CONGRUENCES FOR ℓ-REGULAR TRIPARTITIONS FOR $\ell \in \{2, 3\}$

Mohammed L. Nadji
Faculty of Mathematics, University of Science and Technology Houari Boumediene
LMAM laboratory, Jijel, Algeria
and
RECITS laboratory, Algiers, Algeria
m.nadji@usthb.dz

Moussa Ahmia
Department of Mathematics, University of Mohamed Seddik Ben Yahia
LMAM laboratory, Jijel, Algeria
moussa.ahmia@univ-jijel.dz

Received: 5/13/24, Accepted: 9/18/24, Published: 10/9/24

Abstract

In this note, we investigate the arithmetic properties of the function $T_\ell(n)$, which counts the number of ℓ-regular tripartitions of n. We obtain several results concerning the generating function dissections, along with some congruences for $T_\ell(n)$ modulo 2, 3, 6, and 12 for $\ell \in \{2, 3\}$. We prove these results using elementary generating function manipulations and classic results from the theory of partitions.

1. Introduction

A *partition* of a nonnegative integer n is a nonincreasing sequence of natural numbers whose sum is n, i.e., $\lambda \dashv n$, if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. Here and throughout we shall use the standard q-series notation

$$(a;q)_n := \begin{cases} \prod_{i=0}^{n-1}(1 - aq^i) & \text{if } n > 0, \\ 1 & \text{if } n = 0. \end{cases}$$

Moreover,

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n \text{ for } \mid q \mid < 1,$$

and f_k is defined by

$$f_k := (q^k, q^k)_\infty = \prod_{n\geq 1}(1 - q^{nk}).$$

DOI: 10.5281/zenodo.13909146

For an integer $\ell > 1$, an *ℓ-regular partition* λ of n is a partition in which none of the parts are divisible by ℓ. Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n. The generating function for $b_\ell(n)$ is given by

$$\sum_{n\geq 0} b_\ell(n)q^n = \frac{f_\ell}{f_1}.$$

For an integer $\ell > 1$, an *ℓ-regular bipartition* (λ, μ) of n is a pair of partitions (λ, μ) such that the sum of all the parts equals n and none of the parts of λ and μ are divisible by ℓ. Let $B_\ell(n)$ denote the number of ℓ-regular bipartitions of n. The generating function for $B_\ell(n)$ is given by

$$\sum_{n\geq 0} B_\ell(n)q^n = \frac{f_\ell^2}{f_1^2}.$$

The arithmetic properties of both ℓ-regular partitions and bipartitions have been extensively studied by a number of mathematicians. For example, see [4, 5, 8, 9, 12].

An *ℓ-regular tripartition* of n is a triplet of ℓ-regular partitions (λ, μ, β) such that the sum of all the parts of λ, μ, and β is equal to n. For example, let $\lambda = (1, 3)$, $\mu = (5, 7)$, and $\beta = (9, 7, 1)$. Then (λ, μ, β) is a 2-regular tripartition of 33. Let $T_\ell(n)$ denote the number of ℓ-regular tripartitions of n, where the enumerating function of $T_\ell(n)$ is given by

$$\sum_{n\geq 0} T_\ell(n)q^n = \frac{f_\ell^3}{f_1^3}. \tag{1}$$

Adiga and Dasappa [1] studied the arithmetic behavior of the 3-regular tripartitions and established the following infinite families of congruences for $\alpha \geq 1$ and $n \geq 0$:

$$T_3\left(3^{2\alpha}n + \frac{11.3^{2\alpha-1} - 1}{4}\right) \equiv 0 \pmod{3^{2\alpha+3}},$$

and

$$T_3\left(3^{2\alpha}n + \frac{7.3^{2\alpha-1} - 1}{4}\right) \equiv 0 \pmod{3^{2\alpha+2}}.$$

In Section 3, we establish several families of congruences modulo 2 and 12 for the partition function $T_2(n)$, and in Section 4, we derive a family of congruences modulo 3 beside few Ramanujan-type congruences for the partition function $T_3(n)$. Note that the sequences $T_2(n)$ and $T_3(n)$ are known in the OEIS [11] as A022568 and A285927, respectively.

2. Preliminary Results

Ramanujan's general theta-function $f(a,b)$ [2, p. 34, 18.1] is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a,ab)_\infty (-b,ab)_\infty (ab,ab)_\infty, \text{ for } |ab| < 1.$$

From $f(a,b)$ we have

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{f_2^2}{f_1}.$$

Jacobi's identity [3, Thm. 1.3.9] is defined as

$$f_1^3 = \sum_{n\geq 0} (-1)^n (2n+1) q^{n(n+1)/2}. \tag{2}$$

In the following lemmas, we list some dissection formulas that are useful in proving our main results.

Lemma 1. *The following* 3*-dissection holds:*

$$\frac{f_2^3}{f_1^3} = \frac{f_6}{f_3} + 3q\frac{f_6^4 f_9^5}{f_3^8 f_{18}} + 6q^2 \frac{f_6^3 f_9^2 f_{18}^2}{f_3^7} + 12q^3 \frac{f_6^2 f_{18}^5}{f_3^6 f_9}. \tag{3}$$

Identity (3) was proved by Toh [10].

Lemma 2. *The following* 3*-dissection holds:*

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2 \frac{f_3 f_{18}^4}{f_6 f_9^2}. \tag{4}$$

For the proof of (4), see [6].

Lemma 3. *The following* 3*-dissections hold:*

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3 \frac{f_3^2 f_{18}^6}{f_6^2 f_9^3}, \tag{5}$$

$$\begin{aligned} \frac{1}{f_1^3} &= \frac{f_9^3}{f_3^{12}} \left(f_1^6 + 9q f_1^3 f_9^3 + 27q^2 f_9^6 \right) \\ &= \frac{f_6^2 f_9^{15}}{f_3^{14} f_{18}^6} + 3q \frac{f_6 f_9^{12}}{f_3^{13} f_{18}^3} + 9q^2 \frac{f_9^9}{f_3^{12}} + 8q^3 \frac{f_9^6 f_{18}^3}{f_3^{11} f_6} + 12q^4 \frac{f_9^3 f_{18}^6}{f_3^{10} f_6^2} + 16q^6 \frac{f_{18}^{12}}{f_3^8 f_6^4 f_9^3}. \end{aligned} \tag{6}$$

To prove (5), expand equation (14.8.5) via the definition after (14.3.1) in [7]. One may obtain (6) by replacing q with ωq and $\omega^2 q$ in (5) and multiplying the two results.

Lemma 4. *For any odd prime p, we have*

$$\psi(q)=\sum_{k=0}^{(p-3)/2} q^{k(k+1)/2} f(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}) + q^{\frac{p^2-1}{8}}\psi(q^{p^2}). \tag{7}$$

Furthermore, $\frac{m^2+m}{2} \not\equiv \frac{p^2-1}{8} \pmod p$ *for* $0 \le m \le (p-3)/2$.

Lemma 5. *For any prime $p \ge 5$, we have*

$$f_1=\sum_{\substack{k=(1-p)/2\\ k\neq \frac{\pm p-1}{6}}}^{(p-1)/2} (-1)^k q^{k(3k+1)/2} f(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}, \tag{8}$$

where

$$\frac{\pm p-1}{6}=\begin{cases}\dfrac{p-1}{6} & \text{if } p\equiv 1 \pmod 6,\\ \dfrac{-p-1}{6} & \text{if } p\equiv -1 \pmod 6.\end{cases}$$

Lemmas 4 and 5 are due to Cui and Gu [4, Thm. 2.1 & 2.2].

Lemma 6. *For all primes p and all $k, m \ge 1$, we have*

$$f_{pm}^{p^{k-1}} \equiv f_m^{p^k} \pmod{p^k}. \tag{9}$$

Let p be any odd prime and δ be any integer relatively prime to p. Then the Legendre symbol $\left(\frac{\delta}{p}\right)$ is defined by

$$\left(\frac{\delta}{p}\right)=\begin{cases}1, & \text{if } \delta \text{ is a quadratic residue of } p,\\ -1, & \text{if } \delta \text{ is a quadratic non-residue of } p.\end{cases}$$

3. Congruences for $T_2(n)$

In this section, we establish several congruences for the sequence $T_2(n)$.

Lemma 7. *We have*

$$\sum_{n\ge 0} T_2(3n)q^n = \frac{f_2}{f_1} + 12q\frac{f_2^2 f_6^5}{f_1^6 f_3}, \tag{10}$$

$$\sum_{n\ge 0} T_2(3n+1)q^n = 3\frac{f_2^4 f_3^5}{f_1^8 f_6}, \tag{11}$$

$$\sum_{n\ge 0} T_2(3n+2)q^n = 6\frac{f_2^3 f_3^2 f_6^2}{f_1^7}. \tag{12}$$

Proof. Collecting the powers of the form q^{3n+j} for $j = 0, 1, 2$ from both sides of (3), we obtain the desired results. □

Corollary 1. *For all $n \geq 0$,*

$$T_2(3n+1) \equiv 0 \pmod{3},$$
$$T_2(3n+2) \equiv 0 \pmod{6}.$$

Theorem 1. *For any prime $p \geq 5$, $\alpha \geq 0$, and $n \geq 0$, we have*

$$T_2\left(3p^{2\alpha+1}(pn+i) + \frac{p^{2\alpha+2}-1}{8}\right) \equiv 0 \pmod{2}, \tag{13}$$

where i is an integer and $1 \leq i \leq p-1$.

Proof. In view of (9) and (10), with $p = 2$ and $k = 1$, we have

$$\sum_{n\geq 0} T_2(3n)q^n \equiv f_1 \pmod{2}. \tag{14}$$

Define

$$\sum_{n\geq 0} a(n)q^n = f_1. \tag{15}$$

Combining (14) and (15), we find that

$$T_2(3n) \equiv a(n) \pmod{2}. \tag{16}$$

Now, we consider the congruence equation

$$\frac{3m^2+m}{2} \equiv \frac{p^2-1}{24} \pmod{p},$$

which is equivalent to

$$(6m+1)^2 \equiv 0 \pmod{p}, \tag{17}$$

where $-(p-1)/2 \leq m \leq (p-1)/2$ and $p \geq 5$ is a prime. Then, the congruence relation (17) holds if and only if $m = (\pm p - 1)/6$. Therefore, if we substitute (8) into (15) and then extract the terms in which the powers of q are congruent to $\frac{p^2-1}{24}$ modulo p and then divide by $q^{(p^2-1)/24}$, we find that

$$\sum_{n\geq 0} a\left(pn + \frac{p^2-1}{24}\right)q^{pn} = (-1)^{\frac{\pm p-1}{6}} f_{p^2},$$

which implies that

$$\sum_{n\geq 0} a\left(p^2 n + \frac{p^2-1}{24}\right)q^{n} = (-1)^{\frac{\pm p-1}{6}} f_1, \tag{18}$$

and for $n \geq 0$,

$$a\left(p^2n + pi + \frac{p^2-1}{24}\right) = 0, \tag{19}$$

where i is an integer and $1 \leq i \leq p-1$. By induction, we see that for $n \geq 0$ and $\alpha \geq 0$,

$$a\left(p^{2\alpha}n + \frac{p^{2\alpha}-1}{24}\right) = (-1)^{\alpha\frac{\pm p-1}{6}} a(n). \tag{20}$$

Replacing n by $p^2n + pi + \frac{p^2-1}{24}$ in (20) and using (19), we find that for $n \geq 0$ and $\alpha \geq 0$,

$$a\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{24}\right) = 0.$$

Again, replacing n by $p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{24}$ $(1 \leq i \leq p-1)$ in (16), we arrive at (13). □

Theorem 2. *For any odd prime p, $\alpha \geq 0$, and $n \geq 0$, we have*

$$T_2\left(9p^{2\alpha+1}(pn+i) + \frac{9p^{2\alpha+2}-1}{8}\right) \equiv 0 \pmod 2, \tag{21}$$

where i is an integer and $1 \leq i \leq p-1$.

Proof. In view of (9) and (11), with $p = 2$ and $k = 1$, we see that

$$\sum_{n\geq 0} T_2(3n+1)q^n \equiv f_3^3 \pmod 2. \tag{22}$$

If we extract the terms involving q^{3n} from both sides of (22), and then replace q^3 by q, we get

$$\sum_{n\geq 0} T_2(9n+1)q^n \equiv f_1^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{n(n+1)/2} \equiv \sum_{n\geq 0} q^{n(n+1)/2} = \psi(q) \pmod 2. \tag{23}$$

Define

$$\sum_{n\geq 0} b(n)q^n = \psi(q). \tag{24}$$

Combining (23) and (24), we obtain

$$T_2(9n+1) \equiv b(n) \pmod 2. \tag{25}$$

Now, we consider the congruence equation

$$\frac{k^2+k}{2} \equiv \frac{p^2-1}{8} \pmod p,$$

which is equivalent to

$$(2k+1)^2 \equiv 0 \pmod{p}, \tag{26}$$

where $0 \le k \le (p-1)/2$ and p is an odd prime. The congruence relation (26) holds if and only if $k = (p-1)/2$. Therefore, if we substitute (7) into (24) and then extract the terms in which the powers of q are $pn + \frac{p^2-1}{8}$, we arrive at

$$\sum_{n\ge 0} b\left(pn + \frac{p^2-1}{8}\right) q^{pn+\frac{p^2-1}{8}} = q^{pn+\frac{p^2-1}{8}} \psi(q^{p^2}).$$

Dividing $q^{\frac{p^2-1}{8}}$ on both sides of the above equation and then replacing q^p by q, we find that

$$\sum_{n\ge 0} b\left(pn + \frac{p^2-1}{8}\right) q^n = \psi(q^p). \tag{27}$$

Again, by extracting the terms containing q^{pn} from both sides of (27), and then replacing q^p by q, we obtain

$$\sum_{n\ge 0} b\left(p^2 n + \frac{p^2-1}{8}\right) q^n = \psi(q), \tag{28}$$

which implies that for $n \ge 0$,

$$b\left(p^2 n + pi + \frac{p^2-1}{8}\right) = 0, \tag{29}$$

where i is an integer and $1 \le i \le p-1$. By induction, we deduce that for $n \ge 0$ and $\alpha \ge 0$,

$$b\left(p^{2\alpha} n + \frac{p^{2\alpha}-1}{8}\right) = b(n). \tag{30}$$

Replacing n by $p^2 n + pi + \frac{p^2-1}{8}$ in (30) and using (29), we find that for $n \ge 0$ and $\alpha \ge 0$,

$$b\left(p^{2\alpha+2} n + p^{2\alpha+1} i + \frac{p^{2\alpha+2}-1}{8}\right) = 0.$$

Again, replacing n by $p^{2\alpha+2} n + p^{2\alpha+1} i + \frac{p^{2\alpha+2}-1}{8}$ $(1 \le i \le p-1)$ in (25), we obtain (21). □

Theorem 3. *For all $\alpha \geq 0$ and $n \geq 0$, we have*

$$T_2\left(3^{4\alpha+4}n+\sum_{i=0}^{2\alpha+1}3^{2i}+3^{4\alpha+3}\right)\equiv 0 \pmod{12}, \tag{31}$$

$$T_2\left(3^{4\alpha+4}n+\sum_{i=0}^{2\alpha+1}3^{2i}+2\cdot 3^{4\alpha+3}\right)\equiv 0 \pmod{12}, \tag{32}$$

$$T_2\left(3^{4\alpha+2}n+\sum_{i=0}^{2\alpha}3^{2i}+3^{4\alpha+1}\right)\equiv 0 \pmod{12}, \tag{33}$$

$$T_2\left(3^{4\alpha+2}n+\sum_{i=0}^{2\alpha}3^{2i}+2\cdot 3^{4\alpha+1}\right)\equiv 0 \pmod{12}. \tag{34}$$

Proof. In view of (9) and (11), with $p = k = 2$, we see that

$$\sum_{n\geq 0} T_2(3n+1)q^n \equiv 3f_3f_6 \pmod{12}. \tag{35}$$

Collecting the terms of the form q^{3n+j} for $j = 0, 1, 2$ from both sides of equation (35), we get

$$\sum_{n\geq 0} T_2(9n+1)q^n \equiv 3f_1f_2 \pmod{12}, \tag{36}$$

$$T_2(9n+4) \equiv 0 \pmod{12}, \tag{37}$$

$$T_2(9n+7) \equiv 0 \pmod{12}. \tag{38}$$

Substituting (4) into (36), we obtain

$$\sum_{n\geq 0} T_2(9n+1)q^n \equiv 3\frac{f_6f_9^4}{f_3f_{18}^2} - 3qf_9f_{18} - 6q^2\frac{f_3f_{18}^4}{f_6f_9^2} \pmod{12}. \tag{39}$$

If we extract the terms involving q^{3n+1} from both sides of the above equation, divide by q and then replace q^3 by q, we get

$$\sum_{n\geq 0} T_2(27n+10)q^n \equiv 9f_3f_6 \pmod{12}. \tag{40}$$

Collecting the terms containing q^{3n+j} for $j = 0, 1, 2$ from both sides of (40), we get

$$\sum_{n\geq 0} T_2(81n+10)q^n \equiv 9f_1f_2 \pmod{12}, \tag{41}$$

$$T_2(81n+37) \equiv 0 \pmod{12}, \tag{42}$$

$$T_2(81n+64) \equiv 0 \pmod{12}. \tag{43}$$

Again, by substituting (4) into (41) and extracting the powers of the form q^{3n+1} from both sides of the resulting equation, we find that

$$\sum_{n\geq 0} T_2(243n+91)q^n \equiv 3f_3f_6 \pmod{12}. \tag{44}$$

From (35), (40), and (44), we deduce that

$$3T_2(3n+1) \equiv T_2(27n+10) \pmod{12}, \tag{45}$$

$$T_2(3n+1) \equiv T_2(243n+91) \pmod{12}, \tag{46}$$

Utilizing both (45) and (46) and by mathematical induction on $\alpha \geq 0$, we arrive at

$$3T_2(3n+1) \equiv T_2\left(3^{4\alpha+3}n + \sum_{i=0}^{2\alpha+1} 3^{2i}\right) \pmod{12}, \tag{47}$$

$$T_2(3n+1) \equiv T_2\left(3^{4\alpha+1}n + \sum_{i=0}^{2\alpha} 3^{2i}\right) \pmod{12}. \tag{48}$$

Using (47), (42), and (43), we obtain (31) and (32), respecively. Similarly, using (48), (37), and (38), we obtain (33) and (34), respecively. □

4. Congruences for $T_3(n)$

In this section, we derive some congruences for the counting sequence $T_3(n)$.

Theorem 4. *We have*

$$\sum_{n\geq 0} T_3(3n)q^n = \frac{f_2^2 f_3^{15}}{f_1^{11} f_6^6} + 8q\frac{f_3^6 f_6^3}{f_1^8 f_2} + 16q^2\frac{f_6^{12}}{f_1^5 f_2^4 f_3^3}, \tag{49}$$

$$\sum_{n\geq 0} T_3(3n+1)q^n = 3\frac{f_2 f_3^{12}}{f_1^{10} f_6^3} + 12q\frac{f_3^3 f_6^6}{f_1^7 f_2^2}, \tag{50}$$

$$\sum_{n\geq 0} T_3(3n+2)q^n = 9\frac{f_3^9}{f_1^9}. \tag{51}$$

Proof. Setting $\ell = 3$ in (1), we have

$$\sum_{n\geq 0} T_3(n)q^n = \frac{f_3^3}{f_1^3}. \tag{52}$$

Substituting (6) into (52), and then extracting the terms of the form q^{3n+j} for $j = 0, 1, 2$ from both sides of the resulting equation, we obtain (49), (50), and (51). □

Corollary 2. *For all $n \geq 0$,*

$$T_3(3n+1) \equiv 0 \pmod{3}, \tag{53}$$
$$T_3(3n+2) \equiv 0 \pmod{9}. \tag{54}$$

Theorem 5. *For any prime $p \equiv 3 \pmod{4}$, $\alpha \geq 0$, and $n \geq 0$, we have*

$$T_3\left(3p^{2\alpha+1}(pn+i) + \frac{p^{2\alpha+2}-1}{4}\right) \equiv 0 \pmod{3}, \tag{55}$$

where i is an integer and $1 \leq i \leq p-1$.

Proof. In view of (9) and (52), with $p = 3$ and $k = 1$, we see that

$$\sum_{n\geq 0} T_3(n)q^n \equiv f_3^2 \pmod{3}. \tag{56}$$

If we extract the terms containing q^{3n} from both sides of the above equation, and then replace q^3 by q, we get

$$\sum_{n\geq 0} T_3(3n)q^n \equiv f_1^2 \pmod{3}. \tag{57}$$

Define

$$\sum_{n\geq 0} c(n)q^n = f_1^2. \tag{58}$$

Combining (57) and (58), we deduce that

$$T_3(3n) \equiv c(n) \pmod{3}. \tag{59}$$

Now, we consider the congruence equation

$$\frac{3k^2+k}{2} + \frac{3m^2+m}{2} \equiv \frac{p^2-1}{12} \pmod{p},$$

which is equivalent to

$$(6k+1)^2 + (6m+1)^2 \equiv 0 \pmod{p}, \tag{60}$$

where $-(p-1)/2 \leq k, m \leq (p-1)/2$ and p is a prime such that $(\frac{-1}{p}) = -1$. Since $(\frac{-1}{p}) = -1$ for $p \equiv 3 \pmod{4}$, then the congruence relation (60) holds if and only if both $k = m = (\pm p - 1)/6$. Substituting (8) into (58) and then extracting the terms in which the powers of q are congruent to $\frac{p^2-1}{12}$ modulo p and then divide by $q^{\frac{p^2-1}{12}}$, we find that

$$\sum_{n\geq 0} c\left(pn + \frac{p^2-1}{12}\right) q^{pn} = f_{p^2}^2,$$

which implies that

$$\sum_{n\geq 0} c\left(p^2n+\frac{p^2-1}{12}\right)q^n = f_1^2, \tag{61}$$

and for $n \geq 0$,

$$c\left(p^2n+pi+\frac{p^2-1}{12}\right)=0, \tag{62}$$

where i is an integer and $1 \leq i \leq p-1$. By induction we see that for $n \geq 0$ and $\alpha \geq 0$,

$$c\left(p^{2\alpha}n+\frac{p^{2\alpha}-1}{12}\right)=c(n). \tag{63}$$

Replacing n by $p^2n+pi+\frac{p^2-1}{12}$ $(1 \leq i \leq p-1)$ in (63) and using (62), we find that for $n \geq 0$ and $\alpha \geq 0$,

$$c\left(p^{2\alpha+2}n+p^{2\alpha+1}i+\frac{p^{2\alpha+2}-1}{12}\right)=0.$$

Again, replacing n by $p^{2\alpha+2}n+p^{2\alpha+1}i+\frac{p^{2\alpha+2}-1}{12}$ in (59) $(1 \leq i \leq p-1)$, we arrive at (55). □

Corollary 3. *For all $n \geq 0$, we have*

$$T_3(6n+4) \equiv 0 \pmod 6, \tag{64}$$

Proof. Using (9) in (50), with $p=2$ and $k=1$, we find that

$$\sum_{n\geq 0} T_3(3n+1)q^n \equiv \frac{f_6^3}{f_2^4} \pmod 2. \tag{65}$$

If we extract the odd terms from both sides of the above equation, we obatin

$$T_3(6n+4) \equiv 0 \pmod 2. \tag{66}$$

Congruence (64) follows from (53) and (66). □

Acknowledgements. The authors would like to thank the reviewers for their valuable remarks and suggestions to improve the original manuscript. This work was supported by DG-RSDT (Algeria), PRFU Project, No. C00L03UN180120220002.

References

[1] C. Adiga and R. Dasappa, On 3-regular tripartitions, *Acta Math. Sin. (Engl. Ser.)* **35** (2019), 355–368.

[2] B. C. Berndt, *Ramanujan's Notebooks Part III*, Springer-Verlag, New York, 1991.

[3] B. C. Berndt, *Number Theory in the Spirit of Ramanujan*, Amer. Math. Soc., Providence, Rhode Island, 2006.

[4] S. Cui and N. Gu, Arithmetic properties of ℓ-regular partitions, *Adv. Appl. Math.* **51** (2013), 507–523.

[5] M. D. Hirschhorn and J. A. Sellers, Elementary proofs of parity results for 5-regular partitions, *Bull. Aust. Math. Soc.* **81** (2010), 58–63.

[6] M. D. Hirschhorn and J. A. Sellers, A congruence modulo 3 for partitions into distinct non-multiples of four, *J. Integer Seq.* (2014), article 14.9.6.

[7] M. D. Hirschhorn, *The Power of q (Developments in Mathematics 49)*, Springer, 2017.

[8] W. J. Keith, Congruences for 9-regular partitions modulo 3, *Ramanujan J.* **35** (2014), 157–164.

[9] D. Ranganatha, On a Ramanujan-type congruence for bipartitions with 5-cores, *J. Integer Seq.* **19** (2016), article 16.8.1

[10] P. C. Toh, Ramanujan-type identities and congruences for partition pairs, *Discrete Math.* **312** (2012), 1244–1250.

[11] On-Line Encyclopedia of Integer Sequences, http://oeis.org, 2024.

[12] D. Penniston, Arithmetic of ℓ-regular partition functions, *Int. J. Number Theory* **4** (2008), 295–302.