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# THE LOG-BEHAVIOR OF SOME SEQUENCES RELATED TO THE GENERALIZED LEONARDO NUMBERS

Feng-Zhen Zhao Department of Mathematics, Shanghai University, Shanghai, China fengzhenzhao@shu.edu.cn

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#### Abstract

Let  $\{\mathcal{L}_{k,n}\}_{n\geq 0}$  denote the generalized Leonardo sequence, where k is a fixed positive integer. In this paper, we discuss the log-behavior of some sequences related to  $\mathcal{L}_{k,n}$ . For example, we show that the sequences  $\{\sqrt[n]{\mathcal{L}_{k,n}}\}_{n\geq 9}$  and  $\{\mathcal{L}_{k,n}^n\}_{n\geq 3}$  are log-convex.

## 1. Introduction

The Leonardo sequence  $\{Le_n\}_{n\geq 0}$  was introduced by Catarino and Borges [4]. This is sequence A001595 in the OEIS [14] and satisfies the recurrence relation

$$Le_{n+1} = Le_n + Le_{n-1} + 1 \quad (n \ge 1),$$

where  $Le_0 = Le_1 = 1$ . For a fixed positive integer k, Kuhapatanakul and Chobsorn [10] defined the generalized Leonardo sequence  $\{\mathcal{L}_{k,n}\}_{n>0}$  by

$$\mathcal{L}_{k,n} = \mathcal{L}_{k,n-1} + \mathcal{L}_{k,n-2} + k \quad (n \ge 2), \tag{1}$$

where  $\mathcal{L}_{k,0} = \mathcal{L}_{k,1} = 1$ . It is clear that  $\mathcal{L}_{1,n} = Le_n$ . The value of  $\mathcal{L}_{1,n}$  is the number of nodes in the Fibonacci tree of order n. The sequence  $\{\mathcal{L}_{2,n}\}_{n\geq 0}$  is sequence A111314 in the OEIS [14]. Let  $\{a_n\}_{n\geq 0}$  ( $\{b_n\}_{n\geq 0}$ ) denote sequence A192746 (A192750) in the OEIS [14]. For  $n \geq 1$ ,  $\mathcal{L}_{3,n} = a_{n-1}$  and  $\mathcal{L}_{4,n} = b_{n-1}$ . The Leonardo numbers are related to the Fibonacci numbers. It is well known that the Fibonacci sequence  $\{F_n\}_{n\geq 0}$  satisfies the recurrence relation

$$F_{n+1} = F_n + F_{n-1} \quad (n \ge 1), \tag{2}$$

where  $F_0 = 0$  and  $F_1 = 1$ . The Binet formula for  $\{F_n\}_{n>0}$  is

$$F_n = \frac{\alpha^n - (-1)^n \alpha^{-n}}{\sqrt{5}},$$

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where  $\alpha = (1 + \sqrt{5})/2$ . Catarino and Borges [4] proved that  $Le_n = 2F_{n+1} - 1$  for  $n \ge 0$ . Kuhapatanakul and Chobsorn [10] showed that

$$\mathcal{L}_{k,n} = (k+1)F_{n+1} - k, \quad (n \ge 0).$$
(3)

Recently, more properties of Leonardo numbers have been studied (see for instance [1, 4, 5, 10, 11, 13, 16]).

The purpose of this paper is to discuss the log-behavior of some sequences involving  $\mathcal{L}_{k,n}$ . Now we recall some definitions involved in this paper. A positive sequence  $\{z_n\}_{n\geq 0}$  is said to be *log-convex* (*log-concave*) if  $z_n^2 \leq z_{n-1}z_{n+1}$  ( $z_n^2 \geq z_{n-1}z_{n+1}$ ) for each  $n \geq 1$ . A log-convex sequence  $\{z_n\}_{n\geq 0}$  is said to be *log-balanced* if  $\{z_n/n!\}_{n\geq 0}$ is log-concave (Došlić [7] gave this definition). It is evident that a sequence  $\{z_n\}_{n\geq 0}$ is log-convex (log-concave) if and only if its quotient sequence  $\{z_{n+1}/z_n\}_{n\geq 0}$  is nondecreasing (nonincreasing) and a log-convex sequence  $\{z_n\}_{n\geq 0}$  is log-balanced if and only if  $\{z_{n+1}/((n+1)z_n)\}_{n\geq 0}$  is nonincreasing. Log-behavior of sequences is not only a fertile source for inequalities, but also plays an important role in many subjects (see for instance [2, 3, 9, 6, 8, 12, 15]). Hence the log-behavior of sequences deserves to be studied. In the next section, we investigate the log-behavior of a series of sequences related to  $\mathcal{L}_{k,n}$ .

# 2. The Log-Behavior of Some Sequences Related to the Generalized Leonardo Numbers

In what follows, the following lemmas will be used.

**Lemma 1.** ([17]) Let  $\{z_n\}_{n\geq 0}$  be a positive sequence of real numbers defined by the following recurrence relation

$$z_n = c_1 z_{n-1} - c_2 z_{n-2} - \dots - c_k z_{n-k}, \quad n \ge k,$$

where  $k \ge 2$ ,  $c_1 > 0$ ,  $c_j \ge 0$   $(2 \le j \le k)$ . If  $\{z_0, z_1, z_2, \cdots, z_k, z_{k+1}\}$  is log-concave (log-convex), the sequence  $\{z_n\}_{n>0}$  is log-concave (log-convex).

**Lemma 2.** Let  $\{z_n\}_{n>0}$  be a positive sequence defined by the recurrence relation

$$z_{n+1} = c_1 z_n - c_2 z_{n-2}, \quad n \ge 2, \tag{4}$$

where  $c_1 > 0$  and  $c_2 \ge 0$  are constants. For  $n \ge 0$ , let  $x_n = z_{n+1}/z_n$ . If there exists a nonnegative integer  $n_0$  such that  $\{1/z_{n_0}, 1/z_{n_0+1}, 1/z_{n_0+2}, 1/z_{n_0+3}\}$  is log-balanced and  $c_1x_nx_{n+1} - 3c_2 \ge 0$  for  $n \ge n_0$ , then the sequence  $\{1/z_n\}_{n\ge n_0}$  is log-balanced.

*Proof.* Since  $\{1/z_{n_0}, 1/z_{n_0+1}, 1/z_{n_0+2}, 1/z_{n_0+3}\}$  is log-balanced,  $\{z_j\}_{n_0 \le j \le n_0+3}$  is log-concave. It follows from Lemma 1 that  $\{z_n\}_{n>n_0}$  is log-concave. Then  $\{1/z_n\}_{n>n_0}$ 

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is log-convex. For  $n \ge 0$ , let  $x_n = z_{n+1}/z_n$ . It follows from (4) that

$$x_n = c_1 - \frac{c_2}{x_{n-2}x_{n-1}}, \quad n \ge 2.$$
(5)

In order to prove that  $\{1/z_n\}_{n\geq n_0}$  is log-balanced, we need to show that  $\{1/((n+1)x_n)\}_{n\geq n_0}$  is decreasing. Now we prove by induction that  $\{(n+1)x_n\}_{n\geq n_0}$  is increasing. Since  $\{1/z_j\}_{n_0\leq j\leq n_0+3}$  is log-balanced,  $(j+2)x_{j+1} \geq (j+1)x_j$  for  $n_0 \leq j \leq n_0+2$ . Assume that  $(j+2)x_{j+1} \geq (j+1)x_j$  for  $j \geq n_0+2$ . It follows from (5) that

$$(j+3)x_{j+2} - (j+2)x_{j+1} = c_1 + \frac{c_2[(j+2)x_{j+1} - (j+3)x_{j-1}]}{x_{j-1}x_jx_{j+1}}$$

It follows from  $(j+2)x_{j+1} \ge (j+1)x_j \ge jx_{j-1}$  for  $j \ge n_0+2$  that

$$(j+3)x_{j+2} - (j+2)x_{j+1} \ge c_1 - \frac{3c_2}{x_j x_{j+1}} = \frac{c_1 x_j x_{j+1} - 3c_2}{x_j x_{j+1}} \ge 0.$$

Hence  $\{(n+1)x_n\}_{n\geq n_0}$  is increasing. Naturally,  $\{1/((n+1)x_n)\}_{n\geq n_0}$  is decreasing.

Now we give the main results of this paper.

**Theorem 1.** For the generalized Leonardo sequence  $\{\mathcal{L}_{k,n}\}_{n\geq 0}$ ,  $\{1/\mathcal{L}_{k,n}\}_{n\geq 3}$  is log-balanced.

*Proof.* It follows from (1) that

$$\mathcal{L}_{k,n+1} = 2\mathcal{L}_{k,n} - \mathcal{L}_{k,n-2}, \quad n \ge 2.$$

Clearly,  $c_1 = 2$  and  $c_2 = 1$ . By means of (3), we obtain

$$\mathcal{L}_{k,3} = 2k+3, \quad \mathcal{L}_{k,4} = 4k+5, \quad \mathcal{L}_{k,5} = 7k+8, \quad \mathcal{L}_{k,6} = 12k+13.$$

We observe that  $\{\mathcal{L}_{k,j}\}_{3\leq j\leq 6}$  is log-concave. Thus, the sequence  $\{\mathcal{L}_{k,n}\}_{n\geq 3}$  is log-concave by Lemma 1. It is clear that  $\{1/\mathcal{L}_{k,n}\}_{n\geq 3}$  is log-convex. Since  $\{\mathcal{L}_{k,n}\}_{n\geq 3}$  is log-concave and

$$\lim_{n \to +\infty} \frac{\mathcal{L}_{k,n+1}}{\mathcal{L}_{k,n}} = \alpha_{k,n}$$

we have

$$\alpha \le x_j \le x_3 = \frac{4k+5}{2k+3} < 2 \quad (j \ge 3).$$
(6)

By applying (6), we derive  $c_1 x_n x_{n+1} - 3c_2 = 2x_n x_{n+1} - 3 \ge 2\alpha - 3 > 0$  for  $n \ge 3$ . On the other hand, we find that  $\{1/z_3, 1/z_4, 1/z_5, 1/z_6\}$  is log-balanced. It follows from Lemma 2 that the sequence  $\{1/\mathcal{L}_{k,n}\}_{n\ge 3}$  is log-balanced.  $\Box$  **Theorem 2.** For the generalized Leonardo sequence  $\{\mathcal{L}_{k,n}\}_{n\geq 0}$ , let  $R_n = \sum_{i=0}^n \mathcal{L}_{k,i}$  $(n \geq 0)$ . The sequence  $\{R_n\}_{n\geq 0}$  is log-concave.

*Proof.* For  $n \ge 0$ , Kuhapatanakul and Chobsorn [10] proved that

$$\sum_{i=0}^{n} \mathcal{L}_{k,i} = \mathcal{L}_{k,n+2} - k(n+1) - 1.$$
(7)

It follows from (7) and (3) that

$$\begin{aligned} R_n^2 - R_{n-1}R_{n+1} &= [(k+1)(F_{n+3}-1) - k(n+1)]^2 \\ &- [(k+1)(F_{n+2}-1) - kn][(k+1)(F_{n+4}-1) - k(n+2)] \\ &= (k+1)^2[(F_{n+3}-1)^2 - (F_{n+2}-1)(F_{n+4}-1)] + k^2 \\ &+ k(k+1)[n(F_{n+2}+F_{n+4}-2F_{n+3}) + 2F_{n+2}-2F_{n+3}]. \end{aligned}$$

For the Fibonacci sequence  $\{F_n\}_{n\geq 0},$  there is an identity

$$F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1}.$$
(8)

Using (8) and (2), we have

$$R_n^2 - R_{n-1}R_{n+1} = (k+1)^2 [F_n + (-1)^n] + k(k+1)(nF_n - 2F_{n+1}) + k^2.$$

We observe that  $R_j^2 - R_{j-1}R_{j+1} > 0$  for  $1 \le j \le 3$ . For  $n \ge 4$ ,

$$R_n^2 - R_{n-1}R_{n+1} \ge k(k+1)(4F_n - 2F_{n+1}) + k^2 = 2k(k+1)F_{n-2} + k^2 > 0.$$

Hence  $\{R_n\}_{n\geq 0}$  is log-concave.

**Theorem 3.** For the generalized Leonardo sequence 
$$\{\mathcal{L}_{k,n}\}_{n\geq 0}$$
,  $\{\sqrt[n]{\mathcal{L}_{k,n}}\}_{n\geq 9}$  is log-convex.

*Proof.* It is clear that  $\{\sqrt[n]{\mathcal{L}_{k,n}}\}_{n\geq 9}$  is log-convex if and only if

$$2(n^2 - 1) \ln \mathcal{L}_{k,n} - n(n+1) \ln \mathcal{L}_{k,n-1} - n(n-1) \ln \mathcal{L}_{k,n+1} \le 0 \quad (n \ge 10).$$

For  $n \ge 10$ , put

$$S_n = 2(n^2 - 1) \ln \mathcal{L}_{k,n} - n(n+1) \ln \mathcal{L}_{k,n-1} - n(n-1) \ln \mathcal{L}_{k,n+1}$$

and

$$T_n = \frac{\mathcal{L}_{k,n}^3 \mathcal{L}_{k,n+2}}{\mathcal{L}_{k,n-1} \mathcal{L}_{k,n+1}^3}.$$

It is evident that

$$S_n - S_{n+1} = n(n+1)\ln T_n$$

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Owing to (3), (8), and (2),

$$\mathcal{L}_{k,n}^2 - \mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1} = (-1)^n (k+1)^2 + k(k+1)F_{n-2}.$$
(9)

This leads to

$$T_n = \frac{\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1} + k(k+1)F_{n-2} + (-1)^n(k+1)^2}{\mathcal{L}_{k,n}\mathcal{L}_{k,n+2} + k(k+1)F_{n-1} + (-1)^{n+1}(k+1)^2} \cdot \frac{\mathcal{L}_{k,n}\mathcal{L}_{k,n+2}}{\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1}}.$$

For  $j \ge 0$ , set  $x_j = \mathcal{L}_{k,j+1}/\mathcal{L}_{k,j}$ . Then

$$T_{n} = \frac{\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1} + k(k+1)F_{n-2} + (-1)^{n}(k+1)^{2}}{\mathcal{L}_{k,n}\mathcal{L}_{k,n+2} + k(k+1)F_{n-1} + (-1)^{n+1}(k+1)^{2}} \cdot x_{n-1}x_{n+1}$$
  
$$= \frac{\mathcal{L}_{k,n}\mathcal{L}_{k,n+2} + k(k+1)x_{n-1}x_{n+1}F_{n-2} + (-1)^{n}(k+1)^{2}x_{n-1}x_{n+1}}{\mathcal{L}_{k,n}\mathcal{L}_{k,n+2} + k(k+1)F_{n-1} + (-1)^{n+1}(k+1)^{2}}.$$

Now we prove that  $T_n \ge 1$  for  $n \ge 10$ . When  $n \ge 4$ ,

$$\mathcal{L}_{k,n}\mathcal{L}_{k,n+2} + k(k+1)x_{n-1}x_{n+1}F_{n-2} + (-1)^n(k+1)^2x_{n-1}x_{n+1} > 0$$

and

$$\mathcal{L}_{k,n}\mathcal{L}_{k,n+2} + k(k+1)F_{n-1} + (-1)^{n+1}(k+1)^2 > 0$$

hold. For  $n \ge 10$ , let

$$U_n = k(k+1)(x_{n-1}x_{n+1}F_{n-2} - F_{n-1}) + (-1)^n(k+1)^2(x_{n-1}x_{n+1} + 1).$$

It is obvious that  $T_n \ge 1$  if and only if  $U_n \ge 0$ . We note that

$$U_n \ge k(k+1)(x_{n-1}x_{n+1}F_{n-2} - F_{n-1}) - (k+1)^2(x_{n-1}x_{n+1} + 1).$$

By using (6), we get

$$U_n \geq k(k+1)(\alpha^2 F_{n-2} - F_{n-1}) - 5(k+1)^2$$
  
$$\geq k(k+1)\left(\frac{5}{2}F_{n-2} - F_{n-1}\right) - 5(k+1)^2.$$

By using (2), we have

$$U_n \ge k(k+1)\left(\frac{1}{2}F_{n-2} + F_{n-4}\right) - 5(k+1)^2 > 0 \quad (n \ge 10).$$

Naturally,  $T_n \geq 1$  for each  $n \geq 10.$  Thus the sequence  $\{S_n\}_{n \geq 10}$  is decreasing. We observe that

$$S_{10} = 198 \ln(88k + 89) - 110 \ln(54k + 55) - 90 \ln(143k + 144)$$
  
=  $110 \ln\left(\frac{88k + 89}{54k + 55}\right) - 90 \ln\left(\frac{143k + 144}{88k + 89}\right) - 2\ln(88k + 89)$   
 $\leq 110 \ln\frac{44}{27} - 90 \ln\frac{287}{177} - 2\ln 177 < 0.$ 

Hence  $2(n^2 - 1) \ln \mathcal{L}_{k,n} - n(n+1) \ln \mathcal{L}_{k,n-1} - n(n-1) \ln \mathcal{L}_{k,n+1} \leq 0$  holds for  $n \geq 10$ .

**Theorem 4.** For the generalized Leonardo sequence  $\{\mathcal{L}_{k,n}\}_{n\geq 0}$ , there exists a positive integer  $n_k \geq \max\{9, (1+\sqrt{64k-7})/4\}$  such that  $\{\sqrt[n]{\mathcal{L}_{k,n}}\}_{n\geq n_k}$  is log-balanced.

Proof. For  $n \ge 0$ , let

$$\mathcal{U}_n = n(n+1)(n+2)\ln\left(1+\frac{1}{n+1}\right) + 2n(n+2)\ln\mathcal{L}_{k,n+1} - (n+1)(n+2)\ln\mathcal{L}_{k,n} - n(n+1)\ln\mathcal{L}_{k,n+2}.$$

It is clear that  $\{\sqrt[n]{\mathcal{L}_{k,n}}/n!\}_{n\geq n_k}$  is log-concave if and only if  $\mathcal{U}_n \geq 0$  for  $n \geq n_k$ . For  $n \geq 0$ , put  $x_n = \mathcal{L}_{k,n+1}/\mathcal{L}_{k,n}$ . We note that

$$\begin{aligned} \mathcal{U}_n &= n(n+1)(n+2)\ln\left(1+\frac{1}{n+1}\right) + 2n(n+2)\ln\mathcal{L}_{k,n+1} - (n+1)(n+2)\ln\frac{\mathcal{L}_{k,n+1}}{x_n} \\ &-n(n+1)\ln(x_{n+1}\mathcal{L}_{k,n+1}) \\ &= n(n+1)(n+2)\ln\left(1+\frac{1}{n+1}\right) - 2\ln\mathcal{L}_{k,n+1} + (n+1)^2\ln\frac{x_n}{x_{n+1}} \\ &+(n+1)\ln(x_nx_{n+1}) \\ &= n(n+1)(n+2)\ln\left(1+\frac{1}{n+1}\right) - 2\ln(x_3\cdots x_n\mathcal{L}_{k,3}) + (n+1)^2\ln\frac{x_n}{x_{n+1}} \\ &+(n+1)\ln(x_nx_{n+1}). \end{aligned}$$

Since  $\{\mathcal{L}_{k,n}\}_{n\geq 3}$  is log-concave,  $x_{n-1}/x_n \geq 1$  for  $n\geq 4$ . Thus

$$\begin{aligned} \mathcal{U}_n &\geq n(n+1)(n+2)\ln\left(1+\frac{1}{n+1}\right) - 2\ln\mathcal{L}_{k,3} - 2(n-2)\ln x_3 + (n+1)\ln(x_nx_{n+1}) \\ &= n(n+1)(n+2)\ln\left(1+\frac{1}{n+1}\right) - 2\ln(2k+3) - 2(n-2)\ln\left(1+\frac{2k+2}{2k+3}\right) \\ &+ (n+1)\ln(x_nx_{n+1}). \end{aligned}$$

By the inequality

$$\frac{t}{1+t} < \ln(1+t) < t \quad (t>0)$$
(10)

and (6), we obtain

$$\mathcal{U}_n \ge n^2 + n - 2(2k+2) - \frac{2(n-2)(2k+2)}{2k+3} + 2(n+1)\ln\alpha.$$

It is clear that

$$\mathcal{U}_n \ge n^2 + n - 2(2k+2) - 2(n-2) + \frac{n+1}{2} \ge 0 \quad \left(n \ge \frac{1 + \sqrt{64k - 7}}{4}\right).$$

On the other hand, the sequence  $\{\sqrt[n]{\mathcal{L}_{k,n}}\}_{n\geq 9}$  is log-convex. Then there exists a positive integer  $n_k \geq \max\{9, (1+\sqrt{64k-7})/4\}$  such that  $\{\sqrt[n]{\mathcal{L}_{k,n}}\}_{n\geq n_k}$  is log-balanced.

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**Theorem 5.** For the generalized Leonardo sequence  $\{\mathcal{L}_{k,n}\}_{n\geq 0}$ ,  $\{\mathcal{L}_{k,n}^n\}_{n\geq 3}$  is logconvex.

*Proof.* It is clear that  $\mathcal{L}_{k,n}^{2n} - \mathcal{L}_{k,n-1}^{n-1} \mathcal{L}_{k,n+1}^{n+1} \leq 0 \ (n \geq 4)$  is equivalent to

$$2n \ln \mathcal{L}_{k,n} - (n-1) \ln \mathcal{L}_{k,n-1} - (n+1) \ln \mathcal{L}_{k,n+1} \le 0 \quad (n \ge 4).$$

For  $n \ge 0$ , set  $x_n = \mathcal{L}_{k,n+1}/\mathcal{L}_{k,n}$  and

$$V_n = 2n \ln \mathcal{L}_{k,n} - (n-1) \ln \mathcal{L}_{k,n-1} - (n+1) \ln \mathcal{L}_{k,n+1} \quad (n \ge 4).$$

It is evident that

$$V_n = n \ln \frac{\mathcal{L}_{k,n}^2}{\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1}} - \ln x_{n-1} - \ln x_n.$$

By (9), we have

$$V_n = n \ln \frac{\mathcal{L}_{k,n-1} \mathcal{L}_{k,n+1} + (-1)^n (k+1)^2 + k(k+1) F_{n-2}}{\mathcal{L}_{k,n-1} \mathcal{L}_{k,n+1}} - \ln x_{n-1} - \ln x_n.$$

For  $n \ge 4$ , it follows from (10) and (6) that

$$V_n \leq \frac{(-1)^n (k+1)^2 n + nk(k+1)F_{n-2}}{\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1}} - 2\ln\alpha$$
  
$$< \frac{(k+1)^2 n + nk(k+1)F_{n-2} - \frac{1}{2}\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1}}{\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1}}.$$

For convenience, let

$$W_n = (k+1)^2 n + nk(k+1)F_{n-2} - \frac{1}{2}\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1} \quad (n \ge 4)$$

Then  $V_n < W_n/(\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1})$ . One can prove by induction that

$$\mathcal{L}_{k,j} > 2kj \quad (j \ge 6). \tag{11}$$

For  $n \geq 5$ , it follows from (11) that

$$W_n < (k+1)^2 n + nk(k+1)F_{n-2} - k(n+1)\mathcal{L}_{k,n-1}.$$

By using (3) and (2), we derive

$$W_n < (k+1)^2 n + nk(k+1)F_{n-2} - k(n+1)[(k+1)F_n - k]$$
  
=  $(k+1)^2 n + k^2(n+1) - nk(k+1)F_{n-1} - k(k+1)F_n$   
<  $0 \quad (n \ge 5).$ 

Thus  $V_n < 0$  for  $n \ge 5$ . On the other hand, we find that  $V_4 < 0$ . Hence the sequence  $\{\mathcal{L}_{k,n}^n\}_{n\ge 3}$  is log-convex.

## 3. Conclusions

For the generalized Leonardo sequence  $\{\mathcal{L}_{k,n}\}_{n\geq 0}$ , we discussed the log-behavior of some sequences involving  $\mathcal{L}_{k,n}$  in this paper. We mainly proved that  $\{1/\mathcal{L}_{k,n}\}_{n\geq 3}$ is log-balanced and  $\{\sqrt[n]{\mathcal{L}_{k,n}}\}_{n\geq 9}$  and  $\{\mathcal{L}_{k,n}^n\}_{n\geq 3}$  are log-convex. Let  $\{f_n\}_{n\geq 0}$  be a positive sequence. The future work is to study the log-behavior of  $\{\mathcal{L}_{k,n}^{f_n}\}_{n\geq 0}$  under some conditions.

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