RUNS OF INTEGERS WITH CONSTANT VALUES OF THE CARMICHAEL FUNCTION

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Abstract

In 2023, the first author and Vandehey proved that the largest k for which the string of equalities $\lambda(n+1) = \lambda(n+2) = \cdots = \lambda(n+k)$ holds for some $n \leq x$, where λ is the Carmichael λ function, is bounded above by $O\left((\log x \log\log x)^2\right)$. Their method involved bounding the value of $\lambda(n+i)$ from below using the prime factorization of $n+i$ for each $i \leq k$. They then used the fact that *every* $\lambda(n+i)$ had to satisfy this bound. Here we improve their result by incorporating a reverse counting argument on a result of Baker and Harman on the largest prime factor of a shifted prime.

1. Introduction

For a given arithmetic function f, let $F_f(x)$ be the largest k for which the set of equalities $f(n+1) = f(n+2) = \cdots = f(n+k)$ has a solution satisfying $n+k \leq x$. In addition, let $G_f(x)$ be the largest k for which the set of inequalities $f(n+1) \geq f(n+2) \geq \cdots \geq f(n+k)$ has a solution satisfying $n+k \leq x$.

The functions F_f and G_f have been studied for various functions f. Erdős [5] conjectured that $F_\varphi(x) \to \infty$ as $x \to \infty$, where φ is Euler's totient function. To date, however, the only known solution to the equation $\varphi(n+1) = \varphi(n+2) = \varphi(n+3)$ is $n = 5185$. Pollack, Pomerance, and Treviño [9, Theorem 1.5] found an asymptotic formula for $G_\varphi(x)$.

Theorem 1 ([9])**.** *As $x \to \infty$, we have*

$$G_\varphi(x) \sim \log_3 x / \log_6 x,$$

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where (here and below) $\log_k x$ *refers to the* k*th iterate of the logarithm.*

There are also results for other arithmetic functions as well. By modifying the proof of the previous theorem, one can also show that $G_\sigma(x) \sim \log_3 x / \log_6 x$, where σ is the sum-of-divisors function. Spătaru [11] and the first author and Vandehey [8] independently proved that $F_d(x) = \exp(O(\sqrt[3]{\log x \log_2 x}))$, where d is the number of divisors function. The first author and Vandehey [8] also showed that $G_d(x) = O(\sqrt{\log x \log_2 x})$. Their proofs relied on bounding the size of $d(n+1), \ldots, d(n+k)$ from below using the prime factorization of this common size.

In this note, we extend these results to the Carmichael λ function, which we define below.

Definition 1. The *Carmichael function* $\lambda(n)$ refers to the smallest number m for which the congruence $a^m \equiv 1 \ \mathit{mod}\ n$ holds for all a coprime to n.

The Fermat-Euler Theorem implies that $\lambda(n) \leq \varphi(n)$ for all n. Carmichael [3, 4] first defined this function in 1910. He also found a simple formula for computing $\lambda(n)$.

Theorem 2. *For all* n*, we have*

$$\lambda(n) = \begin{cases} \varphi(n), & \text{if } 8 \nmid n, \\ \varphi(n)/2, & \text{if } 8 \mid n. \end{cases}$$

Fermat's Little Theorem states that for a given prime p, we have $a^{p-1} \equiv 1 \bmod p$ for all non-multiples a of p. In particular, for a given prime p, we have $\lambda(p) = p-1$. The number n is a *Carmichael number* if it is composite, but still satisfies $\lambda(n) \mid n-1$. Alford, Granville, and Pomerance [1] showed that there are infinitely many Carmichael numbers. Last year, Larsen proved that for all $C > 1/2$, there is a Carmichael number in the interval $[x, x + x/(\log x)^C]$ for all sufficiently large x. (For a survey of results on Carmichael numbers, see [10].)

The first author and Vandehey proved that $F_\lambda(x) = \exp(O(\log x \log\log x)^2)$. In this note, we obtain a better bound for $F_\lambda(x)$ by incorporating a result of Baker and Harman [2] on the largest prime factor of a shifted prime.

Theorem 3. *As* $x \to \infty$, $F_\lambda(x) = O\left((\log x)^{1/0.677}\right)$.

Note. For notational convenience, we let $c = 0.677$ from this point on. The quantity 0.677 in the previous theorem is not exact and refers to the exponent in [2, Thm. 2].

2. Proof of Theorem 3

Our proof begins with the following result of Baker and Harman [2], quoted as Theorem 1 in [6].

Note. Throughout the rest of the paper, we let p and q denote prime values.

Lemma 1. *For every $a \in \mathbb{Z}$ and $0 < \theta \leq c$ there exists $0 < \delta(\theta) < 1$ such that, for sufficiently large $x > X(a, \theta)$, we have*

$$\sum_{\substack{p \leq x \\ P(p+a) > x^{\theta}}} 1 > \delta(\theta) \frac{x}{\log x},$$

where $P(n)$ is the largest prime factor of n.

We use Lemma 1 to derive the following result.

Lemma 2. *There exists a constant $C > 0$ such that for sufficiently large x, we have*

$$\frac{x^{0.677}}{\log x} \leq C \cdot \#\{q : q > x^c, \text{ there exists } p \leq x \text{ such that } p \equiv 1 \pmod{q}\}.$$

Proof. Fix a positive integer a and some $\theta \leq c$. The previous lemma implies that there exists $\delta := \delta(\theta) \in (0, 1)$ such that if x is sufficiently large, then

$$\sum_{\substack{p \leq x \\ P(p+a) > x^{\theta}}} 1 > \delta \frac{x}{\log x}.$$

In particular, setting $a = -1$ and $\theta = c$, we may choose δ so that

$$\sum_{\substack{p \leq x \\ P(p-1) > x^{c}}} 1 > \frac{\delta x}{\log x},$$

for all sufficiently large x. Put another way, we have

$$\#\{p \leq x : \text{ there exists } q > x^c \text{ such that } p \equiv 1 \pmod{q}\} > \frac{\delta x}{\log x}.$$

For any p in the set on the left-hand of the above inequality, the existence of q is unique since if there were two such values of q, say q_1 and q_2, for any p we would have $p - 1 > q_1 q_2 > x^{2c} = x^{1.354} > p^{1.354}$, a contradiction. Therefore, we may partition the above set as

$$\bigcup_{x^c < q < x} \{p : p \leq x, p \equiv 1 \pmod{q}\}.$$

Therefore,

$$\begin{aligned} &\#\{p : p \leq x, \text{ there exists } q > x^c, p \equiv 1 \pmod{q}\} \\ &\quad = \sum_{\substack{x^c < q \leq x \\ \text{there exists } p \leq x \text{ such that } p \equiv 1 \pmod{q}}} \#\{p \leq x : p \equiv 1 \pmod{q}\}. \end{aligned}$$

Since for each $q > x^c$

$$\#\{p \leq x : p \equiv 1 \pmod{q}\} \leq \left\lfloor \frac{x-1}{q} \right\rfloor \ll \frac{x}{q} < x^{1-c},$$

we have

$$\frac{x}{\log x} \ll \#\{q : q > x^c, \text{ there exists } p \leq x \text{ such that } p \equiv 1 \pmod{q}\} \cdot x^{1-c}.$$

Thus,

$$\frac{x^c}{\log x} \ll \#\{q : q > x^c, \text{ there exists } p \leq x \text{ such that } p \equiv 1 \pmod{q}\}. \qquad \square$$

Using this result, we can bound the length of a sequence of numbers with the same value on the Carmichael function.

Lemma 3. *Let*

$$T = \lambda(n+1) = \lambda(n+2) = \cdots = \lambda(n+k).$$

Let C be the constant in Lemma 1*. Then,*

$$\exp\left(\frac{c}{C} \cdot \left(\frac{k}{2}\right)^c\right) \leq T.$$

Proof. We may assume that k is sufficiently large so that Lemma 1 holds with $x = k/2$. Consider a prime $q > \left(\frac{k}{2}\right)^c$ such that there exists a prime $p \leq \frac{k}{2}$ with $p \equiv 1 \pmod{q}$. Then for any integer n, there exists $1 \leq i \leq k$ such that $p \parallel n+i$, where $p^a \parallel n$ means p^a is the highest power of p dividing n. Therefore, $p-1 \mid T$ and hence $q \mid T$. Therefore, T is bounded below by the product of all such q. Lemma 2 therefore implies

$$\exp\left(\frac{c}{C} \cdot \left(\frac{k}{2}\right)^c\right) = \left(\frac{k}{2}\right)^{c \cdot \frac{\left(\frac{k}{2}\right)^c}{C(\log k - \log 2)}} \leq T. \qquad \square$$

We now show that Lemma 3 directly implies Theorem 3.

Proof of Theorem 3. By Lemma 3, there exists a constant $D > 0$ such that

$$T \geq \exp\left(Dk^c\right).$$

By Mertens' Theorem, we have $T \ll x \log\log x$. Therefore, $k^c \ll \log x$. Thus, $k \ll (\log x)^{1/c}$. $\square$

Remark 1. The above proof also works with λ replaced with φ, σ, or σ_d, the sum of the dth powers of all the divisors function, where d is any positive odd integer. The proof is identical in the case of φ. For σ, one just has to take $a = 1$ in applying Lemma 1 and replacing the congruence $p \equiv 1 \pmod{q}$ in Lemma 2 with $p \equiv -1 \pmod{q}$. Of course, in both of these cases, the resulting bound is still much weaker than Pollack, Pomerance, and Treviño's result [9] as their method and result also holds for σ. For σ_d, where d is any positive odd integer, one simply makes the observation in the proof of Lemma 3 that $p \mid n + i$ also implies that $p^d + 1 \mid T$, where $T = \sigma_d(n + i)$ so that $p + 1 \mid T$ since $p + 1 \mid p^d + 1$. Unfortunately, this proof does not work for positive even values of d.

Remark 2. The Elliott-Halberstam Conjecture [12, pg. 403] implies that we can increase the range of θ to $0 < \theta < 1$ in Lemma 1. If this is true, we can replace tse exponent of $1/c$ in Theorem 3 with $1 + o(1)$.

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