integers which are elements of $4^\wedge_\pm A$:

$$\gamma_1 = -a_1 + a_2 + a_3 - a_4, \qquad z_5 = -a_1 + a_2 + a_3 + a_5,$$
$$x_1 = -a_1 + a_2 - a_3 + a_4,$$
$$x_6 = a_1 + a_2 + a_3 + a_5,$$
$$x_2 = a_1 + a_2 - a_3 + a_4,$$
$$x_7 = a_1 + a_2 + a_4 + a_5,$$
$$\beta_1 = -a_1 - a_2 + a_3 + a_4,$$
$$y_7 = -a_1 + a_3 + a_4 + a_5,$$
$$x_3 = a_1 - a_2 + a_3 + a_4,$$
$$x_8 = a_1 + a_3 + a_4 + a_5,$$
$$x_4 = -a_1 + a_2 + a_3 + a_4,$$
$$x_5 = a_1 + a_2 + a_3 + a_4, \qquad x_9 = a_2 + a_3 + a_4 + a_5.$$

Clearly,

$$x_1 < \beta_1 < x_3, \ x_4 < z_5 < x_6, \ x_5 \neq z_5, \ x_7 \neq y_7, \ \text{and } \ y_7 < x_8.$$

Since $a_4 - a_2 > a_4 - a_3 = 2a_1$, we have $x_6 < y_7$. Therefore from (11), we have at least 12 distinct positive integers in $4\hat{}_{\pm}A$ as follows:

$$0 < \{\gamma_1 \text{ or } -\gamma_1\} < x_1 < x_2 < x_3 < x_4 < \{x_5, z_5\} < x_6 < \{x_7, y_7\} < x_8 < x_9. \quad (31)$$

Next, we show the existence of at least one more positive integer in $4\hat{}_{\pm}A$ which is different from the elements listed in (31). Consider the following subcases.
(i) If $x_2 \neq \beta_1$, then we have at least 13 positive integers as follows:

$$0 < \{\gamma_1 \text{ or } -\gamma_1\} < x_1 < \{x_2, \beta_1\} < x_3 < x_4 < \{x_5, z_5\} < x_6 < \{x_7, y_7\} < x_8 < x_9.$$

(ii) If $x_2 = \beta_1$, then $a_3 - a_2 = a_1$. Therefore, $a_5 > a_4 = a_3 + 2a_1 = a_2 + 3a_1 > 4a_1$. Consider the following increasing sequence of elements of $4\hat{}_{\pm}A$:

$$\begin{aligned}
0 &< -4a_1 + a_5 = -a_1 + a_2 - a_4 + a_5 < -a_1 + a_3 - a_4 + a_5 < -a_1 + a_2 - a_3 + a_5 \\
&< a_1 + a_3 - a_4 + a_5 < -a_1 - a_2 + a_3 + a_5 < -a_1 - a_3 + a_4 + a_5 < a_1 - a_2 + a_3 + a_5 \\
&< a_1 - a_3 + a_4 + a_5 < a_1 - a_2 + a_4 + a_5 < -a_1 + a_2 + a_4 + a_5 < a_1 + a_2 + a_4 + a_5 \\
&< a_1 + a_3 + a_4 + a_5 < a_2 + a_3 + a_4 + a_5.
\end{aligned}$$

Thus, we have at least 26 elements in $4\hat{}_{\pm}A$.

`Case D`: $a_5 - a_4 = 2a_1$, $a_4 - a_3 \neq 2a_1$, $a_3 - a_2 \neq 2a_1$, and $a_2 - a_1 \neq 2a_1$. Consider the following list of positive integers which are elements of $4\hat{}_{\pm}A$:

$$\begin{aligned}
z_1 &= a_1 + a_2 + a_4 - a_5, & y_5 &= a_1 - a_2 + a_4 + a_5, \\
x_1 &= -a_1 + a_2 - a_3 + a_4, & y_6 &= -a_1 + a_2 + a_4 + a_5, \\
y_2 &= -a_1 + a_2 - a_4 + a_5, & x_7 &= a_1 + a_2 + a_4 + a_5, \\
\delta_2 &= -a_1 + a_2 - a_3 + a_5, & y_7 &= -a_1 + a_3 + a_4 + a_5, \\
\beta_2 &= -a_1 - a_2 + a_3 + a_5, & x_8 &= a_1 + a_3 + a_4 + a_5, \\
y_4 &= -a_1 - a_2 + a_4 + a_5, & x_9 &= a_2 + a_3 + a_4 + a_5. \\
z_4 &= a_1 - a_2 + a_3 + a_5,
\end{aligned}$$

Note that $0 < z_1 < \{x_1, y_2\} < \delta_2 < \beta_2 < \{y_4, z_4\} < y_5 < y_6 < \{x_7, y_7\} < x_8 < x_9$. Thus, we have at least 26 elements in $4\hat{}_{\pm}A$. □

Lemma 6. *Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ be a set of positive integers with $a_1 < a_2 < a_3 < a_4 < a_5$. If $a_i - a_{i-1} \neq 2a_1$ for all $i \in [2, 5]$ and $a_4 - a_3 \neq a_2 - a_1$, then $\left|4\hat{}_{\pm}A\right| \geq 26$.*

Proof. Consider the following list of necessarily distinct integers:

$$\gamma_1 = -a_1 + a_2 + a_3 - a_4, \qquad x_7 = a_1 + a_2 + a_4 + a_5,$$
$$x_1 = -a_1 + a_2 - a_3 + a_4, \qquad x_8 = a_1 + a_3 + a_4 + a_5,$$
$$x_2 = a_1 + a_2 - a_3 + a_4, \qquad x_9 = a_2 + a_3 + a_4 + a_5,$$
$$\beta_1 = -a_1 - a_2 + a_3 + a_4, \qquad y_5 = a_1 - a_2 + a_4 + a_5,$$
$$x_3 = a_1 - a_2 + a_3 + a_4, \qquad y_6 = -a_1 + a_2 + a_4 + a_5,$$
$$x_4 = -a_1 + a_2 + a_3 + a_4, \qquad y_7 = -a_1 + a_3 + a_4 + a_5,$$
$$x_5 = a_1 + a_2 + a_3 + a_4, \qquad \epsilon_1 = -a_1 + a_3 + a_4 - a_5,$$
$$z_5 = -a_1 + a_2 + a_3 + a_5, \qquad \delta_2 = -a_1 + a_2 - a_3 + a_5,$$
$$x_6 = a_1 + a_2 + a_3 + a_5, \qquad z_3 = a_1 + a_2 - a_3 + a_5.$$

Since $a_i - a_{i-1} \neq 2a_1$ for all $i \in [2, 5]$ and $a_4 - a_3 \neq a_2 - a_1$, we have

$$y_6 \neq x_6,\ y_7 \neq x_7,\ x_5 \neq z_5,\ x_2 \neq \delta_2,\ \text{and } \gamma_1 \neq 0. \tag{32}$$

Therefore, from (11) and (32), we have

$$0 < \{\gamma_1 \text{ or } -\gamma_1\} < x_1 < x_2 < x_3 < x_4 < \{z_5, x_5\} < x_6 < x_7 < x_8 < x_9.$$

Also

$$x_4 < z_5 < y_6 < x_7,\ x_4 < z_5 < y_7 < x_8,\ x_1 < \beta_1 < x_3,\ \text{and } x_1 < \delta_2.$$

Consider the following cases.

`Case A`: $a_3 \neq a_2 + a_1$. In this case, $x_2 = a_1 + a_2 - a_3 + a_4 \neq -a_1 - a_2 + a_3 + a_4 = \beta_1$. Now, consider the following subcases.
(i) If $y_6 \neq x_5$, then we have at least 13 positive integers in $4^{\wedge}_{\pm}A$ as follows:

$$0 < \{\gamma_1 \text{ or } -\gamma_1\} < x_1 < \{x_2, \beta_1\} < x_3 < x_4 < \{z_5, x_5, y_6, x_6\} < x_7 < x_8 < x_9.$$

(ii) If $y_6 = x_5$ and $y_7 \neq x_6$, then we have at least 13 positive integers in $4^{\wedge}_{\pm}A$ as follows:

$$0 < \{\gamma_1 \text{ or } -\gamma_1\} < x_1 < \{x_2, \beta_1\} < x_3 < x_4 < \{z_5, y_6, y_7, x_6\} < x_7 < x_8 < x_9.$$

(iii) If $y_6 = x_5$ and $y_7 = x_6$, then $a_5 - a_3 = 2a_1$ and $a_4 - a_2 = 2a_1$. Therefore,

$$\epsilon_1 = -a_1 + a_3 + a_4 - a_5 = -a_1 + a_4 - 2a_1 = -a_1 + a_2 < x_1,$$

$$x_1 < \delta_2 = -a_1 + a_2 - a_3 + a_5 = a_1 + a_2 < a_1 + a_2 - a_3 + a_4 = x_2,$$

and

$$\beta_1 = -a_1 - a_2 + a_3 + a_4 = a_1 + a_3 > a_1 + a_2 = \delta_2.$$

It follows that

$$0 < \epsilon_1 < x_1 < \delta_2 < \{x_2, \beta_1\} < x_3 < x_4 < \{z_5, x_5\} < x_6 < x_7 < x_8 < x_9.$$

Therefore, in each of the above subcases, we have at least 13 positive integers in $4^{\wedge}_{\pm}A$.

`Case B`: $a_3 = a_2 + a_1$. From (11) and (32), we have

$$0 < \{\gamma_1 \text{ or } -\gamma_1\} < x_1 < \{x_2, \delta_2\} < z_3 < z_4 < z_5 < \{x_6, y_6\} < x_7 < x_8 < x_9.$$

Also

$$x_2 < x_3 < z_4 < y_5 < y_6 < x_7, \; z_5 < y_6 < y_7 < x_8, \text{ and } y_7 \neq x_7.$$

Now, we have the following observations.

1. If $y_7 = x_6$, then $a_4 - a_2 = 2a_1$. So $y_5 < x_6$.
2. If $y_7 = x_6$ and $y_5 = z_5$, then $a_4 - a_2 = 2a_1, a_4 - a_3 = a_1$, and $a_4 - a_3 = 2a_2 - 2a_1$. This gives $2a_2 = 3a_1, 2a_3 = 5a_1, 2a_4 = 7a_1$, and $x_3 \neq z_3$.

Now, consider the following subcases.
(*i*) If $y_7 \neq x_6$, then we have at least 13 positive integers in $4^{\wedge}_{\pm}A$ as follows:

$$0 < \{\gamma_1 \text{ or } -\gamma_1\} < x_1 < \{x_2, \delta_2\} < z_3 < z_4 < z_5 < \{x_6, y_6, x_7, y_7\} < x_8 < x_9.$$

(*ii*) If $y_7 = x_6$ and $y_5 \neq z_5$, then we have at least 13 positive integers in $4^{\wedge}_{\pm}A$ as follows:

$$0 < \{\gamma_1 \text{ or } -\gamma_1\} < x_1 < \{x_2, \delta_2\} < z_3 < z_4 < \{z_5, y_5, y_6, x_6, x_7\} < x_8 < x_9.$$

(*iii*) If $y_7 = x_6, y_5 = z_5$, and $x_3 \neq \delta_2$, then we have at least 13 positive integers in $4^{\wedge}_{\pm}A$ as follows:

$$0 < \{\gamma_1 \text{ or } -\gamma_1\} < x_1 < \{x_2, \delta_2, z_3, x_3\} < z_4 < z_5 < \{x_6, y_6\} < x_7 < x_8 < x_9.$$

(*iv*) If $y_7 = x_6, y_5 = z_5$, and $x_3 = \delta_2$, then $2a_2 = 3a_1, 2a_3 = 5a_1, 2a_4 = 7a_1$, and $2a_5 = 15a_1$. In this subcase, $2 * A = a_1 * \{2, 3, 5, 7, 15\}$. So $|4^{\wedge}_{\pm}A| = |4^{\wedge}_{\pm}(2 * A)| \geq 26$. Therefore, in each of these subcases, we have at least 13 positive integers in $4^{\wedge}_{\pm}A$. □

Lemma 7. *Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ be a set of positive integers with $a_1 < a_2 < a_3 < a_4 < a_5$. If $a_4 - a_3 = a_2 - a_1$, $a_5 - a_3 \neq 2a_1$, and $a_i - a_{i-1} \neq 2a_1$ for all $i \in [2, 5]$, then $\left|4^{\wedge}_{\pm}A\right| \geq 26$.*

Proof. Consider the following list of necessarily distinct positive integers in $4^{\wedge}_{\pm}A$:

$$\gamma_1 = -a_1 + a_2 + a_3 - a_4,$$
$$x_1 = -a_1 + a_2 - a_3 + a_4,$$
$$x_2 = a_1 + a_2 - a_3 + a_4,$$
$$\beta_1 = -a_1 - a_2 + a_3 + a_4,$$
$$x_3 = a_1 - a_2 + a_3 + a_4,$$
$$x_4 = -a_1 + a_2 + a_3 + a_4,$$
$$x_5 = a_1 + a_2 + a_3 + a_4,$$
$$z_5 = -a_1 + a_2 + a_3 + a_5,$$
$$x_6 = a_1 + a_2 + a_3 + a_5,$$
$$x_7 = a_1 + a_2 + a_4 + a_5,$$
$$x_8 = a_1 + a_3 + a_4 + a_5,$$
$$x_9 = a_2 + a_3 + a_4 + a_5,$$
$$y_6 = -a_1 + a_2 + a_4 + a_5,$$
$$y_7 = -a_1 + a_3 + a_4 + a_5,$$
$$\gamma_2 = a_1 + a_2 + a_3 - a_4,$$
$$z_4 = a_1 - a_2 + a_3 + a_5,$$
$$y_4 = -a_1 - a_2 + a_4 + a_5,$$
$$y_5 = a_1 - a_2 + a_4 + a_5,$$
$$z_3 = a_1 + a_2 - a_3 + a_5.$$

Since $a_5 - a_3 \neq 2a_1$ and $a_i - a_{i-1} \neq 2a_1$ for all $i \in [2, 5]$, we have

$$x_5 \neq y_6,\ y_6 \neq x_6,\ y_7 \neq x_7, \text{ and } x_5 \neq z_5. \tag{33}$$

Therefore, from (11) we have

$$0 < x_1 < x_2 < x_3 < x_4 < \{x_5, z_5, x_6, y_6\} < x_7 < x_8 < x_9. \tag{34}$$

Consider the following cases.

`Case A`: $a_4 \neq a_2 + a_1$. We have, $\gamma_2 \neq \beta_1$. Since $0 < \gamma_2 < x_2$ and $x_1 < \beta_1 < x_3$, we have the following subcases.

(i) If $\gamma_2 \neq x_1$ and $\beta_1 \neq x_2$, then we can add β_1 and γ_2 in the list (34) to get 13 positive integers in $4^{\wedge}_{\pm}A$ as follows:

$$0 < \{x_1, x_2, \beta_1, \gamma_2\} < x_3 < x_4 < \{x_5, z_5, x_6, y_6\} < x_7 < x_8 < x_9.$$

(ii) If $\gamma_2 = x_1$ and $\beta_1 = x_2$, then $a_4 - a_3 = a_1$ and $a_3 - a_2 = a_1$. Since $a_4 - a_3 = a_2 - a_1$, we have $a_2 = 2a_1, a_3 = 3a_1$, and $a_4 = 4a_1$. Therefore, $x_2 = 4a_1$, $x_3 = 6a_1$, $x_4 = 8a_1$, $x_5 = 10a_1$, $x_6 = 6a_1 + a_5$, $z_5 = 4a_1 + a_5$, $y_5 = 3a_1 + a_5$, $y_4 = a_1 + a_5$, and $a_5 \neq 5a_1$, $a_5 \neq 6a_1$. Hence $x_2 < y_4 < y_5 < z_5 < x_6$. It is easy to verify that $y_5 \notin \{x_3, x_4, x_6, z_5, y_6\}$ and $y_4 \notin \{x_3, x_6, z_5, y_6\}$. Now, we have the following observations.

1. If $y_4 \neq x_4$, $y_4 \neq x_5$, and $y_5 \neq x_5$, then we have

$$0 < x_1 < x_2 < \{x_3, x_4, x_5, z_5, x_6, y_4, y_5, y_6\} < x_7 < x_8 < x_9.$$

2. If $y_5 = x_5$ or $y_4 = x_4$, then $a_5 = 7a_1$. So $A = a_1 * \{1, 2, 3, 4, 7\}$.

3. If $y_4 = x_5$, then $a_5 = 9a_1$. So $A = a_1 * \{1, 2, 3, 4, 9\}$.

In each of these observations, we have at least 13 positive integers in $4\hat{\underset{\pm}{}}A$.
(iii) If $\gamma_2 = x_1$ and $\beta_1 \neq x_2$, then $a_4 - a_3 = a_1$ and $a_3 - a_2 \neq a_1$. Since $a_4 - a_3 = a_2 - a_1$, so $a_2 = 2a_1$ and $a_3 \neq 3a_1$. Consider y_5 and y_7. Then we have the following observations.

1. If $y_5 = a_1 - a_2 + a_4 + a_5 = -a_1 + a_2 + a_3 + a_4 = x_4$, then $a_5 - a_3 = 2a_2 - 2a_1 = 2(a_4 - a_3) = 2a_1$. But $a_5 - a_3 \neq 2a_1$. Therefore, $y_5 \neq x_4$.

2. If $y_5 = a_1 - a_2 + a_4 + a_5 = -a_1 + a_2 + a_3 + a_5 = z_5$, then $a_4 - a_3 = 2a_2 - 2a_1$. But $a_4 - a_3 = a_2 - a_1$. Therefore, $y_5 \neq z_5$.

3. If $y_5 = a_1 - a_2 + a_4 + a_5 = a_1 + a_2 + a_3 + a_5 = x_6$, then $a_4 - a_3 = 2a_2$. But $a_4 - a_3 = a_1$. Therefore, $y_5 \neq x_6$.

Since $x_3 < y_5 < y_6 < x_7$, $x_4 < z_5 < y_6 < y_7 < x_8$ and $y_7 \neq x_7$, we have the following situations.

1. If $y_5 \neq x_5$, then we have
$$0 < x_1 < \{x_2, \beta_1\} < x_3 < \{x_4, x_5, z_5, x_6, y_5, y_6\} < x_7 < x_8 < x_9.$$

2. If $y_5 = x_5$ and $y_7 \neq x_6$, then we have
$$0 < x_1 < \{x_2, \beta_1\} < x_3 < \{x_4, x_5, z_5, x_6, y_6, y_7, x_7\} < x_8 < x_9.$$

3. If $y_5 = x_5$ and $y_7 = x_6$, then $a_5 = a_3 + 4a_1$ and $a_4 = a_2 + 2a_1 = 4a_1$. This gives $A = a_1 * \{1, 2, 3, 4, 7\}$.

In each of these situations, we have at least 13 positive integers in $4\hat{\underset{\pm}{}}A$.
(iv) If $\gamma_2 \neq x_1$ and $\beta_1 = x_2$, then $a_3 = a_2 + a_1$ and so $a_4 = a_3 + a_2 - a_1 = 2a_2$. Consider z_3 and y_7. Then we have
$$\begin{array}{ll} x_2 < z_3 < x_6 < x_7, & z_3 < z_4 < y_6 < y_7 < x_8, \\ x_4 < z_5 < y_7 < x_8, & y_7 \neq x_7. \end{array}$$

Now consider the following subcases.

1. If $y_7 \neq x_5$ and $y_7 \neq x_6$, then we have
$$0 < \{\gamma_2, x_1\} < x_2 < x_3 < x_4 < \{x_5, z_5, x_6, y_6, y_7, x_7\} < x_8 < x_9.$$

2. If $y_7 = x_5$, then $a_5 = 2a_1 + a_2$. So $z_3 = a_1 + a_2 - a_3 + a_5 = 2a_1 + a_2 < 2a_2 + 2a_1 = a_1 - a_2 + a_3 + a_4 = x_3$. Therefore, we have
$$0 < \{\gamma_2, x_1\} < x_2 < z_3 < x_3 < x_4 < \{x_5, z_5, x_6, y_6\} < x_7 < x_8 < x_9.$$

3. If $y_7 = x_6$, then $a_4 = a_2 + 2a_1$. Since $a_4 = 2a_2$, we have $a_2 = 2a_1$, $a_3 = a_1 + a_2 = 3a_1$, and $a_4 = 2a_2 = 4a_1$. Clearly, $z_3 \notin \{z_5, y_6, x_7\}$ and $x_2 < z_3 < x_6$. Now, consider the following situations.

   (a) If $z_3 \notin \{x_3, x_4, x_5\}$, then

   $$0 < \{\gamma_2, x_1\} < x_2 < \{z_3, x_3, x_4, x_5, z_5, x_6, y_6\} < x_7 < x_8 < x_9.$$

   (b) If $z_3 = x_3$, then $a_5 = a_4 + 2a_3 - 2a_2 = 6a_1$. So $A = a_1 * \{1, 2, 3, 4, 6\}$.

   (c) If $z_3 = x_4$, then $a_5 = a_4 + 2a_3 - 2a_1 = 8a_1$. So $A = a_1 * \{1, 2, 3, 4, 8\}$.

   (d) If $z_3 = x_5$, then $a_5 = a_4 + 2a_3 = 10a_1$. So $A = a_1 * \{1, 2, 3, 4, 10\}$.

   In each of these situations, we have $\left|4^{\wedge}_{\pm}A\right| \geq 26$.

Therefore, in the case $a_4 \neq a_2 + a_1$, we have at least 13 positive integers in $4^{\wedge}_{\pm}A$.

`Case B`: $a_4 = a_2 + a_1$. In this case, $a_3 = 2a_1$ and $a_2 < 2a_1$. Therefore,

$$x_1 = -a_1 + a_2 - a_3 + a_4 = 2a_2 - 2a_1 < 2a_1 = a_1 + a_2 + a_3 - a_4 = \gamma_2 \text{ and } y_6 < x_6.$$

It follows that

$$0 < x_1 < \gamma_2 < x_2 < x_3 < x_4 < \{x_5, z_5, y_6\} < x_6 < x_7 < x_8 < x_9. \tag{35}$$

In (35), we have 12 distinct positive integers of $4^{\wedge}_{\pm}A$. Next, we show the existence of at least one more positive integer in $4^{\wedge}_{\pm}A$ which is different from the elements listed in (35). Note that $y_7 = -a_1 + a_3 + a_4 + a_5 = a_2 + a_3 + a_5 < a_1 + a_2 + a_3 + a_5 = x_6$. Since $x_4 < z_5 < y_6 < y_7$, we have the following subcases.
(i) If $y_7 \neq x_5$, then we have

$$0 < x_1 < \gamma_2 < x_2 < x_3 < x_4 < \{x_5, z_5, y_6, y_7\} < x_6 < x_7 < x_8 < x_9.$$

(ii) If $y_7 = x_5$ and $z_4 \neq x_4$, then $a_5 - a_2 = 2a_1$. Therefore $z_4 = 5a_1 < 3a_1 + a_2 + a_4 = x_5$. Since $x_3 < z_4$, we have

$$0 < x_1 < \gamma_2 < x_2 < x_3 < \{z_4, x_4, x_5, z_5, y_6\} < x_6 < x_7 < x_8 < x_9.$$

(iii) If $y_7 = x_5$ and $z_4 = x_4$, then $a_5 - a_2 = 2a_1$ and $2a_2 = 3a_1$. This gives $2 * A = \{2, 3, 4, 5, 7\}$. In this case also, $\left|4^{\wedge}_{\pm}A\right| = \left|4^{\wedge}_{\pm}(2 * A)\right| \geq 26$. Therefore, we have at least 13 positive integers in $4^{\wedge}_{\pm}A$. □

Lemma 8. *Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ be a set of positive integers with $a_1 < a_2 < a_3 < a_4 < a_5$. If $a_4 - a_3 = a_2 - a_1$, $a_5 - a_3 = 2a_1$, and $a_i - a_{i-1} \neq 2a_1$ for all $i \in [2, 5]$, then $\left|4^{\wedge}_{\pm}A\right| \geq 26$.*

Proof. Consider the following list of nonnegative integers:

$$x_1 = -a_1+a_2-a_3+a_4 = 2a_2-2a_1, \qquad x_5 = a_1+a_2+a_3+a_4,$$

$$y_2 = -a_1+a_2-a_4+a_5 = 2a_1, \qquad x_6 = a_1+a_2+a_3+a_5,$$

$$\delta_2 = -a_1+a_2-a_3+a_5 = a_1+a_2, \qquad x_7 = a_1+a_2+a_4+a_5,$$

$$x_2 = a_1+a_2-a_3+a_4 = 2a_2, \qquad x_8 = a_1+a_3+a_4+a_5,$$

$$x_3 = a_1-a_2+a_3+a_4 = 2a_3, \qquad x_9 = a_2+a_3+a_4+a_5,$$

$$x_4 = -a_1+a_2+a_3+a_4 = 2a_4, \qquad z_4 = a_1-a_2+a_3+a_5,$$

$$z_5 = -a_1+a_2+a_3+a_5 = a_4+a_5, \qquad y_7 = -a_1+a_3+a_4+a_5.$$

Clearly,

$$0 < y_2 < \delta_2 < x_2 < x_3 < x_4 < \{z_5, x_5\} < x_6 < x_7 < x_8 < x_9.$$

Since $a_5 - a_3 = a_5 - a_4 + a_4 - a_3 = 2a_1$, we have $a_5 - a_4 = 2a_1 - a_4 + a_3 = 2a_1 - a_2 + a_1 = 3a_1 - a_2$ and $a_5 - a_4 < 2a_1$. Therefore, $z_5 < x_5$. Consider the following cases.

`Case A`: $a_2 \neq 2a_1$. In this case, $x_1 \neq y_2$. Therefore, we have

$$0 < \{x_1, y_2\} < \delta_2 < x_2 < x_3 < x_4 < z_5 < x_5 < x_6 < x_7 < x_8 < x_9.$$

Also

$$x_3 < z_4 < z_5 < x_5 < y_7 < x_8 \text{ and } y_7 \neq x_7.$$

Now, consider the following subcases.
(i) If $z_4 \neq x_4$, then we have at least 13 positive integers in $4\hat{}_{\pm}A$ as follows:

$$0 < \{x_1, y_2\} < \delta_2 < x_2 < x_3 < \{x_4, z_4\} < z_5 < x_5 < x_6 < x_7 < x_8 < x_9.$$

(ii) If $y_7 \neq x_6$, we have at least 13 positive integers in $4\hat{}_{\pm}A$ as follows:

$$0 < \{x_1, y_2\} < \delta_2 < x_2 < x_3 < x_4 < z_5 < x_5 < \{x_6, x_7, y_7\} < x_8 < x_9.$$

(iii) If $z_4 = x_4$ and $y_7 = x_6$, then $a_5 - a_4 = 2a_2 - 2a_1$ and $a_4 - a_2 = 2a_1$. This gives $3a_2 = 5a_1$, $a_3 = 3a_1$, $a_5 = 5a_1$, and $3a_4 = 11a_1$. Therefore, we have $3 * A = a_1 * \{3, 5, 9, 11, 15\}$. So $\left|4\hat{}_{\pm}A\right| = \left|4\hat{}_{\pm}(3 * A)\right| \geq 26$.

`Case B`: $a_2 = 2a_1$. In this case, $a_5 - a_4 = 3a_1 - a_2 = a_1$ and $a_4 - a_3 = a_2 - a_1 = 2a_1 - a_1 = a_1$. Consider the following list of nonnegative integers:

$$x_1 = -a_1 + a_2 - a_3 + a_4 = 2a_1,$$

$$\delta_2 = -a_1 + a_2 - a_3 + a_5 = 3a_1,$$

$$x_2 = a_1 + a_2 - a_3 + a_4 = 4a_1,$$

$$z_3 = a_1 + a_2 - a_3 + a_5 = 5a_1,$$

$$z_4 = a_1 - a_2 + a_3 + a_5 = a_1 + 2a_3,$$

$$x_4 = -a_1 + a_2 + a_3 + a_4 = a_1 + a_3 + a_4,$$

$$z_5 = -a_1 + a_2 + a_3 + a_5 = a_2 + a_3 + a_4,$$

$$x_5 = a_1 + a_2 + a_3 + a_4,$$

$$x_3 = a_1 - a_2 + a_3 + a_4 = 2a_3.$$

Clearly,

$$0 < x_1 < \delta_2 < x_2 < z_3 < z_4 < x_4 < z_5 < x_5 < x_6 < x_7 < x_8 < x_9 \text{ and } x_2 < x_3 < z_4.$$

Now, consider the following subcases.
(i) If $x_3 \neq z_3$, then we have at least 13 positive integers in $4^{\wedge}_{\pm}A$ as follows:

$$0 < x_1 < \delta_2 < x_2 < \{z_3, x_3\} < z_4 < x_4 < z_5 < x_5 < x_6 < x_7 < x_8 < x_9.$$

(ii) If $x_3 = z_3$, then $2a_3 = a_1 + 2a_2 = 5a_1$. Thus $2 * A = a_1 * \{2, 4, 5, 7, 9\}$. So $\left|4^{\wedge}_{\pm}A\right| = \left|4^{\wedge}_{\pm}(2 * A)\right| \geq 26$. □

Combining Lemma 6, Lemma 7, and Lemma 8, we have the following lemma.

Lemma 9. *Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ be a set of positive integers with $a_1 < a_2 < a_3 < a_4 < a_5$. If $a_i - a_{i-1} \neq 2a_1$ for all $i \in [2, 5]$, then $\left|4^{\wedge}_{\pm}A\right| \geq 26$.*

Now, we give a proof of Theorem 10.

Proof of Theorem 10. Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ be a set of positive integers with $0 < a_1 < a_2 < a_3 < a_4 < a_5$. If $a_i - a_{i-1} = 2a_1$ for all $i \in [2, 5]$, then by Theorem 3, $\left|4^{\wedge}_{\pm}A\right| = 25$. If $a_i - a_{i-1} \neq 2a_1$ for some $i \in [2, 5]$, then by Lemma 3, Lemma 4, Lemma 5, and Lemma 9, $\left|4^{\wedge}_{\pm}A\right| \geq 26$.

Conversely, if $\left|4^{\wedge}_{\pm}A\right| = 25$, then $a_i - a_{i-1} = 2a_1$ for all $i \in [2, 5]$. Otherwise, Lemma 3, Lemma 4, Lemma 5, and Lemma 9 imply that $\left|4^{\wedge}_{\pm}A\right| \geq 26$. Thus, $A = a_1 * \{1, 3, 5, 7, 9\}$. This completes the proof of the theorem. □

By arguments similar to those used in Theorem 9, we have verified and proved the following theorem.

Theorem 11. *Let $k \geq 5$ be a positive integer and A be set of k nonnegative integers with $0 \in A$. Then*

$$\left|4^{\wedge}_{\pm}A\right| \geq 8k - 19.$$

Furthermore, if $\left|4^{\wedge}_{\pm}A\right| = 8k - 19$, then $A = d * [0, k - 1]$.

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