#A85 INTEGERS 24 (2024)

ANALYTIC CONTINUATION OF THE ℓ-GENERALIZED FIBONACCI ZETA FUNCTION

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Received: 11/13/23, Accepted: 9/17/24, Published: 10/9/24

Abstract

In this paper, for any positive integer $\ell \geq 2$, we define the ℓ-generalized Fibonacci zeta function. We then study its analytic continuation to the whole complex plane $\mathbb{C}$. Further, we compute a possible list of singularities and residues of the function at these simple poles. Moreover, we deduce that the special values of the ℓ-generalized Fibonacci zeta function at negative integer arguments are rational.

1. Introduction

Let $\ell \geq 2$ be an integer. The n^{th} *ℓ-generalized Fibonacci sequence* $\left(F_n^{(\ell)}\right)_{n \geq 2-\ell}$ is defined as

$$F_n^{(\ell)} = F_{n-1}^{(\ell)} + F_{n-2}^{(\ell)} + \cdots + F_{n-\ell}^{(\ell)}$$

with the initial conditions

$$F_{-(\ell-2)}^{(\ell)} = F_{-(\ell-3)}^{(\ell)} = \cdots = F_0^{(\ell)} = 0, \quad \text{and} \quad F_1^{(\ell)} = 1.$$

Also, $F_n^{(\ell)}$ is called the n^{th} *ℓ-generalized Fibonacci number.* From [3], we obtain that

$$F_n^{(\ell)} = 2^{n-2} \quad \text{for all} \quad 2 \leq n \leq \ell + 1, \quad \text{and} \quad F_n^{(\ell)} < 2^{n-2} \quad \text{for all} \quad n \geq \ell + 2.$$

The characteristic polynomial of the ℓ-generalized Fibonacci sequence is given by

$$\phi_\ell(x) = x^\ell - x^{\ell-1} - \cdots - x - 1. \tag{1}$$

DOI: 10.5281/zenodo.13909110

It is irreducible over $\mathbb{Q}[x]$ and has one root outside the unit circle. Let $\alpha = \alpha_1$ be that single root which lies between $2\left(1-2^{-\ell}\right)$ and 2 (see [10]), which is the dominant root of $\phi_\ell(x)$. Let the other roots of the polynomial (1) be $\alpha_2, \ldots, \alpha_\ell$. When ℓ is an even integer, $\phi_\ell(x)$ has one negative real root which lies in the interval $(-1,0)$. In 2014, Dresden and Du [4] gave the "Binet-like formula" for the terms $F_n^{(\ell)}$ which is given by

$$F_n^{(\ell)} = \sum_{i=1}^{\ell} \frac{\alpha_i - 1}{2+(\ell+1)(\alpha_i - 2)} \alpha_i^{n-1}. \tag{2}$$

From [4], it is also known that

$$\left| F_n^{(\ell)} - \frac{\alpha - 1}{2+(\ell+1)(\alpha-2)} \alpha^{n-1} \right| < \frac{1}{2} \quad \text{for all} \quad n \geq 2-\ell.$$

In 2013, Bravo and Luca [2] obtained that

$$\alpha^{n-2} \leq F_n^{(\ell)} \leq \alpha^{n-1} \quad \text{holds for all} \quad n \geq 1 \quad \text{and} \quad \ell \geq 2. \tag{3}$$

When $\ell = 2$, $F_n^{(\ell)}$ is same as the Fibonacci number F_n, and when $\ell = 3$, it coincides with the Tribonacci number T_n.

The *Fibonacci zeta function* is defined by the series

$$\zeta_F(s) = \sum_{n=1}^{\infty} \frac{1}{F_n^s}, \quad \operatorname{Re}(s) > 0.$$

The analytic continuation of the Fibonacci zeta function was studied by Navas [8] in 2001. The arithmetic nature of the special values of the Fibonacci zeta function and of the Riemann zeta function $\zeta(s)$ behave similarly. The irrationality of $\zeta_F(1)$ was proved by André-Jeannin [1] in 1989, while the transcendence of $\zeta_F(2m)$, for $m \in \mathbb{N}$, was given by Duverney et al. [5] in 1997. Furthermore, in 2007, Elsner et al. [6] showed that $\zeta_F(2), \zeta_F(4), \zeta_F(6)$ are algebraically independent over $\mathbb{Q}$. Murty [7] deduced that $\zeta_F(2m)$, for $m \in \mathbb{N}$, is transcendental by using the theory of modular forms and a result of Nesterenko [9].

In this paper, we introduce the *ℓ-generalized Fibonacci zeta function* which is defined by

$$\zeta_{F^{(\ell)}}(s) = \sum_{n=1}^{\infty} \frac{1}{\left(F_n^{(\ell)}\right)^s}.$$

When $\ell = 2$, it is same as the Fibonacci zeta function $\zeta_F(s)$. In this paper, we study the analytic continuation of the ℓ-generalized Fibonacci zeta function. We also give a list of possible singularities of the function $\zeta_{F^{(\ell)}}(s)$ and calculate their residues. Moreover, we discuss the arithmetic nature of the ℓ-generalized Fibonacci zeta function at negative integer arguments.

The paper is organized as follows. In Section 2, we prove that $\zeta_{F^{(\ell)}}(s)$ is absolutely convergent in $\mathrm{Re}(s) > 0$. In Section 3, we obtain the analytic continuation of the ℓ-generalized Fibonacci zeta function $\zeta_{F^{(\ell)}}(s)$, and compute a list of possible poles and their residues. In Section 4, we prove that the special values of the ℓ-generalized Fibonacci zeta function at negative integer arguments are rational.

2. Preliminaries

Proposition 1. *The infinite series* $\sum_{n>0}\left(F_n^{(\ell)}\right)^{-s}$ *converges absolutely in the right half plane* $\{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$.

Proof. From (3), we get

$$\left|\left(F_n^{(\ell)}\right)^{-s}\right| = \left(F_n^{(\ell)}\right)^{-\sigma} \leq \left(\alpha^{n-2}\right)^{-\sigma} = \alpha^{2\sigma}\left(\alpha^{-n\sigma}\right). \tag{4}$$

Since $\sigma = \mathrm{Re}(s) > 0$, from (4), we obtain

$$\sum_{n=1}^{\infty}\left|\left(F_n^{(\ell)}\right)^{-s}\right| \leq \alpha^{2\sigma}\sum_{n=1}^{\infty}\left(\alpha^{-n\sigma}\right) = \frac{\alpha^{2\sigma}}{\alpha^{\sigma}-1} < \infty.$$

□

The next lemma tells us that the integer closest to the first term of the Binet-like formula is the ℓ-generalized Fibonacci number.

Lemma 1 (Dresden and Du [4]). *Let* $F_n^{(\ell)}$ *be the* n^{th} ℓ*-generalized Fibonacci number. Then*

$$F_n^{(\ell)} = \mathrm{rnd}\left(\frac{\alpha-1}{2+(\ell+1)(\alpha-2)}\alpha^{n-1}\right) \quad \textit{for all} \quad n \geq 2-\ell,$$

where α *is the unique positive dominant root and* $\mathrm{rnd}(x) = \lfloor x + \frac{1}{2} \rfloor$ *denotes the value of* x *rounded to the nearest integer.*

3. Analytic Continuation of the ℓ-Generalized Fibonacci Zeta Function

Theorem 1. *The* ℓ*-generalized Fibonacci zeta function* $\zeta_{F^{(\ell)}}(s)$ *can be meromorphically continued to the whole complex plane* $\mathbb{C}$ *with possible simple poles at*

$$s = s_{k,k_2,\ldots,k_\ell,n} = -k + \frac{2ni\pi + k_2\log\alpha_2 + \cdots + k_\ell\log\alpha_\ell}{\log\alpha}, \; n \in \mathbb{Z}, k, k_2, k_3, ..., k_\ell \in \mathbb{N}_0 \tag{5}$$

such that $k = k_2 + k_3 + \cdots + k_\ell$*, where* $\mathbb{N}_0$ *denotes the set of non-negative integers and* $\log(\alpha_i)$ $(i = 1, 2, \ldots, \ell)$ *are the principal values whose imaginary parts lie in the interval* $(-\pi, \pi]$*. Moreover, the residue of* $\zeta_{F^{(\ell)}}(s)$ *at* $s = s_{k,k_2,\ldots,k_\ell,n}$ *is*

$$\left(\frac{\alpha-1}{2+(\ell+1)(\alpha-2)}\alpha^{-1}\right)^{-s_{k,k_2,\ldots,k_\ell,n}} \binom{-s_{k,k_2,\ldots,k_\ell,n}}{k} \left(\frac{\alpha-1}{2+(\ell+1)(\alpha-2)}\alpha^{-1}\right)^{-k} \times \frac{k!}{k_2! \cdots k_\ell!} \prod_{i=2}^{\ell} \left(\frac{\alpha_i-1}{2+(\ell+1)(\alpha_i-2)}\alpha_i^{-1}\right)^{k_i} \frac{1}{\log\alpha}.$$

Proof. Because of Lemma 1 and the fact that $F_n^{(\ell)} \in \mathbb{Z}_{>0}$, we have

$$\left|\left(\frac{\alpha-1}{2+(\ell+1)(\alpha-2)}\alpha^{n-1}\right)\right| \geq \frac{1}{2}.$$

Therefore, we get that

$$\left|\frac{\sum_{i=2}^{\ell}\left(\frac{\alpha_i-1}{2+(\ell+1)(\alpha_i-2)}\alpha_i^{n-1}\right)}{\left(\frac{\alpha-1}{2+(\ell+1)(\alpha-2)}\alpha^{n-1}\right)}\right| = \frac{\left|F_n^{(\ell)} - \frac{\alpha-1}{2+(\ell+1)(\alpha-2)}\alpha^{n-1}\right|}{\left|\left(\frac{\alpha-1}{2+(\ell+1)(\alpha-2)}\alpha^{n-1}\right)\right|} < \frac{1/2}{1/2} = 1.$$

Now, we express

$$\begin{aligned}
&\left(F_n^{(\ell)}\right)^{-s} \\
&\quad= \left(\left(\frac{(\alpha-1)\alpha^{n-1}}{2+(\ell+1)(\alpha-2)}\right) + \sum_{i=2}^{\ell}\left(\frac{\alpha_i-1}{2+(\ell+1)(\alpha_i-2)}\alpha_i^{n-1}\right)\right)^{-s} \\
&\quad= \left(\frac{(\alpha-1)\alpha^{n-1}}{2+(\ell+1)(\alpha-2)}\right)^{-s} \left(1 + \frac{\sum_{i=2}^{\ell}\left(\frac{\alpha_i-1}{2+(\ell+1)(\alpha_i-2)}\alpha_i^{n-1}\right)}{\left(\frac{\alpha-1}{2+(\ell+1)(\alpha-2)}\alpha^{n-1}\right)}\right)^{-s} \\
&\quad= \left(\frac{(\alpha-1)\alpha^{n-1}}{2+(\ell+1)(\alpha-2)}\right)^{-s} \sum_{k=0}^{\infty}\binom{-s}{k}\left(\frac{\sum_{i=2}^{\ell}\left(\frac{\alpha_i-1}{2+(\ell+1)(\alpha_i-2)}\alpha_i^{n-1}\right)}{\left(\frac{\alpha-1}{2+(\ell+1)(\alpha-2)}\alpha^{n-1}\right)}\right)^{k}.
\end{aligned}$$

Using Proposition 1, observe that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left|\left(\frac{(\alpha-1)\alpha^{n-1}}{2+(\ell+1)(\alpha-2)}\right)^{-s} \sum_{k=0}^{\infty}\binom{-s}{k}\left(\frac{\sum_{i=2}^{\ell}\left(\frac{\alpha_i-1}{2+(\ell+1)(\alpha_i-2)}\alpha_i^{n-1}\right)}{\left(\frac{\alpha-1}{2+(\ell+1)(\alpha-2)}\alpha^{n-1}\right)}\right)^{k}\right| \\
&\quad= \sum_{n=1}^{\infty}\left|\left(F_n^{(\ell)}\right)^{-s}\right| < \infty.
\end{aligned}$$

On interchanging the order of summation and using the multinomial expansion formula, we get that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(F_n^{(\ell)}\right)^{-s}\\
&\quad=\sum_{n=1}^{\infty}\left(\frac{(\alpha-1)\alpha^{n-1}}{2+(\ell+1)(\alpha-2)}\right)^{-s}\sum_{k=0}^{\infty}\binom{-s}{k}\left(\frac{\sum_{i=2}^{\ell}\left(\frac{\alpha_i-1}{2+(\ell+1)(\alpha_i-2)}\alpha_i^{n-1}\right)}{\left(\frac{\alpha-1}{2+(\ell+1)(\alpha-2)}\alpha^{n-1}\right)}\right)^{k}\\
&\quad=\sum_{k=0}^{\infty}\binom{-s}{k}\sum_{n=1}^{\infty}\left(\frac{(\alpha-1)\alpha^{n-1}}{2+(\ell+1)(\alpha-2)}\right)^{-s}\left(\frac{\sum_{i=2}^{\ell}\left(\frac{\alpha_i-1}{2+(\ell+1)(\alpha_i-2)}\alpha_i^{n-1}\right)}{\left(\frac{\alpha-1}{2+(\ell+1)(\alpha-2)}\alpha^{n-1}\right)}\right)^{k}\\
&\quad=\left(\frac{\alpha-1}{2+(\ell+1)(\alpha-2)}\alpha^{-1}\right)^{-s}\sum_{k=0}^{\infty}\binom{-s}{k}\Bigg\{\left(\frac{\alpha-1}{2+(\ell+1)(\alpha-2)}\alpha^{-1}\right)^{-k}\\
&\qquad\times\sum_{\substack{k_2+\cdots\\+k_\ell=k}}\frac{k!}{k_2!\cdots k_\ell!}\prod_{i=2}^{\ell}\left(\frac{(\alpha_i-1)\alpha_i^{-1}}{2+(\ell+1)(\alpha_i-2)}\right)^{k_i}\sum_{n=1}^{\infty}\left(\frac{\alpha_2^{k_2}\cdots\alpha_\ell^{k_\ell}}{\alpha^{s+k}}\right)^{n}\Bigg\}.
\end{aligned}$$

Note that $\left|\frac{\alpha_2^{k_2}\cdots\alpha_\ell^{k_\ell}}{\alpha^{s+k}}\right| \le \frac{\alpha^{k_2+\cdots+k_\ell}}{\alpha^{\sigma+k}} = \frac{1}{\alpha^{\sigma}} < 1$ for $\sigma > 0$. Therefore, the above series becomes

$$\begin{aligned}
\zeta_{F^{(\ell)}}(s)&=\left(\frac{\alpha-1}{2+(\ell+1)(\alpha-2)}\alpha^{-1}\right)^{-s}\sum_{k=0}^{\infty}\binom{-s}{k}\Bigg\{\left(\frac{\alpha-1}{2+(\ell+1)(\alpha-2)}\alpha^{-1}\right)^{-k}\\
&\quad\times\sum_{\substack{k_2+\cdots\\+k_\ell=k}}\frac{k!}{k_2!\cdots k_\ell!}\prod_{i=2}^{\ell}\left(\frac{(\alpha_i-1)\alpha_i^{-1}}{2+(\ell+1)(\alpha_i-2)}\right)^{k_i}\frac{1}{\alpha^{s+k}\alpha_2^{-k_2}\cdots\alpha_\ell^{-k_\ell}-1}\Bigg\}.
\end{aligned}\tag{6}$$

From (2), it is clear that for any integer i with $1 \le i \le \ell$, we have $2+(\ell+1)(\alpha_i-2) \neq 0$. Note that the function

$$h_{k,k_2,\ldots,k_\ell}(s) = \frac{1}{\alpha^{s+k}\alpha_2^{-k_2}\cdots\alpha_\ell^{-k_\ell}-1}$$

has poles at

$$s = -k + \frac{2ni\pi + k_2\log\alpha_2 + \cdots + k_\ell\log\alpha_\ell}{\log\alpha},\; n \in \mathbb{Z},\; k, k_2, k_3, \ldots, k_\ell \in \mathbb{N}_0$$

with $k = k_2 + k_3 + \cdots + k_\ell$.
Consider the function

$$g_{k,k_2,\ldots,k_\ell}(s) = \alpha^{s+k}\alpha_2^{-k_2}\cdots\alpha_\ell^{-k_\ell} - 1.$$

Then, we have

$$g'_{k,k_2,\ldots,k_\ell}(s) = \alpha^{s+k}\alpha_2^{-k_2}\cdots\alpha_\ell^{-k_\ell}\log\alpha.$$

Thus, the value of $g'_{k,k_2,\ldots,k_\ell}(s)$ at

$$s = s_{k,k_2,\ldots,k_\ell,n} = -k + \frac{2ni\pi + k_2\log\alpha_2 + \cdots + k_\ell\log\alpha_\ell}{\log\alpha}$$

is $\log\alpha$, which is non-zero. Therefore, the poles of $h_{k,k_2,\ldots,k_\ell}(s)$ are simple.

The series (6) determines a holomorphic function on $\mathbb{C}$ except for the poles derived from the function $h_{k,k_2,\ldots,k_\ell}(s)$. Hence, the function $\zeta_{F^{(\ell)}}(s)$ can be meromorphically continued to the whole s-plane and it has the possible simple poles at

$$s = s_{k,k_2,\ldots,k_\ell,n} = -k + \frac{2ni\pi + k_2\log\alpha_2 + \cdots + k_\ell\log\alpha_\ell}{\log\alpha}.$$

Since the residue of

$$h_{k,k_2,\ldots,k_\ell}(s) = \frac{1}{\alpha^{s+k}\alpha_2^{-k_2}\cdots\alpha_\ell^{-k_\ell} - 1}$$

at

$$s = s_{k,k_2,\ldots,k_\ell,n} = -k + \frac{2ni\pi + k_2\log\alpha_2 + \cdots + k_\ell\log\alpha_\ell}{\log\alpha}$$

is

$$\mathrm{Res}_{s=s_{k,k_2,\ldots,k_\ell,n}} h_{k,k_2,\ldots,k_\ell}(s) = \left.\frac{1}{\frac{d}{ds}\left(\alpha^{s+k}\alpha_2^{-k_2}\cdots\alpha_\ell^{-k_\ell} - 1\right)}\right|_{s=s_{k,k_2,\ldots,k_\ell,n}} = \frac{1}{\log\alpha},$$

the residue of $\zeta_{F^{(\ell)}}(s)$ at $s = s_{k,k_2,\ldots,k_\ell,n}$ is

$$\begin{aligned}
&\mathrm{Res}_{s=s_{k,k_2,\ldots,k_\ell,n}}\zeta_{F^{(\ell)}}(s)\\
&= \left(\frac{(\alpha-1)\alpha^{-1}}{2+(\ell+1)(\alpha-2)}\right)^{-s_{k,k_2,\ldots,k_\ell,n}}\binom{-s_{k,k_2,\ldots,k_\ell,n}}{k}\left(\frac{(\alpha-1)\alpha^{-1}}{2+(\ell+1)(\alpha-2)}\right)^{-k}\\
&\qquad\times\frac{k!}{k_2!\cdots k_\ell!}\prod_{i=2}^{\ell}\left(\frac{\alpha_i-1}{2+(\ell+1)(\alpha_i-2)}\alpha_i^{-1}\right)^{k_i}\frac{1}{\log\alpha}.
\end{aligned}$$

□

Remark 1. When $\ell = 2$, we have $\alpha\alpha_2 = -1$, and thus $\alpha_2 = \frac{-1}{\alpha}$. Since α is a positive real number, $\arg\alpha_2 = \pi$. From (5), we get that

$$\begin{aligned}
s = s_{k,k_2,n} &= -k + \frac{2ni\pi + k\log\alpha_2}{\log\alpha} = -k + \frac{2ni\pi + k\log|\alpha_2| + ik\arg\alpha_2}{\log\alpha}\\
&= -k + \frac{2ni\pi - k\log|\alpha| + ik\arg\alpha_2}{\log\alpha} = -2k + i\frac{2n\pi + k\arg\alpha_2}{\log\alpha}\\
&= -2k + \frac{i\pi(2n+k)}{\log\alpha}.
\end{aligned}$$

This is the same list of poles as obtained by Navas [8].

Next, we discuss the special cases of singularities for our better understanding.

Corollary 1. *Let $n \in \mathbb{Z}$, $k \in \mathbb{N}_0$ such that $(\ell-1)|k$. Then, the possible poles of $\zeta_{F^{(\ell)}}(s)$ are given by*

$$s_{k,\frac{k}{\ell-1},\ldots,\frac{k}{\ell-1},n} = \begin{cases} -(k+\frac{k}{\ell-1}) + i\frac{\pi(2n+\frac{k}{\ell-1})}{\log\alpha} & \text{if } \ell \text{ is even},\\ -(k+\frac{k}{\ell-1}) + i\frac{2n\pi}{\log\alpha} & \text{if } \ell \text{ is odd}. \end{cases} \tag{7}$$

Proof. Note that $\alpha\alpha_2\cdots\alpha_\ell = (-1)^{\ell+1}$. Therefore, we have $|\alpha\alpha_2\cdots\alpha_\ell| = 1$. This implies that

$$\log\alpha + \log|\alpha_2| + \cdots + \log|\alpha_\ell| = 0. \tag{8}$$

Observe that $\alpha > 1$ and for $1 \le j \le \ell$, we can write $\log\alpha_j = \log|\alpha_j| + i\arg\alpha_j$, where $i = \sqrt{(-1)}$ and $\arg\alpha_j \in (-\pi,\pi]$. Since the $\log\alpha_j$'s are principal values of complex logarithms, if ℓ is odd,

$$\arg\alpha_2 + \cdots + \arg\alpha_\ell = 0, \tag{9}$$

and if ℓ is even,

$$\arg\alpha_2 + \cdots + \arg\alpha_\ell = \pi. \tag{10}$$

Now using (8), (9), and (10) in (5) with $k_2 = k_3 = \cdots = k_\ell = \frac{k}{\ell-1} \in \mathbb{N}_0$, we get (7). □

Remark 2. From Corollary 1, we can see that some of the possible poles lie on the lines $\mathrm{Re}(s) = -(k+\frac{k}{\ell-1})$ spaced at intervals of length $\frac{2\pi i}{\log\alpha}$. In particular, for $k \in \mathbb{N}_0$, the possible simple poles of $\zeta_{F^{(\ell)}}(s)$ are $s = -(k+\frac{k}{\ell-1})$, when ℓ is odd, and $s = -(k+\frac{k}{\ell-1}) + i\frac{k\pi}{(\ell-1)\log\alpha}$, when ℓ is even. In this section, when ℓ is an even integer and $k = -2n(\ell-1)$, then from (7), we obtain $s = 2n\ell,\ n \in \mathbb{Z}_{\le 0}$, which are the possible simple negative integer poles.

4. Special Values at Negative Integer Arguments

Furthermore, we will discuss the values of $\zeta_{F^{(\ell)}}(s)$ at negative integers.

Theorem 2. *Let m be a positive integer such that $-m$ is not a pole of $\zeta_{F^{(\ell)}}(s)$. Then $\zeta_{F^{(\ell)}}(-m) \in \mathbb{Q}$.*

Proof. Let m be a positive integer such that $-m$ is not a pole of $\zeta_{F^{(\ell)}}(s)$. Then from (6), we have

$$\zeta_{F^{(\ell)}}(-m) = \sum_{k=0}^{m} \binom{m}{k} \left(\frac{\alpha - 1}{2 + (\ell+1)(\alpha - 2)} \alpha^{-1} \right)^{m-k} \\ \times \left(\sum_{k_2+\cdots+k_\ell = k} \frac{k!}{k_2! \cdots k_\ell!} \prod_{i=2}^{\ell} \left(\frac{(\alpha_i - 1)\alpha_i^{-1}}{2 + (\ell+1)(\alpha_i - 2)} \right)^{k_i} \frac{1}{\alpha^{-(m-k)} \alpha_2^{-k_2} \cdots \alpha_\ell^{-k_\ell} - 1} \right). \tag{11}$$

Note that, since $0 \le k \le m$, we have $0 \le m - k \le m$. Let b be a fixed integer with $0 \le b \le m$. If we choose $m - k = b$, then $k = m - b$ and $-m + k = -b$. Then, the coefficient of $\left(\frac{\alpha-1}{2+(\ell+1)(\alpha-2)} \alpha^{-1} \right)^b$ in equation (11) is

$$\binom{m}{m-b} \sum_{\substack{k_2+\cdots+k_\ell \\ = m-b}} \frac{(m-b)!}{k_2! \cdots k_\ell!} \prod_{i=2^\ell} \left(\frac{(\alpha_i - 1)\alpha_i^{-1}}{2 + (\ell+1)(\alpha_i - 2)} \right)^{k_i} \frac{1}{\alpha^{-b} \alpha_2^{-k_2} \cdots \alpha_\ell^{-k_\ell} - 1}.$$

Let $\sigma : \mathbb{Q}(\alpha, \alpha_2, \cdots, \alpha_\ell) \to \mathbb{Q}(\alpha, \alpha_2, \cdots, \alpha_\ell)$ be a non-trivial field automorphism. We know that σ permutes the roots of the irreducible polynomial $\phi_\ell(x)$. We choose r and j ($1 \le j, r \le \ell$) such that $\sigma(\alpha) = \alpha_r$ and $\sigma(\alpha_j) = \alpha$. Just for abbreviation, we denote the term $\frac{\alpha_r - 1}{2+(\ell+1)(\alpha_r-2)} \alpha_r^{-1}$ by a_r. Then, we have

$$\begin{aligned} &\sigma(\zeta_{F^{(\ell)}}(-m)) \\ &= \sum_{k=0}^{m} \binom{m}{k} a_r^{m-k} \left(\sum_{k_2+\cdots+k_\ell = k} \frac{k!}{k_2! \cdots k_\ell!} \prod_{i=2}^{\ell} \left(\frac{\sigma(\alpha_i) - 1}{2 + (\ell+1)(\sigma(\alpha_i) - 2)} \sigma(\alpha_i)^{-1} \right)^{k_i} \right. \\ &\qquad \left. \times \frac{1}{\alpha_r^{-(m-k)} \sigma(\alpha_2)^{-k_2} \cdots \sigma(\alpha_\ell)^{-k_\ell} - 1} \right) \\ &= \sum_{k=0}^{m} \binom{m}{k} a_r^{m-k} \left(\sum_{k_2+\cdots+k_\ell = k} \frac{k!}{k_2! \cdots k_\ell!} \prod_{\substack{i=2 \\ i \ne j}}^{\ell} \left(\frac{\sigma(\alpha_i) - 1}{2 + (\ell+1)(\sigma(\alpha_i) - 2)} \sigma(\alpha_i)^{-1} \right)^{k_i} \right. \\ &\qquad \left. \times \left(\frac{(\alpha - 1)\alpha^{-1}}{2 + (\ell+1)(\alpha - 2)} \right)^{k_j} \frac{1}{\alpha_r^{-(m-k)} \sigma(\alpha_2)^{-k_2} \cdots \sigma(\alpha_\ell)^{-k_\ell} - 1} \right). \end{aligned} \tag{12}$$

In (12), the coefficient of $\left(\frac{\alpha-1}{2+(\ell+1)(\alpha-2)}\alpha^{-1}\right)^b$ is

$$\sum_{k=b}^{m}\binom{m}{k}a_r^{m-k}\Bigg(\sum_{k_2+\cdots+k_{j-1}+k_{j+1}\cdots+k_\ell=k-b}\frac{k!}{\left(\prod_{\substack{i=2\\i\neq j}}^{\ell}k_i!\right)b!}$$
$$\times\prod_{\substack{i=2\\i\neq j}}^{\ell}\left(\frac{(\alpha_i-1)\alpha_i^{-1}}{2+(\ell+1)(\alpha_i-2)}\right)^{k_i}\frac{1}{\alpha^{-b}\alpha_r^{-(m-k)}\prod_{\substack{i=2\\i\neq j}}^{\ell}\alpha_i^{-k_i}-1}\Bigg)$$
$$=\binom{m}{m-b}\sum_{k=b}^{m}a_r^{m-k}\frac{1}{(m-k)!}\Bigg(\sum_{k_2+\cdots+k_{j-1}+k_{j+1}\cdots+k_\ell=k-b}\frac{(m-b)!}{\left(\prod_{\substack{i=2\\i\neq j}}^{\ell}k_i!\right)}$$
$$\times\prod_{\substack{i=2\\i\neq j}}^{\ell}\left(\frac{(\alpha_i-1)\alpha_i^{-1}}{2+(\ell+1)(\alpha_i-2)}\right)^{k_i}\frac{1}{\alpha^{-b}\alpha_r^{-(m-k)}\prod_{\substack{i=2\\i\neq j}}^{\ell}\alpha_i^{-k_i}-1}\Bigg).$$

Note that $(m-k)+(k-b)=m-b$. Since k varies from b to m, we have $0\leq m-k\leq m-b$, and $0\leq k_i\leq m-b$. Thus, the above sum will be

$$\binom{m}{m-b}\sum_{\substack{k_2+\cdots+k_\ell\\=m-b}}\frac{(m-b)!}{k_2!\cdots k_\ell!}\prod_{i=2}^{\ell}\left(\frac{(\alpha_i-1)\alpha_i^{-1}}{2+(\ell+1)(\alpha_i-2)}\right)^{k_i}\frac{1}{\alpha^{-b}\alpha_2^{-k_2}\cdots\alpha_\ell^{-k_\ell}-1}.$$

For any integer b with $0\leq b\leq m$, we have proved that the coefficient of $\left(\frac{\alpha-1}{2+(\ell+1)(\alpha-2)}\alpha^{-1}\right)^b$ in $\zeta_{F^{(\ell)}}(-m)$ is equal to that in $\sigma(\zeta_{F^{(\ell)}}(-m))$. Therefore, for any field automorphism σ, we have $\sigma(\zeta_{F^{(\ell)}}(-m))=\zeta_{F^{(\ell)}}(-m)$. Hence $\zeta_{F^{(\ell)}}(-m)\in\mathbb{Q}$. □

5. Concluding Remark

Analogous to the Fibonacci zeta function, it is interesting to find the zeros of the ℓ-generalized Fibonacci zeta function. The study of arithmetic nature of special values of the ℓ-generalized Fibonacci zeta function at positive even integer arguments is another important object for future discussion.

Acknowledgements. The authors are thankful to the anonymous referees for going through the paper meticulously and giving their suggestions which improved the quality and presentation of the paper. This work started when the first author was in BITS, Hyderabad and the second was in IISER, Berhampur. The authors are thankful to both the institutes for providing nice research facilities. Further, both the authors would like to thank Dr. G. K. Viswanadham for going through this paper. The Second author is supported by an HRI institute postdoctoral fellowship.

References

[1] R. André-Jeannin, Irrationalité de la somme des inverses de certaines suites récurrentes, *C. R. Acad. Sci. Paris, Ser. I* **308** (1989), 539–541.

[2] J. J. Bravo and F. Luca, On a conjecture about repdigits in k-generalized Fibonacci sequences, *Publ. Math. Debrecen* **82** (2013), 623–639.

[3] M. Ddamulira and F. Luca, On the problem of Pillai with k-generalized Fibonacci numbers and powers of 3, *Int. J. Number Theory* **16** (2020), 1643–1666.

[4] G. P. B. Dresden and Z. Du, A simplified binet formula for k-generalized Fibonacci numbers, *J. Integer Seq.* **17** (2014), Art. 14.4.7, 9pp.

[5] D. Duverney, Ke. Nishioka, Ku. Nishioka and I. Shiokawa, Transcendence of Rogers-Ramanujan continued fraction and reciprocal sums of Fibonacci numbers, *Proc. Japan Acad. Ser. A Math. Sci.* **73** (1997), 140–142.

[6] C. Elsner, S. Shimomura and I. Shiokawa, Algebraic relations for reciprocal sums of Fibonacci numbers, *Acta Arith.* **130** (2007), 37–60.

[7] M. R. Murty, The Fibonacci zeta function, in Automorphic representations and L-functions, (Mumbai, 2012), *Tata Inst. Fundam. Res. Stud. Math.* **22** (2013), 409–425, Hindustan Book Agency, Gurgaon.

[8] L. Navas, Analytic continuation of the Fibonacci Dirichlet series, *Fibonacci Quart.* **39** (2001), 409–418.

[9] Yu. V. Nesterenko, Modular functions and transcendence questions, *Mat. Sb.* **187** (1996), 65–96.

[10] D. A. Wolfram, Solving generalized Fibonacci recurrences, *Fibonacci Quart.* **36** (1998), 129–145.