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ON THE BROCARD–RAMANUJAN EQUATION WITH TRIBONACCI AND TETRANACCI NUMBERS

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Abstract

We prove that the Brocard–Ramanujan equation $m! + 1 = u^2$ has no integer solution where u is a Tribonacci number. This is motivated by a previously published incorrect proof of this statement, which followed a sound idea but was based on an incorrect formula for the 2-adic valuation of shifted Tribonacci numbers. Here we resolve the question using the same idea with respect to the more convenient prime $p = 47$. We then return to the original idea using $p = 2$ to furnish a valid proof for the corresponding statement involving Tetranacci numbers.

– *Dedicated to the memory of Tamás Lengyel*

1. Introduction

One of the most famous Diophantine equations involving factorials is the *Brocard–Ramanujan equation*

$$m! + 1 = u^2, \tag{1.1}$$

which was initially proposed by Brocard [5] in 1876, and independently by Ramanujan [11] in 1913. It is conjectured that there are no positive integer solutions other than $(m, u) \in \{(4, 5), (5, 11), (7, 71)\}$, and it has been shown [2] that there are no others with $m < 10^9$.

Among the many variant problems is to prove that (1.1) has no integer solutions m where u has a particular form. For example, Marques [9] has shown that there are no solutions, excepting $(m, u) = (4, 5)$, where u is a Fibonacci number, and Ismail et al. [8] have shown that there are no solutions where u is a Narayana number. In this note we take u to be either a *Tribonacci number* or a *Tetranacci number* (see Section 2) and prove that there are no solutions to (1.1) for such u.

Theorem 1. *There are no solutions (m, u) to the Brocard–Ramanujan equation $m! + 1 = u^2$, where u is a Tribonacci number.*

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Theorem 2. *There are no solutions* (m, u) *to the Brocard–Ramanujan equation* $m! + 1 = u^2$*, where* u *is a Tetranacci number.*

Our primary motivation arises from a published proof [7] of Theorem 1, which was subsequently discovered [14, Corollary 2] to be based on an incorrect formula for the 2-adic valuation of shifted Tribonacci numbers. As detailed in Remark 3 in Section 4 below, it is not clear how to make the original argument work using the 2-adic valuation. In Sections 3 and 4 we provide a proof of Theorem 1 using the same idea as in [7] but replacing the troublesome 2-adic valuation with the better behaved 47-adic valuation. Then in Section 5 we return to the 2-adic valuation as in [7] to give a proof of Theorem 2.

Since several recent papers [1, 7, 8, 9, 10, 13] have employed p-adic valuations in the study of Diophantine problems, we consider it important to provide a correct proof of Theorem 1, while emphasizing that the basic idea of the proof appearing in [7] was sound. Indeed, in [8] the same argument as in [7] was applied, with 3-adic valuations, to a different linear recurrent sequence. The basic idea is to show that the p-adic valuation of $u_n^2 - 1$ grows only logarithmically in n, where (u_n) is a linear recurrence and p is a convenient prime. With respect to (1.1), such growth is much slower than the linear growth of the valuation of $m!$, forcing an upper bound on m and n for any solutions.

2. Preliminaries

For $k \geqslant 2$, the *generalized Fibonacci sequence* $(F_n^{(k)})$ of order k is defined [1, 4, 6] by the recurrence $F_n^{(k)} = F_{n-1}^{(k)} + \cdots + F_{n-k}^{(k)}$, with initial conditions $F_1^{(k)} = 1$ and $F_n^{(k)} = 0$ for $2 - k \leqslant n \leqslant 0$. For order $k = 2$ we have the usual Fibonacci sequence, while for $k = 3$ we have the *Tribonacci sequence* $(T_n) := (F_n^{(3)})$, and for $k = 4$ we we have the *Tetranacci sequence* $(\tau_n) := (F_n^{(4)})$. When considering the p-adic valuations of these sequences it is useful to include negative indices, while for Diophantine equations as in Theorems 1 and 2 we consider only positive indices; thus we use *Tribonacci number* and *Tetranacci number* to refer to values of $F_n^{(3)}$ and $F_n^{(4)}$, respectively, with $n > 0$.

For any order $k \geqslant 2$ we have the Binet formula [6]

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{1 - \alpha_i^{-1}}{2 + (k+1)(\alpha_i - 2)} \alpha_i^n = \sum_{i=1}^{k} a_i \alpha_i^n \tag{2.1}$$

for all $n \in \mathbb{Z}$, where $\{\alpha_i\}_{i=1}^k$ are the zeros of the characteristic polynomial

$$f_k(x) = x^k - x^{k-1} - \cdots - x - 1.$$

For any order $k \geqslant 2$, there is precisely one zero α_k of $f_k(x)$ which lies outside the unit circle, which is real and lies in $(2-2^{1-k}, 2)$. For this dominant root we have the estimate

$$\alpha_k^{n-2} \leqslant F_n^{(k)} \leqslant \alpha_k^{n-1} \tag{2.2}$$

[4, Lemma 1] for all $n > 0$.

For a prime p, the p-adic valuation $\nu_p(n)$ of an integer n denotes the highest power of p which divides n, with convention $\nu_p(0) = +\infty$. This valuation extends uniquely to the field $\mathbb{Q}_p$ of p-adic numbers, on which it takes integer values. We require the well-known formula

$$\nu_p(m!) = \frac{m - S_p(m)}{p-1} \geqslant \frac{m}{p-1} - \log_p(m+1) \tag{2.3}$$

where $S_p(m)$ denotes the sum of the base p digits of m, and $\log_p$ denotes the real base p logarithm.

3. The 47-adic Valuation of Shifted Tribonacci Numbers

For $-10 \leqslant n \leqslant 10$, the values of T_n are

$$\ldots, 5, -8, 4, \mathbf{1}, -3, 2, 0, -\mathbf{1}, \mathbf{1}, 0, 0^*, \mathbf{1}, \mathbf{1}, 2, 4, 7, 13, 24, 44, 81, 149, \ldots$$

where the boldface values are $T_{-3} = -1$ and $T_1 = T_2 = T_{-2} = T_{-7} = 1$, and 0^* represents $T_0 = 0$. Throughout this section we take $p = 47$ and determine the p-adic valuation of $T_n \pm 1$. There are two important properties that make the choice of prime $p = 47$ convenient for our purposes. First, it happens that $p = 47$ is the smallest prime for which the characteristic polynomial $f_3(x) = x^3 - x^2 - x - 1$ factors into distinct linear factors in $\mathbb{F}_p[x]$. By the Binet formula (2.1), this means that for all integers j, the lacunary subsequences $r \mapsto T_{(p-1)r+j} \pm 1$ on congruence classes modulo $p-1$ are interpolated by p-adic analytic functions on large disks containing the ring of p-adic integers $\mathbb{Z}_p$. Second, and more importantly, it happens that the only zeros in $\mathbb{Z}_p$ of these analytic functions correspond to the zeros $\{1, 2, -2, -3, -7\}$ of the sequences $(T_n \pm 1)$ indicated above. This results in a formula for $\nu_p(T_n^2 - 1)$ which may be conveniently bounded.

Proposition 1. *Let $p = 47$. For $j \in \{1, 2, -2, -3, -7\}$ we have $T_j \in \{\pm 1\}$ and*

$$\nu_p(T_{(p-1)r+j} - T_j) = 1 + \nu_p(r) \qquad \textit{for all} \quad r \in \mathbb{Z},$$

and we have $\nu_p(T_n \pm 1) = 0$ for all other integers n.

Remark 1. This formula for the 47-adic valuation shows that $n = -3$ is the only integer solution to $T_n = -1$, and $n \in \{1, 2, -2, -7\}$ are the only solutions to $T_n = 1$.

Proof. Modulo $p = 47$, we find that f_3 factors into distinct linear factors, as

$$f_3(x) = x^3 - x^2 - x - 1 \equiv (x-5)(x-17)(x-26) \pmod{p\mathbb{Z}[x]},$$

which means that in $\mathbb{Z}_p[x]$ we have the factorization

$$f_3(x) = x^3 - x^2 - x - 1 = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$$

with p-adic zeros $\alpha_i \equiv 5, 17, 26 \pmod{p\mathbb{Z}_p}$ for $i = 1, 2, 3$. Using the Binet formula (2.1), we may write $T_n = \sum_{i=1}^{3} a_i \alpha_i^n$ with p-adic unit coefficients $a_i \equiv 39, 29, 26 \pmod{p\mathbb{Z}_p}$ for $i = 1, 2, 3$, respectively. By Fermat's little theorem, for $i = 1, 2, 3$ we have $\alpha_i^{p-1} = 1 + \lambda_i$ with $\nu_p(\lambda_i) \geqslant 1$. Therefore the p-adic logarithm

$$\eta_i := \mathrm{Log}_p(1+\lambda_i) := \sum_{m=1}^{\infty} (-1)^{m-1} \frac{\lambda_i^m}{m} \quad \text{in} \quad \mathbb{Z}_p$$

is convergent and satisfies $\nu_p(\eta_i) = \nu_p(\lambda_i) \geqslant 1$ for $i = 1, 2, 3$, so that by (2.3) each p-adic exponential

$$\alpha_i^{(p-1)x} := \mathrm{Exp}_p(x\,\mathrm{Log}_p(1+\lambda_i)) := \sum_{m=0}^{\infty} \frac{\eta_i^m}{m!} x^m = \sum_{m=0}^{\infty} b_{i,m} x^m$$

converges to an analytic function [12] on a p-adic disk containing $\mathbb{Z}_p$, with coefficients $b_{i,m} = \eta_i^m/m!$ satisfying $\nu_p(b_{i,m}) \geqslant (p-2)m/(p-1) + S_p(m)/(p-1)$. By the Binet formula (2.1), for any $j \in \mathbb{Z}$ there is an analytic function

$$g_j(x) := \sum_{i=1}^{3} a_i \alpha_i^j \alpha_i^{(p-1)x} = \sum_{m=0}^{\infty} a_{j,m} x^m \tag{3.1}$$

for $x \in \mathbb{Z}_p$, satisfying $g_j(x) = T_{(p-1)x+j}$ for $x \in \mathbb{Z}$, with coefficients $a_{j,m} = \sum_{i=1}^{3} a_i \alpha_i^j b_{i,m}$ satisfying $\nu_p(a_{j,m}) \geqslant 1$ for $m \geqslant 1$, and $\nu_p(a_{j,m}) \geqslant 2$ for $m \geqslant 2$.

Recalling that $T_j \in \{\pm 1\}$ for $j \in \{1, 2, -2, -3, -7\}$, by (3.1) we have

$$T_{(p-1)x+j} - T_j = \sum_{m=1}^{\infty} a_{j,m} x^m$$

for such j. Taking $x = 1$, we check numerically that $\nu_p(T_{p-1+j} - T_j) = 1$ for $j \in \{1, 2, -2, -3, -7\}$. Since $\nu_p(\sum_{m=1}^{\infty} a_{j,m}) = 1$ and $\nu_p(a_{j,m}) \geqslant 2$ for $m \geqslant 2$, we conclude that $\nu_p(a_{j,1}) = 1$ and thus

$$\nu_p(T_{(p-1)x+j} - T_j) = \nu_p(a_{j,1}x) = 1 + \nu_p(x)$$

for $x \in \mathbb{Z}$ and $j \in \{1, 2, -2, -3, -7\}$. If $-23 < j \leqslant 23$ and $j \notin \{1, 2, -2, -3, -7\}$, we check numerically that $\nu_p(T_j \pm 1) = 0$. Therefore for such j we have by (3.1)

$$T_{(p-1)x+j} \pm 1 = \sum_{m=0}^{\infty} a_{j,m} x^m \qquad \text{with} \quad \nu_p(a_{j,0}) = 0$$

and $\nu_p(a_{j,m}) \geqslant 1$ for $m \geqslant 1$, so that $\nu_p(T_{(p-1)x+j} \pm 1) = 0$ for all such j and all $x \in \mathbb{Z}$. □

Remark 2. It appears that simple formulas for p-adic valuations as in Proposition 1 above should be viewed as exceptional rather than typical. Marques and Lengyel [10, Conjecture 8] had conjectured that for some $N \in \mathbb{N}$, the p-adic valuation of T_n on each congruence class $n \equiv i \pmod{N}$ should always be a linear function of $\nu_p(n - a_i)$ for some integer a_i; however, in [3] it was shown that this conjecture fails for infinitely many primes, and in fact holds for at most seven of the 109 primes $p < 600$; in particular, it fails for $p = 47$ [3, Theorems 1.5, 1.7]. In this context it is rather surprising that the analogous statement with $p = 47$ actually holds for both *shifted* sequences $(T_n \pm 1)$, according to Proposition 1. For this purpose the sequences $(T_n \pm 1)$ may also be viewed as fourth order linear recurrences satisfying $a_n = 2a_{n-1} - a_{n-4}$ with initial conditions $(1, 2, 2, 3)$ or $(-1, 0, 0, 1)$, respectively.

4. Proof of Theorem 1

Proof of Theorem 1. Take $p = 47$. By Proposition 1, if we can write the index $n = u(p-1)p^e + j$ with $j \in \{1, 2, -2, -3, -7\}$ and $(u, p) = 1$, then $\nu_p(T_n^2 - 1) = 1 + e$; for all other $n > 2$ the valuation is zero. Thus for $n = u(p-1)p^e + j$ as described, we have

$$\begin{aligned} \nu_p(T_n^2 - 1) &= 1 + \log_p\left(\frac{n-j}{u(p-1)}\right) \\ &\leqslant 1 + \log_p(n+7) - \log_p(p-1) \\ &< \log_p(n+7) + 0.006, \end{aligned} \tag{4.1}$$

which implies $\nu_p(T_n^2 - 1) < \log_p(n+7) + 0.006$ for every integer $n > 2$.

Thus if $T_n^2 - 1 = m!$ is a solution with $n > 2$, we must have $\nu_p(m!) = \nu_p(T_n^2 - 1)$, so that by (2.3) and (4.1) we must have

$$\frac{m}{p-1} - \log_p(m+1) < \log_p(n+7) + 0.006. \tag{4.2}$$

Since the dominant zero of $f_3(x)$ is $\alpha_3 = 1.8392867552\cdots$, the estimate (2.2) implies $(1.83)^{2n-4} < T_n^2 = m! + 1 < m^m$ for $m > 1$. Taking base 2 logarithms gives

$$n < 0.6m \log_2 m + 2. \tag{4.3}$$

Therefore any solution $m! = T_n^2 - 1$ with $m > 1$, $n > 2$ requires

$$\frac{m}{p-1} - \log_p(m+1) < \log_p(0.6m \log_2 m + 9) + 0.006,$$

which we rearrange to

$$m < 46\log_{47}(m+1) + 46\log_{47}(0.6m\log_2 m + 9) + 0.28. \tag{4.4}$$

Since the right side of (4.4) is concave down as a function of real $m > 1$, a routine computation shows that this inequality requires $m \leqslant 134$. Since there are no solutions to (1.1) with $m < 10^9$ except $(m, u) \in \{(4, 5), (5, 11), (7, 71)\}$ [2], and none of 5, 11, or 71 are Tribonacci numbers, we conclude that there are no solutions to $m! + 1 = T_n^2$ with $m, n > 0$. □

Remark 3. In [14, Corollary 2] we find the 2-adic valuation of $T_n + 1$ given as

$$\nu_2(T_n+1) = \begin{cases} 0, & \text{if } n \equiv 0, 3 \pmod 4, \\ 1, & \text{if } n \equiv 1, \pm 2 \pmod 8, \\ 3 + \nu_2(m) + \nu_2(m-z), & \text{if } n = 8m-3, \end{cases} \tag{4.5}$$

where z is some 2-adic integer satisfying $z \equiv -601592 \pmod{2^{20}\mathbb{Z}_2}$; and [1, Proposition 6.3] gave the refined estimate $z \equiv 4635424790827476707644672 8 \pmod{2^{92}\mathbb{Z}_2}$ for this value. It will be observed that the term $\nu_2(m)$ in the formula (4.5) accounts for the value $T_{-3} = -1$; however, by Remark 1, $n = -3$ is the only solution $n \in \mathbb{Z}$ to $T_n = -1$, so the value z in (4.5) cannot be an integer. The argument in [7] used an incorrect formula [7, Lemma 2] for $\nu_2(T_n + 1)$, which essentially replaced the term $\nu_2(m - z)$ with $\nu_2(m - 8)$, to deduce a bound of the form $\nu_2(T_n + 1) \leqslant A + B \log n$. Such a bound would not be valid, for example, if z were a p-adic Liouville number ([12, Section 66,67]). As the nature of z remains unknown, it does not seem that the proof of [7] can be easily fixed using 2-adic valuations. Our modified proof using 47-adic valuations works because of the absence of such "extraneous" z values.

5. Proof of Theorem 2

In this final section we employ the 2-adic valuation as in [7] to show that there are no solutions to $m! + 1 = u^2$ where u is a Tetranacci number $\tau_n = F_n^{(4)}$ with $n > 0$. For $-15 \leqslant n \leqslant 10$ the values of τ_n are

$$\ldots, 8, -\mathbf{1}, 5, -8, 4, 0, \mathbf{1}, -3, 2, 0, 0, -\mathbf{1}, \mathbf{1}, 0, 0, 0^*, \mathbf{1}, \mathbf{1}, 2, 4, 8, 15, 29, 56, 108, 208, \ldots$$

where the boldface values are $\tau_{-4} = \tau_{-14} = -1$ and $\tau_1 = \tau_2 = \tau_{-3} = \tau_{-9} = 1$, and 0^* represents $\tau_0 = 0$.

Our proof uses the 2-adic valuations of $(\tau_n \pm 1)$ which were determined in [1, Propositions 4.1, 4.2] as

$$\nu_2(\tau_n - 1) = \begin{cases} 0, & n \equiv 0, 3, 4 \pmod 5, \\ 1, & n \equiv 6 \pmod{10}, \\ 2 + \nu_2(r), & n = 10r + 2, n = 10r - 3, \\ 4 + \nu_2\left(\binom{2r+3}{4}\right), & n = 10r + 1, \end{cases} \tag{5.1}$$

and

$$\nu_2(\tau_n + 1) = \begin{cases} 0, & n \equiv 0, 3, 4 \pmod 5, \\ 1, & n \equiv 1, 2, 7 \pmod{10}, \\ 4 + \nu_2\left(\binom{2r+2}{4}\right), & n = 10r - 4. \end{cases} \tag{5.2}$$

Considering a binomial coefficient $\binom{x}{4} = x(x-1)(x-2)(x-3)/4$ as a polynomial in x, we see that these formulas account for each value $n \in \{1, 2, -3, -4, -9, -14\}$ where $\tau_n = \pm 1$, with no "extraneous" zeros as in (4.5); this is crucial to the proof.

Proof of Theorem 2. For an integer $b \geqslant 4$, the 2-adic valuation of the binomial coefficient $\binom{b}{4}$ is equal to the number of carries in the binary addition of $b - 4$ and 4, and is therefore bounded above by $\log_2 b$. Therefore, for $n = 5a + 1$ with $a > 0$, we find from (5.1) and (5.2) that

$$\nu_2(\tau_n^2 - 1) = 5 + \nu_2\left(\binom{a+3}{4}\right) < 5 + \log_2(n/5 + 3), \tag{5.3}$$

and a check of the remaining cases shows that this bound (5.3) actually holds for all integers $n > 2$. Thus if $\tau_n^2 - 1 = m!$ with $n > 2$ then $\nu_2(m!) = \nu_2(\tau_n^2 - 1)$, so by (2.3) and (5.3) we have

$$m - \log_2(m+1) < 5 + \log_2(n/5 + 3).$$

Since the dominant zero α_4 of $f_4(x)$ is $\alpha_4 = 1.9275619754\cdots$, the bound (2.2) gives $(1.92)^{2n-4} < \tau_n^2 = m! + 1 < m^m$, so we have $n < 0.6m \log_2 m + 2$ as in (4.3). Thus ,any solutions require

$$m < 5 + \log_2(m+1) + \log_2(0.12m \log_2 m + 3.4). \tag{5.4}$$

Since the right side of (5.4) is concave down as a function of real $m > 1$, a routine computation shows that $m \leqslant 11$. Thus we are reduced to the known solutions $(m, u) \in \{(4, 5), (5, 11), (7, 71)\}$, but none of 5, 11, or 71 are Tetranacci numbers. Thus there are no solutions to $m! + 1 = \tau_n^2$ with $m, n > 0$. □

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