#A82 INTEGERS 24 (2024)

THE LOG-BEHAVIOR OF SOME SEQUENCES RELATED TO THE GENERALIZED LEONARDO NUMBERS

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Abstract

Let $\{\mathcal{L}_{k,n}\}_{n\geq 0}$ denote the generalized Leonardo sequence, where k is a fixed positive integer. In this paper, we discuss the log-behavior of some sequences related to $\mathcal{L}_{k,n}$. For example, we show that the sequences $\{\sqrt[n]{\mathcal{L}_{k,n}}\}_{n\geq 9}$ and $\{\mathcal{L}_{k,n}^n\}_{n\geq 3}$ are log-convex.

1. Introduction

The Leonardo sequence ${Le_n}_{n>0}$ was introduced by Catarino and Borges [4]. This is sequence A001595 in the OEIS [14] and satisfies the recurrence relation

$$
Le_{n+1} = Le_n + Le_{n-1} + 1 \quad (n \ge 1),
$$

where $Le_0 = Le_1 = 1$. For a fixed positive integer k, Kuhapatanakul and Chobsorn [10] defined the generalized Leonardo sequence $\{\mathcal{L}_{k,n}\}_{n\geq0}$ by

$$
\mathcal{L}_{k,n} = \mathcal{L}_{k,n-1} + \mathcal{L}_{k,n-2} + k \quad (n \ge 2),\tag{1}
$$

where $\mathcal{L}_{k,0} = \mathcal{L}_{k,1} = 1$. It is clear that $\mathcal{L}_{1,n} = Le_n$. The value of $\mathcal{L}_{1,n}$ is the number of nodes in the Fibonacci tree of order n. The sequence ${\{\mathcal{L}_{2,n}\}}_{n>0}$ is sequence A111314 in the OEIS [14]. Let $\{a_n\}_{n\geq 0}$ $(\{b_n\}_{n\geq 0})$ denote sequence A192746 (A192750) in the OEIS [14]. For $n \geq 1$, $\mathcal{L}_{3,n} = a_{n-1}$ and $\mathcal{L}_{4,n} = b_{n-1}$. The Leonardo numbers are related to the Fibonacci numbers. It is well known that the Fibonacci sequence $\{F_n\}_{n\geq 0}$ satisfies the recurrence relation

$$
F_{n+1} = F_n + F_{n-1} \quad (n \ge 1), \tag{2}
$$

where $F_0 = 0$ and $F_1 = 1$. The Binet formula for $\{F_n\}_{n>0}$ is

$$
F_n = \frac{\alpha^n - (-1)^n \alpha^{-n}}{\sqrt{5}},
$$

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where $\alpha = (1 + \sqrt{5})/2$. Catarino and Borges [4] proved that $Le_n = 2F_{n+1} - 1$ for $n \geq 0$. Kuhapatanakul and Chobsorn [10] showed that

$$
\mathcal{L}_{k,n} = (k+1)F_{n+1} - k, \quad (n \ge 0). \tag{3}
$$

Recently, more properties of Leonardo numbers have been studied (see for instance $[1, 4, 5, 10, 11, 13, 16]$.

The purpose of this paper is to discuss the log-behavior of some sequences involving $\mathcal{L}_{k,n}$. Now we recall some definitions involved in this paper. A positive sequence $\{z_n\}_{n\geq 0}$ is said to be *log-convex* (*log-concave*) if $z_n^2 \leq z_{n-1}z_{n+1}$ ($z_n^2 \geq z_{n-1}z_{n+1}$) for each $n \geq 1$. A log-convex sequence $\{z_n\}_{n\geq 0}$ is said to be *log-balanced* if $\{z_n/n!\}_{n\geq 0}$ is log-concave (Došlić [7] gave this definition). It is evident that a sequence $\{z_n\}_{n>0}$ is log-convex (log-concave) if and only if its quotient sequence $\{z_{n+1}/z_n\}_{n>0}$ is nondecreasing (nonincreasing) and a log-convex sequence $\{z_n\}_{n\geq 0}$ is log-balanced if and only if $\{z_{n+1}/((n+1)z_n)\}_n>0$ is nonincreasing. Log-behavior of sequences is not only a fertile source for inequalities, but also plays an important role in many subjects (see for instance $[2, 3, 9, 6, 8, 12, 15]$). Hence the log-behavior of sequences deserves to be studied. In the next section, we investigate the log-behavior of a series of sequences related to $\mathcal{L}_{k,n}$.

2. The Log-Behavior of Some Sequences Related to the Generalized Leonardo Numbers

In what follows, the following lemmas will be used.

Lemma 1. ([17]) Let $\{z_n\}_{n>0}$ be a positive sequence of real numbers defined by the following recurrence relation

$$
z_n = c_1 z_{n-1} - c_2 z_{n-2} - \dots - c_k z_{n-k}, \quad n \ge k,
$$

where $k \geq 2$, $c_1 > 0$, $c_j \geq 0$ $(2 \leq j \leq k)$. If $\{z_0, z_1, z_2, \dots, z_k, z_{k+1}\}$ is log-concave (log-convex), the sequence $\{z_n\}_{n>0}$ is log-concave (log-convex).

Lemma 2. Let $\{z_n\}_{n>0}$ be a positive sequence defined by the recurrence relation

$$
z_{n+1} = c_1 z_n - c_2 z_{n-2}, \quad n \ge 2,
$$
\n⁽⁴⁾

where $c_1 > 0$ and $c_2 \geq 0$ are constants. For $n \geq 0$, let $x_n = z_{n+1}/z_n$. If there exists a nonnegative integer n_0 such that $\{1/z_{n_0}, 1/z_{n_0+1}, 1/z_{n_0+2}, 1/z_{n_0+3}\}$ is logbalanced and $c_1x_nx_{n+1} - 3c_2 \ge 0$ for $n \ge n_0$, then the sequence $\{1/z_n\}_{n \ge n_0}$ is log-balanced.

Proof. Since $\{1/z_{n_0}, 1/z_{n_0+1}, 1/z_{n_0+2}, 1/z_{n_0+3}\}$ is log-balanced, $\{z_j\}_{n_0 \leq j \leq n_0+3}$ is log-concave. It follows from Lemma 1 that $\{z_n\}_{n\geq n_0}$ is log-concave. Then $\{1/z_n\}_{n\geq n_0}$

INTEGERS: 24 (2024) 3

is log-convex. For $n \geq 0$, let $x_n = z_{n+1}/z_n$. It follows from (4) that

$$
x_n = c_1 - \frac{c_2}{x_{n-2}x_{n-1}}, \quad n \ge 2.
$$
 (5)

In order to prove that $\{1/z_n\}_{n\geq n_0}$ is log-balanced, we need to show that $\{1/(n+1)\}$ $(1)x_n$ } $_{n\geq n_0}$ is decreasing. Now we prove by induction that $\{(n+1)x_n\}_{n\geq n_0}$ is increasing. Since $\{1/z_j\}_{n_0\leq j\leq n_0+3}$ is log-balanced, $(j+2)x_{j+1} \geq (j+1)x_j$ for $n_0 \le j \le n_0 + 2$. Assume that $(j + 2)x_{j+1} \ge (j + 1)x_j$ for $j \ge n_0 + 2$. It follows from (5) that

$$
(j+3)x_{j+2} - (j+2)x_{j+1} = c_1 + \frac{c_2[(j+2)x_{j+1} - (j+3)x_{j-1}]}{x_{j-1}x_jx_{j+1}}
$$

It follows from $(j + 2)x_{j+1} ≥ (j + 1)x_j ≥ jx_{j-1}$ for $j ≥ n_0 + 2$ that

$$
(j+3)x_{j+2} - (j+2)x_{j+1} \ge c_1 - \frac{3c_2}{x_jx_{j+1}} = \frac{c_1x_jx_{j+1} - 3c_2}{x_jx_{j+1}} \ge 0.
$$

Hence $\{(n+1)x_n\}_{n\geq n_0}$ is increasing. Naturally, $\{1/((n+1)x_n)\}_{n\geq n_0}$ is decreasing. \Box

Now we give the main results of this paper.

Theorem 1. For the generalized Leonardo sequence $\{\mathcal{L}_{k,n}\}_{n\geq 0}$, $\{1/\mathcal{L}_{k,n}\}_{n\geq 3}$ is log-balanced.

Proof. It follows from (1) that

$$
\mathcal{L}_{k,n+1} = 2\mathcal{L}_{k,n} - \mathcal{L}_{k,n-2}, \quad n \ge 2.
$$

Clearly, $c_1 = 2$ and $c_2 = 1$. By means of (3), we obtain

$$
\mathcal{L}_{k,3} = 2k + 3
$$
, $\mathcal{L}_{k,4} = 4k + 5$, $\mathcal{L}_{k,5} = 7k + 8$, $\mathcal{L}_{k,6} = 12k + 13$.

We observe that $\{\mathcal{L}_{k,j}\}_{3\leq j\leq 6}$ is log-concave. Thus, the sequence $\{\mathcal{L}_{k,n}\}_{n\geq 3}$ is logconcave by Lemma 1. It is clear that $\{1/\mathcal{L}_{k,n}\}_{n\geq 3}$ is log-convex. Since $\{\mathcal{L}_{k,n}\}_{n\geq 3}$ is log-concave and

$$
\lim_{n \to +\infty} \frac{\mathcal{L}_{k,n+1}}{\mathcal{L}_{k,n}} = \alpha,
$$

we have

$$
\alpha \le x_j \le x_3 = \frac{4k+5}{2k+3} < 2 \quad (j \ge 3). \tag{6}
$$

By applying (6), we derive $c_1x_nx_{n+1} - 3c_2 = 2x_nx_{n+1} - 3 \ge 2\alpha - 3 > 0$ for $n \ge 3$. On the other hand, we find that $\{1/z_3, 1/z_4, 1/z_5, 1/z_6\}$ is log-balanced. It follows from Lemma 2 that the sequence $\{1/\mathcal{L}_{k,n}\}_{n\geq 3}$ is log-balanced. \Box

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Theorem 2. For the generalized Leonardo sequence $\{\mathcal{L}_{k,n}\}_{n\geq0}$, let $R_n = \sum_{i=0}^n \mathcal{L}_{k,i}$ ($n \geq 0$). The sequence $\{R_n\}_{n \geq 0}$ is log-concave.

Proof. For $n \geq 0$, Kuhapatanakul and Chobsorn [10] proved that

$$
\sum_{i=0}^{n} \mathcal{L}_{k,i} = \mathcal{L}_{k,n+2} - k(n+1) - 1.
$$
 (7)

It follows from (7) and (3) that

$$
R_n^2 - R_{n-1}R_{n+1} = [(k+1)(F_{n+3} - 1) - k(n+1)]^2
$$

$$
-[(k+1)(F_{n+2} - 1) - kn][(k+1)(F_{n+4} - 1) - k(n+2)]
$$

$$
= (k+1)^2[(F_{n+3} - 1)^2 - (F_{n+2} - 1)(F_{n+4} - 1)] + k^2
$$

$$
+k(k+1)[n(F_{n+2} + F_{n+4} - 2F_{n+3}) + 2F_{n+2} - 2F_{n+3}].
$$

For the Fibonacci sequence $\{F_n\}_{n\geq 0}$, there is an identity

$$
F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1}.
$$
\n(8)

Using (8) and (2) , we have

$$
R_n^2 - R_{n-1}R_{n+1} = (k+1)^2 [F_n + (-1)^n] + k(k+1)(nF_n - 2F_{n+1}) + k^2.
$$

We observe that $R_j^2 - R_{j-1}R_{j+1} > 0$ for $1 \le j \le 3$. For $n \ge 4$,

$$
R_n^2 - R_{n-1}R_{n+1} \ge k(k+1)(4F_n - 2F_{n+1}) + k^2 = 2k(k+1)F_{n-2} + k^2 > 0.
$$

Hence ${R_n}_{n\geq 0}$ is log-concave.

Theorem 3. For the generalized Leonardo sequence
$$
\{\mathcal{L}_{k,n}\}_{n\geq 0}
$$
, $\{\sqrt[n]{\mathcal{L}_{k,n}}\}_{n\geq 9}$ is log-convex.

Proof. It is clear that $\{\sqrt[n]{\mathcal{L}_{k,n}}\}_{n\geq 9}$ is log-convex if and only if

$$
2(n^2 - 1) \ln \mathcal{L}_{k,n} - n(n+1) \ln \mathcal{L}_{k,n-1} - n(n-1) \ln \mathcal{L}_{k,n+1} \le 0 \quad (n \ge 10).
$$

For $n \geq 10$, put

$$
S_n = 2(n^2 - 1) \ln \mathcal{L}_{k,n} - n(n+1) \ln \mathcal{L}_{k,n-1} - n(n-1) \ln \mathcal{L}_{k,n+1}
$$

and

$$
T_n = \frac{\mathcal{L}_{k,n}^3 \mathcal{L}_{k,n+2}}{\mathcal{L}_{k,n-1} \mathcal{L}_{k,n+1}^3}.
$$

It is evident that

$$
S_n - S_{n+1} = n(n+1) \ln T_n.
$$

 \Box

INTEGERS: 24 (2024) 5

Owing to (3), (8), and (2),

$$
\mathcal{L}_{k,n}^2 - \mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1} = (-1)^n (k+1)^2 + k(k+1)F_{n-2}.
$$
 (9)

This leads to

$$
T_n = \frac{\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1} + k(k+1)F_{n-2} + (-1)^n(k+1)^2}{\mathcal{L}_{k,n}\mathcal{L}_{k,n+2} + k(k+1)F_{n-1} + (-1)^{n+1}(k+1)^2} \cdot \frac{\mathcal{L}_{k,n}\mathcal{L}_{k,n+2}}{\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1}}
$$

For $j \geq 0$, set $x_j = \mathcal{L}_{k,j+1}/\mathcal{L}_{k,j}$. Then

$$
T_n = \frac{\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1} + k(k+1)F_{n-2} + (-1)^n(k+1)^2}{\mathcal{L}_{k,n}\mathcal{L}_{k,n+2} + k(k+1)F_{n-1} + (-1)^{n+1}(k+1)^2} \cdot x_{n-1}x_{n+1}
$$

=
$$
\frac{\mathcal{L}_{k,n}\mathcal{L}_{k,n+2} + k(k+1)x_{n-1}x_{n+1}F_{n-2} + (-1)^n(k+1)^2x_{n-1}x_{n+1}}{\mathcal{L}_{k,n}\mathcal{L}_{k,n+2} + k(k+1)F_{n-1} + (-1)^{n+1}(k+1)^2}.
$$

Now we prove that $T_n \geq 1$ for $n \geq 10$. When $n \geq 4$,

$$
\mathcal{L}_{k,n}\mathcal{L}_{k,n+2} + k(k+1)x_{n-1}x_{n+1}F_{n-2} + (-1)^n(k+1)^2x_{n-1}x_{n+1} > 0
$$

and

$$
\mathcal{L}_{k,n}\mathcal{L}_{k,n+2} + k(k+1)F_{n-1} + (-1)^{n+1}(k+1)^2 > 0
$$

hold. For $n \geq 10$, let

$$
U_n = k(k+1)(x_{n-1}x_{n+1}F_{n-2} - F_{n-1}) + (-1)^n(k+1)^2(x_{n-1}x_{n+1} + 1).
$$

It is obvious that $T_n\geq 1$ if and only if $U_n\geq 0.$ We note that

$$
U_n \ge k(k+1)(x_{n-1}x_{n+1}F_{n-2} - F_{n-1}) - (k+1)^2(x_{n-1}x_{n+1} + 1).
$$

By using (6), we get

$$
U_n \ge k(k+1)(\alpha^2 F_{n-2} - F_{n-1}) - 5(k+1)^2
$$

$$
\ge k(k+1)\left(\frac{5}{2}F_{n-2} - F_{n-1}\right) - 5(k+1)^2.
$$

By using (2) , we have

$$
U_n \ge k(k+1)\left(\frac{1}{2}F_{n-2} + F_{n-4}\right) - 5(k+1)^2 > 0 \quad (n \ge 10).
$$

Naturally, $T_n \ge 1$ for each $n \ge 10$. Thus the sequence $\{S_n\}_{n \ge 10}$ is decreasing. We observe that

$$
S_{10} = 198 \ln(88k + 89) - 110 \ln(54k + 55) - 90 \ln(143k + 144)
$$

=
$$
110 \ln \left(\frac{88k + 89}{54k + 55}\right) - 90 \ln \left(\frac{143k + 144}{88k + 89}\right) - 2 \ln(88k + 89)
$$

$$
\leq 110 \ln \frac{44}{27} - 90 \ln \frac{287}{177} - 2 \ln 177 < 0.
$$

Hence $2(n^2 - 1) \ln \mathcal{L}_{k,n} - n(n+1) \ln \mathcal{L}_{k,n-1} - n(n-1) \ln \mathcal{L}_{k,n+1} \leq 0$ holds for $n \geq 10$. \Box

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Theorem 4. For the generalized Leonardo sequence $\{\mathcal{L}_{k,n}\}_{n\geq0}$, there exists a posi-**Theorem 4.** For the generalized Leonardo sequence $\{\mathcal{L}_{k,n}\}_{n\geq 0}$, there exists a positive integer $n_k \geq \max\{9, (1+\sqrt{64k-7})/4\}$ such that $\{\sqrt[n]{\mathcal{L}_{k,n}}\}_{n\geq n_k}$ is log-balanced.

Proof. For $n \geq 0$, let

$$
\mathcal{U}_n = n(n+1)(n+2)\ln\left(1+\frac{1}{n+1}\right) + 2n(n+2)\ln\mathcal{L}_{k,n+1} - (n+1)(n+2)\ln\mathcal{L}_{k,n}
$$

$$
-n(n+1)\ln\mathcal{L}_{k,n+2}.
$$

It is clear that $\{\sqrt[n]{\mathcal{L}_{k,n}}/n!\}_{n\geq n_k}$ is log-concave if and only if $\mathcal{U}_n \geq 0$ for $n \geq n_k$. For $n \geq 0$, put $x_n = \mathcal{L}_{k,n+1}/\overline{\mathcal{L}}_{k,n}$. We note that

$$
\mathcal{U}_n = n(n+1)(n+2)\ln\left(1+\frac{1}{n+1}\right) + 2n(n+2)\ln\mathcal{L}_{k,n+1} - (n+1)(n+2)\ln\frac{\mathcal{L}_{k,n+1}}{x_n}
$$

$$
-n(n+1)\ln(x_{n+1}\mathcal{L}_{k,n+1})
$$

$$
= n(n+1)(n+2)\ln\left(1+\frac{1}{n+1}\right) - 2\ln\mathcal{L}_{k,n+1} + (n+1)^2\ln\frac{x_n}{x_{n+1}}
$$

$$
+ (n+1)\ln(x_nx_{n+1})
$$

$$
= n(n+1)(n+2)\ln\left(1+\frac{1}{n+1}\right) - 2\ln(x_3\cdots x_n\mathcal{L}_{k,3}) + (n+1)^2\ln\frac{x_n}{x_{n+1}}
$$

$$
+ (n+1)\ln(x_nx_{n+1}).
$$

Since $\{\mathcal{L}_{k,n}\}_{n\geq 3}$ is log-concave, $x_{n-1}/x_n \geq 1$ for $n \geq 4$. Thus

$$
\mathcal{U}_n \ge n(n+1)(n+2)\ln\left(1+\frac{1}{n+1}\right) - 2\ln\mathcal{L}_{k,3} - 2(n-2)\ln x_3 + (n+1)\ln(x_nx_{n+1})
$$

= $n(n+1)(n+2)\ln\left(1+\frac{1}{n+1}\right) - 2\ln(2k+3) - 2(n-2)\ln\left(1+\frac{2k+2}{2k+3}\right)$
+ $(n+1)\ln(x_nx_{n+1}).$

By the inequality

$$
\frac{t}{1+t} < \ln(1+t) < t \quad (t > 0) \tag{10}
$$

and (6), we obtain

$$
u_n \ge n^2 + n - 2(2k + 2) - \frac{2(n-2)(2k+2)}{2k+3} + 2(n+1)\ln \alpha.
$$

It is clear that

$$
u_n \ge n^2 + n - 2(2k + 2) - 2(n - 2) + \frac{n + 1}{2} \ge 0 \quad \left(n \ge \frac{1 + \sqrt{64k - 7}}{4}\right).
$$

On the other hand, the sequence $\{\sqrt[n]{\mathcal{L}_{k,n}}\}_{n\geq 9}$ is log-convex. Then there exists a positive integer $n_k \geq \max\{9, (1 + \sqrt{64k - 7})/4\}$ such that $\{\sqrt[n]{\mathcal{L}_{k,n}}\}_{n \geq n_k}$ is logbalanced. \Box INTEGERS: 24 (2024) 7

Theorem 5. For the generalized Leonardo sequence $\{\mathcal{L}_{k,n}\}_{n\geq0}$, $\{\mathcal{L}_{k,n}^n\}_{n\geq3}$ is logconvex.

Proof. It is clear that $\mathcal{L}_{k,n}^{2n} - \mathcal{L}_{k,n-1}^{n-1} \mathcal{L}_{k,n+1}^{n+1} \leq 0$ ($n \geq 4$) is equivalent to

$$
2n \ln \mathcal{L}_{k,n} - (n-1) \ln \mathcal{L}_{k,n-1} - (n+1) \ln \mathcal{L}_{k,n+1} \le 0 \quad (n \ge 4).
$$

For $n \geq 0$, set $x_n = \mathcal{L}_{k,n+1}/\mathcal{L}_{k,n}$ and

$$
V_n = 2n \ln \mathcal{L}_{k,n} - (n-1) \ln \mathcal{L}_{k,n-1} - (n+1) \ln \mathcal{L}_{k,n+1} \quad (n \ge 4).
$$

It is evident that

$$
V_n = n \ln \frac{\mathcal{L}_{k,n}^2}{\mathcal{L}_{k,n-1} \mathcal{L}_{k,n+1}} - \ln x_{n-1} - \ln x_n.
$$

By (9) , we have

$$
V_n = n \ln \frac{\mathcal{L}_{k,n-1} \mathcal{L}_{k,n+1} + (-1)^n (k+1)^2 + k(k+1) F_{n-2}}{\mathcal{L}_{k,n-1} \mathcal{L}_{k,n+1}} - \ln x_{n-1} - \ln x_n.
$$

For $n \geq 4$, it follows from (10) and (6) that

$$
V_n \leq \frac{(-1)^n (k+1)^2 n + nk(k+1) F_{n-2}}{\mathcal{L}_{k,n-1} \mathcal{L}_{k,n+1}} - 2 \ln \alpha
$$

$$
< \frac{(k+1)^2 n + nk(k+1) F_{n-2} - \frac{1}{2} \mathcal{L}_{k,n-1} \mathcal{L}_{k,n+1}}{\mathcal{L}_{k,n-1} \mathcal{L}_{k,n+1}}.
$$

For convenience, let

$$
W_n = (k+1)^2 n + nk(k+1)F_{n-2} - \frac{1}{2}\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1} \quad (n \ge 4).
$$

Then $V_n < W_n/(\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1})$. One can prove by induction that

$$
\mathcal{L}_{k,j} > 2kj \quad (j \ge 6). \tag{11}
$$

For $n \geq 5$, it follows from (11) that

$$
W_n < (k+1)^2 n + nk(k+1)F_{n-2} - k(n+1)L_{k,n-1}.
$$

By using (3) and (2) , we derive

$$
W_n \le (k+1)^2 n + nk(k+1)F_{n-2} - k(n+1)[(k+1)F_n - k]
$$

= $(k+1)^2 n + k^2(n+1) - nk(k+1)F_{n-1} - k(k+1)F_n$
< 0 \quad (n \ge 5).

Thus $V_n < 0$ for $n \geq 5$. On the other hand, we find that $V_4 < 0$. Hence the sequence $\{\mathcal{L}_{k,n}^n\}_{n\geq 3}$ is log-convex. \Box

3. Conclusions

For the generalized Leonardo sequence $\{\mathcal{L}_{k,n}\}_{n\geq 0}$, we discussed the log-behavior of some sequences involving $\mathcal{L}_{k,n}$ in this paper. We mainly proved that $\{1/\mathcal{L}_{k,n}\}_{n\geq 3}$ is log-balanced and $\{\sqrt[n]{\mathcal{L}_{k,n}}\}_{n\geq 9}$ and $\{\mathcal{L}_{k,n}^n\}_{n\geq 3}$ are log-convex. Let $\{f_n\}_{n\geq 0}$ be a positive sequence. The future work is to study the log-behavior of $\{\mathcal{L}_{k,n}^{f_n}\}_{n\geq 0}$ under some conditions.

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