ALTERNATING SUMS IN THE GENERALIZED HYPERBOLIC PASCAL TRIANGLE

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Abstract

Recently, a generalization of hyperbolic Pascal triangles was introduced. These triangles are based on the regular mosaics in the hyperbolic plane and the number of shortest paths in a graph-theoretical sense. A previous paper by the authors describes a generalization with several properties, such as the recurrence relation satisfied by the row sums. In the present paper, we describe the alternating sums of rows linked to the regular square mosaic $\{4, q\}$ with $q \geq 5$, providing the corresponding recurrence relations, explicit formula, and generating function.

1. Introduction

In the hyperbolic plane, there exist infinitely many types of regular mosaics denoted by Schläfli's symbol $\{p, q\}$, where the positive integers p and q satisfy the condition $(p-2)(q-2) > 4$ (see [4]). The parameters p and q signify that exactly q regular p-gons meet at each node of the mosaic. Each regular mosaic induces a so-called hyperbolic Pascal triangle, following and generalizing the connection between the classical Pascal triangle and the Euclidean regular square mosaic $\{4, 4\}$.

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The hyperbolic Pascal triangle based on the mosaic $\{p, q\}$ can be represented as a directed graph, where the vertices and the edges correspond to the vertices and edges of a well-defined part of the lattice $\{p, q\}$, respectively. For $\{4, q\}$, the base vertex has two edges, while the leftmost and rightmost vertices have three edges each, and the remaining vertices have q edges. The square-shaped cells, surrounded by appropriate edges, correspond to the squares in the mosaic. Apart from the leftmost and the rightmost elements, referred to as *winger elements*, specific vertices, designated as *Type A vertices*, have two ascendants and $q-2$ descendants; while the other vertices, termed *Type B vertices*, have one ascendant and $q-1$ descendants.

The general method of drawing is the following. Going along the vertices of the j^{th} row, according to the type of the elements (winger, A, B), we draw the appropriate number of edges downwards (2, $q-2$, $q-1$, respectively). The edges of two neighboring vertices in the j^{th} row meet in the $(j+1)^{th}$ row, forming a vertex of type A. Apart from the wingers, the other descendants of row j in row $(j+1)$ are of type B (see [1] for more details). Some interesting properties of hyperbolic Pascal triangles have been investigated, including power sums, alternating sums, and the connection between Fibonacci words and hyperbolic Pascal triangles (see, for instance, [1, 5, 6, 7]).

Recently, generalized hyperbolic Pascal triangles ($\mathcal{GHPT}$) were introduced in [2], which are structurally identical to the hyperbolic Pascal triangles associated with $\{4, q\}$. In this generalization, two arbitrary sequences, $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n\}_{n\geq 0}$, define the leg-sequences. Figure 1 illustrates the $\mathcal{GHPT}$ for $q = 5$ with $\alpha_n = 3n$ and $\beta_n = 3^n$. In the figure, vertices of type A are represented by red circles, vertices of type B by cyan diamonds, and wing vertices by white diamonds. The vertices that are n edges away from the base vertex belong to row n.

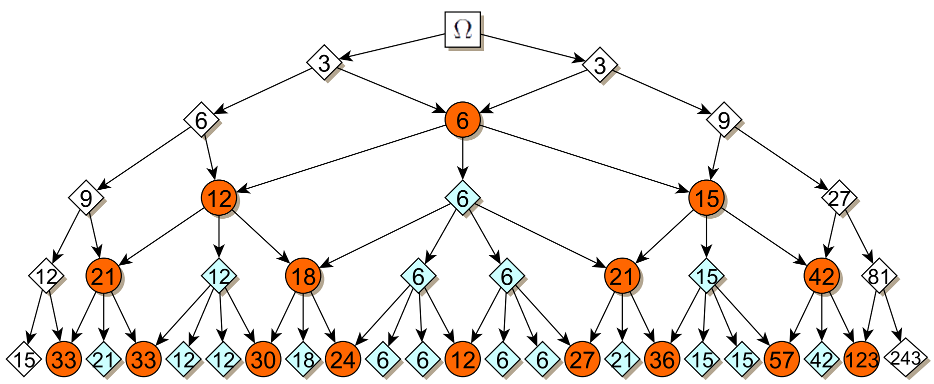

Figure 1: Rows $0, 1, \ldots, 5$ of a $\mathcal{GHPT}$ linked to $\{4, 5\}_{\{3n, 3^n\}}$

In the sequel, $\left|\begin{smallmatrix} n \\ k \end{smallmatrix}\right|_{\{\alpha_n,\beta_n\}}$ denotes the k^{th} element in row n of $\mathcal{GHPT}$ generated by the sequences $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n\}_{n\geq 0}$, which are located on the left and right leg, respectively. This element is either the sum of the values of its two ascendants or the value of its unique ascendant.

Regardless of whether $\alpha_0 \neq \beta_0$, we replace all terms with Ω as an indeterminate object. For convenience, we define $\sigma_n = \alpha_n + \beta_n$ and $\delta_n = \alpha_n - \beta_n$.

In the next section, we will summarize some key results from [1] and [2], which are essential for proving the main result of this paper.

2. Previous Results

Fixing q, we consider the hyperbolic Pascal triangle $\{4, q\}$. Let s_n denote the number of vertices in row n. The sequence s_n is detailed in [1]. According to the types of the entries, we write

$$s_n = a_n + b_n + 2,$$

where a_n and b_n denote the number of vertices of type A and B of the n^{th} row, respectively.

The three sequences $\{a_n\}$, $\{b_n\}$, and $\{s_n\}$ satisfy the same ternary recurrence relation for $n \geq 4$:

$$x_n = (q-1)x_{n-1} - (q-1)x_{n-2} + x_{n-3}, \tag{1}$$

with initial values $a_1 = 0, a_2 = 1, a_3 = 2$; $b_1 = 0, b_2 = 0, b_3 = q-4$; $s_1 = 2, s_2 = 3, s_3 = q$.

As previously noted, a generalized hyperbolic Pascal triangle is structurally identical to the hyperbolic Pascal triangle $\{4, q\}$. Therefore, the arguments presented above remain applicable to $\mathcal{GHPT}$.

Now, we state two consequences of Relation (1), upon which the proof of the main result in this paper partially relies. For $n \geq 1$, if q is even, then

$$s_n \equiv \begin{cases} 0 \pmod 2, & \text{for} \quad n = 2t+1, \\ 1 \pmod 2, & \text{for} \quad n = 2t. \end{cases} \tag{2}$$

Otherwise, if q is odd, then

$$s_n \equiv \begin{cases} 0 \pmod 2, & \text{for} \quad n = 3t+1, \\ 1 \pmod 2, & \text{for} \quad n \neq 3t+1. \end{cases} \tag{3}$$

Moreover, let $\widehat{a}_n$, $\widehat{b}_n$, and $\widehat{s}_n$ denote the sum of elements of type A, the sum of elements of type B, and the sum of all elements of the n^{th} row in $\mathcal{GHPT}$, respectively. In [2], it was established that each of the three sequences $\{\widehat{a}_n\}_{\{\alpha_n,\beta_n\}}$, $\{\widehat{b}_n\}_{\{\alpha_n,\beta_n\}}$, and $\{\widehat{s}_n\}_{\{\alpha_n,\beta_n\}}$ satisfy the recurrence relation:

$$\widehat{x}_n = (q-1)\widehat{x}_{n-1} - 2\widehat{x}_{n-2} + w_{n-2}, \qquad n \geq 3, \tag{4}$$

where the initial values and the sequence $\{w_n\}$ are given, respectively, by

1. $\widehat{a}_1 = 0, \widehat{a}_2 = \sigma_1$, $w_n = \sigma_{n+1} - (q-3)\sigma_n$ for $n \geq 1$,
2. $\widehat{b}_1 = 0, \widehat{b}_2 = 0$, $w_n = (q-4)\sigma_n$ for $n \geq 1$,
3. $\widehat{s}_1 = \sigma_1, \widehat{s}_2 = \sigma_1 + \sigma_2$, $w_n = \sigma_{n+2} - (q-2)\sigma_{n+1} + \sigma_n$ for $n \geq 1$.

We also proved a practical general theorem in [2], which will be used later under specific circumstances. Given a positive integer k and complex numbers $f_0, f_1, \ldots, f_{k-1}$, define

$$f_n = A_1 f_{n-1} + A_2 f_{n-2} + \cdots + A_k f_{n-k} \qquad (n \geq k), \tag{5}$$

where the coefficients $A_1, \ldots, A_{k-1}, 0 \neq A_k$ are fixed complex numbers. Moreover, suppose that $\{w_n\}_{n\geq 0} \in \mathbb{C}^\infty$ is an arbitrary sequence. Based on the notation above, we construct the linear recurrence

$$G_n = A_1 G_{n-1} + A_2 G_{n-2} + \cdots + A_k G_{n-k} + w_{n-k} \qquad (n \geq k), \tag{6}$$

assuming that the complex initial values $G_0, G_1, \ldots, G_{k-1}$ are also given. Note that formulae (5) and (6) essentially differ only in the diverting sequence $\{w_n\}$.

Theorem 1. *For $n \geq k$ the terms of the sequences $\{f_n\}$, $\{w_n\}$, and $\{G_n\}$ satisfy the identity*

$$\sum_{j=0}^{k-1} f_j G_{n+k-j} = \sum_{j=0}^{k-1} \sum_{i=0}^{k-1-j} f_{n-j} A_{j+1+i} G_{k-1-i} + \sum_{j=0}^{k-2} \sum_{i=1}^{k-1-j} f_j A_i G_{n+k-j-i} + \sum_{j=0}^{n} f_{n-j} w_j. \tag{7}$$

The coefficients $A_1, \ldots, A_k$ in the definition of $\{f_n\}$ are important in the sense that they, together with $\{w_n\}$ also establish the sequence $\{G_n\}$. But, generally, the initial values $f_0, \ldots, f_{k-1}$ can be chosen as simply as possible. Therefore it is natural, if there is no other reason, to put $f_0 = \cdots = f_{k-1} = 0$, $f_{k-1} = 1$. The next corollary describes this situation.

Corollary 1. *Assume that $f_0 = \cdots = f_{k-2} = 0$, $f_{k-1} = 1$. Then (7) simplifies to*

$$G_{n+1} = \sum_{j=0}^{k-1} \sum_{i=0}^{k-1-j} f_{n-j} A_{j+1+i} G_{k-1-i} + \sum_{j=0}^{n} f_{n-j} w_j. \tag{8}$$

The next section is the main focus of this paper, where we describe the alternating sums of $\mathcal{GHPT}$ based on the $\{4, q\}$ mosaic with $q \geq 5$. We provide a recurrence relation, an explicit formula, and the generating function.

3. Main Results

Let $\tilde{s_n}$ denote the alternating row sum

$$\tilde{s_n} = \sum_{i=0}^{s_n-1} (-1)^i \left)\begin{smallmatrix} n \\ k \end{smallmatrix}\right(_{\{\alpha_n,\beta_n\}}$$

of elements of the generalized hyperbolic Pascal triangle in row n. Put

$$\tilde{q} = \begin{cases} 1, & \text{if } q \text{ is odd,} \\ 0, & \text{if } q \text{ is even,} \end{cases}$$

and

$$\chi_n = \begin{cases} \sigma_n, & \text{if } (q = 2k \text{ and } n = 2t) \\ & \quad \text{or } (q = 2k+1 \text{ and } n \neq 3t+1), \\ \delta_n, & \text{if } (q = 2k \text{ and } n = 2t+1) \\ & \quad \text{or } (q = 2k+1 \text{ and } n = 3t+1). \end{cases} \tag{9}$$

Note that the two branches formulate the two cases s_n even and odd, respectively.

Theorem 2. *Assume $n \geq 4$ if q is even, and $n \geq 6$ otherwise. The sequence $(\tilde{s}_n)_n$ satisfies the recurrence relation*

$$\tilde{s}_n = 2^{\tilde{q}}(5-q)\tilde{s}_{n-2-\tilde{q}} + w_{n-2-\tilde{q}}, \tag{10}$$

where

$$w_{n-2-\tilde{q}} = \chi_n - \chi_{n-1} - \tilde{q}\chi_{n-2} - 2^{\tilde{q}}(5-q)\chi_{n-2-\tilde{q}} + 2^{\tilde{q}}\chi_{n-3-\tilde{q}},$$

and the initial values are as follows:

- *For q even:* $\tilde{s}_1 = \delta_1$, $\tilde{s}_2 = \sigma_2 - \sigma_1$, $\tilde{s}_3 = \delta_3 - \delta_2$;
- *For q odd:* $\tilde{s}_1 = \delta_1$, $\tilde{s}_2 = \sigma_2 - \sigma_1$, $\tilde{s}_3 = \sigma_3 - \sigma_2 - \sigma_1$, $\tilde{s}_4 = \delta_4 - \delta_3 - \delta_2$, $\tilde{s}_5 = 2(q-4)\sigma_1 - \sigma_3 - \sigma_4 + \sigma_5$.

Proof. The basis of the proof is to consider the vertices of types A and B of row n and to observe their influence either on $\tilde{s}_{n+2}$ or on $\tilde{s}_{n+3}$, depending on the parity of q. We separate the contribution of each $\left)\begin{smallmatrix} n \\ k \end{smallmatrix}\right(_{\{\alpha_n,\beta_n\}}$ individually, and then take their superposition. Let $\tilde{s}_n^{(A)}$ and $\tilde{s}_n^{(B)}$ be the subsum of $\tilde{s}_n$ restricted only to the elements of type A and B, respectively. Since $)_0^n(= \alpha_n$ and $)_{s_n-1}^{\;n}(= \beta_n$, we see that:

$$\tilde{s}_n = \tilde{s}_n^{(A)} + \tilde{s}_n^{(B)} + \delta_n \tag{11}$$

if s_n is even. Otherwise,

$$\tilde{s}_n = \tilde{s}_n^{(A)} + \tilde{s}_n^{(B)} + \sigma_n. \tag{12}$$

Keeping the notation of [1], we use x_A and x_B to denote the value of an element of type A and B, respectively. Their contributions to $\tilde{s}_{n+k} \quad (k \geq 1)$ are denoted by $\mathcal{H}_k(x_A)$ and $\mathcal{H}_k(x_B)$, respectively. We also use the notation $\mathcal{H}_k^{(A)}(x_A)$ and $\mathcal{H}_k^{(B)}(x_A)$ to represent the contribution of the type A element x_A from row n to the alternating sum of row $n+k$ restricted to the elements of types A and B, respectively. In accordance with [1], we express this as:

$$\begin{aligned}
\mathcal{H}_k(x_A) =& \mathcal{H}_k^{(A)}(x_A) + \mathcal{H}_k^{(B)}(x_A),\\
\mathcal{H}_k(x_B) =& \mathcal{H}_k^{(A)}(x_B) + \mathcal{H}_k^{(B)}(x_B).
\end{aligned}$$

As the alternating sum begins with a positive coefficient, the contribution of the leftmost element from row n to the alternating sum of row $n+k$ is given by the equation

$$\mathcal{H}_k(\alpha_n) = \mathcal{H}_k^{(A)}(\alpha_n) + \mathcal{H}_k^{(B)}(\alpha_n) + \alpha_{n+k}. \tag{13}$$

However, the contribution of the rightmost element from row n to the alternating sum of the $(n+k)^{th}$ row depends on the parity of s_n the number of elements of the n^{th} row. Clearly,

$$\mathcal{H}_k(\beta_n) = \mathcal{H}_k^{(A)}(\beta_n) + \mathcal{H}_k^{(B)}(\beta_n) - (-1)^{s_n}\beta_{n+k}. \tag{14}$$

It is easy to see that $\tilde{s}_0 = \Omega$, $\tilde{s}_1 = \delta_1$, and $\tilde{s}_2 = \sigma_2 - \sigma_1$ hold for all $q \geq 5$.

Now, let us consider two cases:

`Case 1:` q `is even`. In this case, there is a connection between the alternating sums of rows n and $n+2$. Specifically, as indicated in [5], the contributions of x_A and x_B from row n to the alternating sum of row $n+2$ are given by:

$$\mathcal{H}_2^{(A)}(x_A) = -2(q-4)\varepsilon_1 x_A, \qquad \mathcal{H}_2^{(B)}(x_A) = (q-4)\varepsilon_1 x_A,$$

and

$$\mathcal{H}_2^{(A)}(x_B) = -2(q-3)\varepsilon_2 x_B, \qquad \mathcal{H}_2^{(B)}(x_B) = (q-3)\varepsilon_2 x_B,$$

respectively. Here, $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$ when considering the vertex of type A and the vertex of type B, respectively.

Figure 2 shows the contributions of the influence of the leftmost element of row n to row $n+2$. Thus,

$$\mathcal{H}_2^{(A)}(\alpha_n) = -\alpha_{n+1}, \qquad \mathcal{H}_2^{(B)}(\alpha_n) = 0.$$

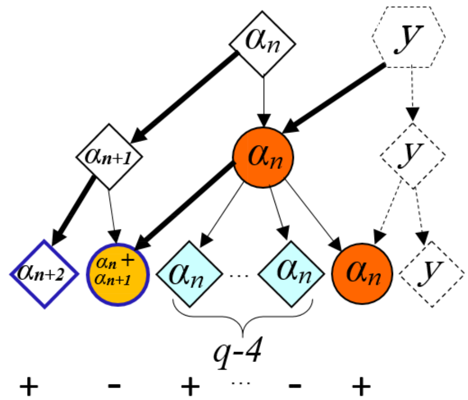

Figure 2: The influence of $\mathcal{H}_2(\alpha_n)$

Similarly, the contributions of the influence of the rightmost element of row n to row $n+2$, which depend on the parity of s_n, are given by:

$$\mathcal{H}_2^{(A)}(\beta_n) = (-1)^{s_n}\beta_{n+1}, \qquad \mathcal{H}_2^{(B)}(\beta_n) = 0.$$

We have described the influence of an element x (whether of type A, of type B, or a winger element) located in row n on row $n+2$. Let us suppose that y is the value of the neighboring element of x in row n. The signs of x and y in the alternating sum in row n are different, and the signs on the left hand side of their influence structures are also different in row $n+2$. The change in the sign from x to y entails the same change in the sign from $\mathcal{H}_2(x)$ to $\mathcal{H}_2(y)$. In other words, the signs of the alternating sum in row n descend to row $n+2$ in this manner. Thus, according to [1] we can describe the changes in the alternating sums from row n to row $n+2$.

Summarizing the results, we obtain the system of two recurrence equations as follows.

`Subcase (a)`: s_n is even. In this case, for $n \geq 0$, we get

$$\begin{cases} \tilde{s}_{n+2}^{(A)} = -2(q-4)\tilde{s}_n^{(A)} - 2(q-3)\tilde{s}_n^{(B)} - \delta_{n+1}, \\ \tilde{s}_{n+2}^{(B)} = (q-4)\tilde{s}_n^{(A)} + (q-3)\tilde{s}_n^{(B)}. \end{cases} \tag{15}$$

First, we multiply the first equation of the system by $-1/(2(q-3))$ and eliminate the term

$$\tilde{s}_n^{(B)} = \frac{-\tilde{s}_{n+2}^{(A)}}{2(q-3)} - \frac{(q-4)}{(q-3)}\tilde{s}_n^{(A)} - \frac{\delta_{n+1}}{2(q-3)}.$$

Then, by replacing $\tilde{s}_n^{(B)}$ and the shifted term $\tilde{s}_{n+2}^{(B)}$ in the second equation of the system, we immediately obtain

$$\tilde{s}_{n+4}^{(A)} = (5-q)\tilde{s}_{n+2}^{(A)} + (q-3)\delta_{n+1} - \delta_{n+3}, \quad (n \geq 0).$$

To find $\tilde{s}_{n+4}^{(B)}$, we follow the same approach that we used above. We divide the second equation of system (15) by $(q-4)$, eliminate $\tilde{s}_n^{(A)}$, and replace both $\tilde{s}_n^{(A)}$ and $\tilde{s}_{n+2}^{(A)}$ with the appropriate expressions. It leads right away to

$$\tilde{s}_{n+4}^{(B)} = (5-q)\tilde{s}_{n+2}^{(B)} - (q-4)\delta_{n+1}.$$

From (12), we conclude that for $(n \geq 0)$,

$$\begin{aligned} \tilde{s}_{n+4} &= (5-q)\tilde{s}_{n+2}^{(A)} + (5-q)\tilde{s}_{n+2}^{(B)} + \delta_{n+1} - \delta_{n+3} + \delta_{n+4} \\ &= (5-q)\tilde{s}_{n+2} + \delta_{n+1} - (5-q)\delta_{n+2} - \delta_{n+3} + \delta_{n+4}. \end{aligned}$$

`Subcase (b)`: s_n is odd. In this case, for $n \geq 0$, we get

$$\begin{cases} \tilde{s}_{n+2}^{(A)} = -2(q-4)\tilde{s}_n^{(A)} - 2(q-3)\tilde{s}_n^{(B)} - \sigma_{n+1}, \\ \tilde{s}_{n+2}^{(B)} = (q-4)\tilde{s}_n^{(A)} + (q-3)\tilde{s}_n^{(B)} \end{cases} \tag{16}$$

holds. The analogous operations we applied to the system (15) now lead to

$$\tilde{s}_{n+4}^{(A)} = (5-q)\tilde{s}_{n+2}^{(A)} + (q-3)\sigma_{n+1} - \sigma_{n+3},$$

and

$$\tilde{s}_{n+4}^{(B)} = (5-q)\tilde{s}_{n+2}^{(B)} - (q-4)\sigma_{n+1}.$$

For $n \geq 0$, it implies, by (11), that

$$\begin{aligned} \tilde{s}_{n+4} &= (5-q)\tilde{s}_{n+2}^{(A)} + (5-q)\tilde{s}_{n+2}^{(B)} + \sigma_{n+1} - \sigma_{n+3} + \sigma_{n+4} \\ &= (5-q)\tilde{s}_{n+2} + \sigma_{n+1} - (5-q)\sigma_{n+2} - \sigma_{n+3} + \sigma_{n+4}. \end{aligned}$$

For q even and $n \geq 4$ the two cases above can be united as

$$\tilde{s}_n = (5-q)\tilde{s}_{n-2} + (\underbrace{\chi_n - \chi_{n-1} - (5-q)\chi_{n-2} + \chi_{n-3}}_{w_{n-2}}),$$

where χ_n is given in (9), and $\tilde{s}_1 = \delta_1$, $\tilde{s}_2 = \sigma_2 - \sigma_1$, $\tilde{s}_3 = \delta_3 - \delta_2$.

`Case 2:` q `is odd`. In this case, we replicate the treatment from the previous case, with some exceptions to be addressed. The key difference is that now we need to examine the influence of the elements from row n on row $n+3$, as the property regarding the signs first appears three rows later.

According to [5], the contributions of x_A and x_B from row n to the alternating sum of row $n+3$ are given by

$$\mathcal{H}_3^{(A)}(x_A) = -4(q-4)\varepsilon_1 x_A, \qquad \mathcal{H}_3^{(B)}(x_A) = 2(q-4)\varepsilon_1 x_A,$$

and

$$\mathcal{H}_3^{(A)}(x_B) = -4(q-3)\varepsilon_2 x_B, \qquad \mathcal{H}_3^{(B)}(x_B) = 2(q-3)\varepsilon_2 x_B,$$

where $\varepsilon_i \in \{\pm 1\}$ $(i = 1, 2)$.

Figure 3 illustrates the contributions of the influence of the leftmost element in row n to row $n+3$. Thus,

$$\mathcal{H}_3^{(A)}(\alpha_n) = -2\alpha_{n+1} - \alpha_{n+2}, \qquad \mathcal{H}_3^{(B)}(\alpha_n) = \alpha_{n+1}.$$

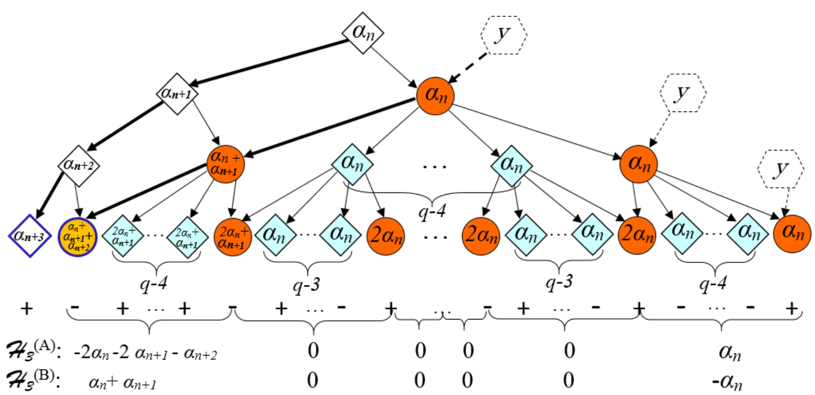

Figure 3: The influence of $\mathcal{H}_3(\alpha_n)$

Then similarly, the contributions of the influence of the rightmost element in row n to row $n+3$ are given by:

$$\mathcal{H}_3^{(A)}(\beta_n) = (-1)^{s_n}(2\beta_{n+1} + \beta_{n+2}), \qquad \mathcal{H}_3^{(B)}(\beta_n) = -(-1)^{s_n}\beta_{n+1}.$$

Combining the information above results in the two systems of recurrence relations as follows. In `Subcase (a)`, s_n is even. In this case, for $n \geq 0$, we get

$$\begin{cases} \tilde{s}_{n+3}^{(A)} = -4(q-4)\tilde{s}_n^{(A)} - 4(q-3)\tilde{s}_n^{(B)} - 2\delta_{n+1} - \delta_{n+2}, \\ \tilde{s}_{n+3}^{(B)} = 2(q-4)\tilde{s}_n^{(A)} + 2(q-3)\tilde{s}_n^{(B)} + \delta_{n+1}. \end{cases} \tag{17}$$

The elimination process leads to

$$\begin{aligned}\tilde{s}^{(A)}_{n+6} &= 2(5-q)\tilde{s}^{(A)}_{n+3} - \delta_{n+5} - 2\delta_{n+4} + 2(q-3)\delta_{n+2}, \quad n \geq 0;\\ \tilde{s}^{(B)}_{n+6} &= 2(5-q)\tilde{s}^{(B)}_{n+3} + \delta_{n+4} - 2(q-3)\delta_{n+2}, \quad n \geq 0.\end{aligned}$$

From (12), we conclude that for $n \geq 0$, we have

$$\begin{aligned}\tilde{s}_{n+6} &= 2(5-q)\tilde{s}^{(A)}_{n+3} + 2(5-q)\tilde{s}^{(B)}_{n+3} + \delta_{n+6} - \delta_{n+5} - \delta_{n+4}\\ &\quad +2\delta_{n+2}\\ &= 2(5-q)\tilde{s}_{n+3} + \delta_{n+6} - \delta_{n+5} - \delta_{n+4} - 2(5-q)\delta_{n+3}\\ &\quad +2\delta_{n+2}.\end{aligned}$$

`Subcase (b)`: s_n is odd. In this case, for $n \geq 0$, we get

$$\begin{cases} \tilde{s}^{(A)}_{n+3} = -4(q-4)\tilde{s}^{(A)}_n - 4(q-3)\tilde{s}^{(B)}_n - 2\sigma_{n+1} - \sigma_{n+2}, \\ \\ \tilde{s}^{(B)}_{n+3} = 2(q-4)\tilde{s}^{(A)}_n + 2(q-3)\tilde{s}^{(B)}_n + \sigma_{n+1}. \end{cases} \tag{18}$$

In this case, The elimination process leads to

$$\begin{aligned}\tilde{s}^{(A)}_{n+6} &= 2(5-q)\tilde{s}^{(A)}_{n+3} - \sigma_{n+5} - 2\sigma_{n+4} + 2(q-3)\sigma_{n+2},\\ \tilde{s}^{(B)}_{n+6} &= 2(5-q)\tilde{s}^{(B)}_{n+3} + \sigma_{n+4} - 2(q-4)\sigma_{n+2}.\end{aligned}$$

Then, we get

$$\begin{aligned}\tilde{s}_{n+6} &= 2(5-q)\tilde{s}^{(A)}_{n+3} + 2(5-q)\tilde{s}^{(B)}_{n+3} + \sigma_{n+6} - \sigma_{n+5} - \sigma_{n+4}\\ &\quad +2\sigma_{n+2}\\ &= 2(5-q)\tilde{s}_{n+3} + \sigma_{n+6} - \sigma_{n+5} - \sigma_{n+4} - 2(5-q)\sigma_{n+3}\\ &\quad +2\sigma_{n+2}.\end{aligned}$$

It follows that for q odd and $n \geq 6$, we have

$$\tilde{s}_n = 2(5-q)\tilde{s}_{n-3} + (\underbrace{\chi_n - \chi_{n-1} - \chi_{n-2} - 2(5-q)\chi_{n-3} + 2\chi_{n-4}}_{w_{n-3}}).$$

The initial values are given by $\tilde{s}_1 = \delta_1$, $\tilde{s}_2 = \sigma_2 - \sigma_1$, $\tilde{s}_3 = \sigma_3 - \sigma_2 - \sigma_1$, $\tilde{s}_4 = \delta_4 - \delta_3 - \delta_2$, and $\tilde{s}_5 = 2(q-4)\sigma_1 - \sigma_3 - \sigma_4 + \sigma_5$.

□

Remark 1. In case $\alpha_n = \beta_n$, the alternating sum would return 0 when s_n is even, since the triangle would be vertically symmetrical.

Remark 2. In the specific case where $\alpha_n = 1$, $\beta_n = 1$, and $q = 4$, the theorem implies that the alternating sum $\tilde{s}_n$ evaluates to 0. This result aligns with the well-known property of the original Pascal's triangle, where $\tilde{s}_n = 0$.

Assume that the root element Ω equals 0. Then it is easy to check that taking

$$\sigma_0 = \delta_0 = 0, \quad \tilde{s}_0 = 0$$

recurrence (10) holds for $n \geq 2$ if q is even, and $n \geq 3$ otherwise. Now

$$w_0 = \begin{cases} \sigma_3 - \sigma_2 - \sigma_1, & \text{if } q \text{ is odd,} \\ \sigma_2 - \sigma_1, & \text{if } q \text{ is even.} \end{cases}$$

These extensions facilitate the work with generating functions. Assume that the generating functions of the sequences $\{\tilde{s}_n\}$ and $\{w_n\}$ are denoted by $S(t) = \sum_{i=0}^{\infty} \tilde{s}_i t^i$ and $W(t) = \sum_{i=0}^{\infty} w_i t^i$, respectively.

Theorem 3. *The generating function of $\{\tilde{s}_n\}$ is given as follows. Let $c_f(t) = t^{2+\tilde{q}} - 2^{\tilde{q}}(5-q)$ denote the generating function of the recursive sequence $f_n = 2^{\tilde{q}}(5-q)f_{n-2-\tilde{q}}$ for $n \geq 2+\tilde{q}$ (with arbitrary initial values).*

- *If q is even, then*
$$S(t) = \frac{\tilde{s}_1 t + t^2 W(t)}{t^2 c_f(1/t)}.$$

- *If q is odd, then*
$$S(t) = \frac{\tilde{s}_1 t + \tilde{s}_2 t^2 + t^3 W(t)}{t^3 c_f(1/t)}.$$

Proof. The standard method provides the identities above. This is the evaluation of the equality

$$\sum_{i=2+\tilde{q}}^{\infty} \widehat{s}_i t^i = 2^{\tilde{q}}(5-q) \sum_{i=2+\tilde{q}}^{\infty} \widehat{s}_{i-2-\tilde{q}} t^i + \sum_{i=2+\tilde{q}}^{\infty} w_{i-2-\tilde{q}} t^i.$$

□

Corollary 1, together with Theorem 2 and the sequence $\{f_n\}$ implies the following explicit formula.

Theorem 4. *The following formula holds for $n \geq 1+\tilde{q}$:*

$$\tilde{s}_{n+1} = 2^{\tilde{q}}(5-q)\left(f_{n-1}\delta_1 + \tilde{q} f_{n-2}(\sigma_2 - \sigma_1)\right) + \sum_{j=0}^{n} f_{n-j} w_j.$$

Proof. The recurrence in (8) is used in a direct manner. $\{f_n\}$ has just been fixed in Theorem 3, and $\{G_n\}$ is $\tilde{s}_n$ with the initial values described in Theorem 2, with the extensions after its proof. The sequence $\{w_n\}$ is also given in Theorem 2. □

The order of the recurrence $\{f_n\}$ is 2 or 3 depending on the parity of q. The (complex) zeros of the characteristic polynomial $c_f(t) = t^{2+\tilde{q}} - 2^{\tilde{q}}(5-q)$ are distinct if $q > 5$, and are given by $\sqrt[2+\tilde{q}]{2^{\tilde{q}}(5-q)}$. Note that in the specific case when $\alpha_n = 1$ and $\beta_n = 1$, the alternating sum has the following explicit formulas (see [5]). For q even:

$$\tilde{s}_n = \begin{cases} 0, & \text{if } n = 2t+1, \quad n \geq 1, \\ -2(5-q)^{t-1} + 2, & \text{if } n = 2t, \quad n \geq 2. \end{cases}$$

For q odd ($q \geq 5$):

$$\tilde{s}_n = \begin{cases} 0, & \text{if } n = 3t+1, \quad n \geq 1, \\ (-2)^t(5-q)^{t-1} + 2, & \text{if } n = 3t-1, \quad n \geq n_1, \\ 2(-2)^t(5-q)^{t-1} + 2, & \text{if } n = 3t, \quad n \geq n_2, \end{cases}$$

where $(n_1, n_2) = (2, 3)$ and $(n_1, n_2) = (5, 6)$ if $n > 5$ and $n = 5$, respectively. In the latter case, $\tilde{s}_2 = 0$ and $\tilde{s}_3 = -2$.

With the help of $\hat{s}_n$ and $\tilde{s}_n$ we can easily determine the alternating sum with the arbitrary weights v and w.

Corollary 2. *We have*

$$\begin{aligned} \tilde{s}_{(v,w),n} &= \sum_{i=0}^{s_n-1} \left(v\delta_{0,i mod 2} + w\delta_{1,i mod 2}\right) \left| \begin{matrix} n \\ k \end{matrix} \right(_{\{\alpha_n,\beta_n\}} \\ &= \frac{\hat{s}_n + \tilde{s}_n}{2} v + \frac{\hat{s}_n - \tilde{s}_n}{2} w \\ &= \frac{v+w}{2} \hat{s}_n + \frac{v-w}{2} \tilde{s}_n, \end{aligned}$$

where $\delta_{i,j}$ is the Kronecker delta.

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