A STUDY ON GENERALIZED ZUMKELLER NUMBERS

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Abstract

The aim of this paper is to generalize the concept of Zumkeller numbers and introduce two new types, namely, s-Zumkeller numbers and unitary s-Zumkeller numbers. Different characteristics exhibited by these numbers are investigated and illustrative examples are provided to support the findings. Additionally, connections are derived between the harmonic mean of the squares of the divisors of a positive integer n and s-Zumkeller numbers.

1. Introduction

In recent years, numerous researchers [1, 10, 12, 13] have been focused on expanding the concept of perfect numbers. A positive integer n is called a *perfect number* if the sum of all proper positive divisors of n is equal to itself. If $\sigma(n)$ denotes the sum of all positive divisors of n, then n is *perfect* if and only if $\sigma(n) - n = n$, that is, $\sigma(n) = 2n$.

In 2003, generalizing the concept of perfect number, R. H. Zumkeller introduced a new type of positive integer what are now known as Zumkeller number [5]. A positive integer n is called a *Zumkeller number* if the set of all positive divisors of n can be partitioned into two disjoint subsets of equal sum. Similarly, n is called a *half Zumkeller number* if the set of all proper positive divisors of n can be partitioned into two disjoint subsets of equal sum.

In 2013, Peng and Bhaskara Rao [9] investigated various properties of Zumkeller numbers and half Zumkeller numbers. Mahanta, Saikia and Yaqubi [7] examined Zumkeller numbers and the harmonic mean of divisors, establishing certain relations between the two. In [4], Cai, Chen and Zhang explored perfect numbers

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and Fibonacci primes, investigating several of their properties. Additionally, many researchers have been working on Zumkeller numbers and their variations, such as layered Zumkeller numbers, unitary Zumkeller numbers, and m-Zumkeller numbers.

Motivated by their work, we attempt to study two new types of numbers, namely, s-Zumkeller numbers and unitary s-Zumkeller numbers. A positive integer n is called an *s-Zumkeller number* if the set of proper positive divisors of n can be partitioned into two disjoint subsets, such that the sum of the squares of the elements of each subset are equal. A divisor d of a positive integer n is said to be a *unitary divisor* of n if $\left(d, \frac{n}{d}\right) = 1$. A positive integer n is called a *unitary s-Zumkeller number* if the set of proper unitary positive divisors of n can be partitioned into two disjoint subsets, such that the sum of the squares of the elements of each subset are equal. It is interesting to note that 6 is a perfect number and also a Zumkeller number, but it is neither an s-Zumkeller number nor a unitary s-Zumkeller number. In Section 2 and Section 3 of this paper, we shall study various properties of s-Zumkeller numbers and unitary s-Zumkeller numbers, respectively.

2. s-Zumkeller Numbers

In this section, we first define some preliminary arithmetic functions and present a few results that are needed for the sequel. We then define s-Zumkeller numbers with suitable examples and obtain some properties of these type of numbers.

The function $\tau(n)$ denotes the number of positive divisors of n and the function $\sigma_2(n)$ denotes the sum of the squares of the positive divisors of n. The function $H(n)$, the harmonic mean of the positive divisors of n, is defined by $\frac{1}{H(n)} = \frac{1}{\tau(n)} \sum_{d|n} \frac{1}{d}$. Then $H(n) = \frac{\tau(n)}{\sum_{d|n} \frac{1}{d}}$ and this implies $H(n) = \frac{n\tau(n)}{\sigma(n)}$. Similarly, the function $H_2(n)$, the harmonic mean of the squares of the positive divisors of n, is defined by $\frac{1}{H_2(n)} = \frac{1}{\tau(n)} \sum_{d|n} \frac{1}{d^2}$. Then $H_2(n) = \frac{n^2\tau(n)}{\sigma_2(n)}$.

We present the following facts about the functions $\sigma(n)$ and $\sigma_2(n)$.

Lemma 1 ([9])**.** *If the prime factorization of n is $p_1^{k_1} p_2^{k_2} ... p_m^{k_m}$, then*

(i)

$$\sigma(n) = \prod_{i=1}^{m} \frac{p_i^{(k_i+1)} - 1}{p_i - 1} \text{ and } \frac{\sigma(n)}{n} = \prod_{i=1}^{m} \frac{p_i^{(k_i+1)} - 1}{p_i^{k_i}(p_i - 1)} < \prod_{i=1}^{m} \frac{p_i}{p_i - 1};$$

(ii)

$$\sigma_2(n) = \prod_{i=1}^{m} \frac{p_i^{2(k_i+1)} - 1}{p_i^2 - 1} \text{ and } \frac{\sigma_2(n)}{n^2} = \prod_{i=1}^{m} \frac{p_i^{2(k_i+1)} - 1}{p_i^{2k_i}(p_i^2 - 1)} < \prod_{i=1}^{m} \frac{p_i^2}{p_i^2 - 1}.$$

We now define s-Zumkeller numbers as follows.

Definition 1. A positive integer n is called an *s-Zumkeller number* if the set D of proper positive divisors of n can be partitioned as $\{A, B\}$, such that

$$\sum_{d\in A} d^2 = \sum_{d\in B} d^2 = \frac{\sigma_2(n) - n^2}{2}.$$

Example 1. The integers 60, 120, 180, 252, 300, 336, 360, 420, 480, 504, 600, 660, 672, 756, 792, 840, 936 are some s-Zumkeller numbers.

2.1. Main Results

Proposition 1. *If n is an even s-Zumkeller number, then*

(i) $\sigma_2(n)$ *is even;*

(ii) the prime factorization of n must include at least one odd prime with an odd exponent;

(iii) $\sigma_2(n) \geq \frac{3}{2}n^2$.

Proof. (i) Let $\sigma_2(n)$ be an odd integer. Then $\sigma_2(n) - n^2$ is also odd. Thus, it is impossible to partition the proper positive divisors of n into two equal squared summed subsets. Therefore, $\sigma_2(n)$ is even.

(ii) Since $\sigma_2(n)$ is even, the number of odd positive divisors of n must be even. Let the prime factorization of n be $2^k p_1^{k_1} p_2^{k_2} ... p_m^{k_m}$. Obviously, the number of odd positive divisors of n is $(k_1 + 1)(k_2 + 1)...(k_m + 1)$ and this number must be even. Therefore, at least one of the k_i's must be odd. Hence, the prime factorization of n must include at least one odd prime with an odd exponent.

(iii) Let n be an even s-Zumkeller number. Then, by definition, the proper positive divisors of n can be partitioned into two disjoint subsets of equal sum of the squares of the divisors. Hence, each subset must contain those divisors, whose sum of the squares is equal to $\frac{\sigma_2(n)-n^2}{2}$. Since n is even, the greatest proper positive divisor is $\frac{n}{2}$. Also, $\frac{n}{2}$ must be contained in one of these two subsets. Thus, $\frac{\sigma_2(n)-n^2}{2} \geq (\frac{n}{2})^2$ which implies $\sigma_2(n) \geq \frac{3n^2}{2}$. □

The following result exhibits a necessary and sufficient condition for an even integer to be an s-Zumkeller number.

Proposition 2. *An even positive integer n is an s-Zumkeller number if and only if $\frac{2\sigma_2(n)-3n^2}{4}$ is a sum (possibly an empty sum) of the squares of distinct proper positive divisors of n, excluding $\frac{n}{2}$.*

Proof. We assume that n is an s-Zumkeller number with Zumkeller partition $\{A, B\}$. Without loss of generality, we may assume that $\frac{n}{2} \in A$. Then, the sum of the squares of the remaining elements of A will be

$$\frac{\sigma_2(n) - n^2}{2} - \left(\frac{n}{2}\right)^2 = \frac{2\sigma_2(n) - 3n^2}{4}.$$

Thus, $\frac{2\sigma_2(n)-3n^2}{4}$ is a sum of the squares of distinct proper positive divisors of n, excluding $\frac{n}{2}$.

Conversely, suppose $\frac{2\sigma_2(n)-3n^2}{4}$ is a sum of the squares of distinct proper positive divisors of n, excluding $\frac{n}{2}$. If we augment this set with $\frac{n}{2}$, we have a set of positive divisors of n whose sum of the squares is equal to

$$\frac{2\sigma_2(n) - 3n^2}{4} + \left(\frac{n}{2}\right)^2 = \frac{\sigma_2(n) - n^2}{2}.$$

Therefore, the sum of the squares of the proper positive divisors of n of the complementary set of the above mentioned augmented set will be equal to $\frac{\sigma_2(n)-n^2}{2}$. Hence, n is an s-Zumkeller number. □

Proposition 3. *If n is an even s-Zumkeller number, then $\tau(n) \geq 12$.*

Proof. Let n be an even s-Zumkeller number. So, by Proposition 1 (iii), we have $\sigma_2(n) \geq \frac{3}{2}n^2$. This implies

$$\sum_{d_i|n} \frac{1}{d_i^2} \geq \frac{3}{2}. \tag{1}$$

Suppose, $\tau(n) < 12$. Let $\tau(n) = k$. (To get $\sum_{d_i|n} \frac{1}{d_i^2} \geq \frac{3}{2}$, we have to consider the smallest possible value of n.)

If $k = 2$, then the smallest possible value of n is 2 and $d_1 = 1, d_2 = 2$. Then $\sum_{d_i|n} \frac{1}{d_i^2} = 1.25 < \frac{3}{2}$, which contradicts (1).

If $k = 3$, then the smallest possible value of n is 4 and $d_1 = 1, d_2 = 2, d_3 = 4$. Then $\sum_{d_i|n} \frac{1}{d_i^2} = \frac{21}{16} < \frac{3}{2}$, which contradicts (1).

If $k = 4$, then the smallest possible value of n is 6 and $d_1 = 1, d_2 = 2, d_3 = 3, d_4 = 6$. Then $\sum_{d_i|n} \frac{1}{d_i^2} = \frac{25}{18} < \frac{3}{2}$, which contradicts (1).

Similarly, for all $k < 12$, $\sum_{d_i|n} \frac{1}{d_i^2} < \frac{3}{2}$, which contradicts (1). But, if $k = 12$, then the smallest possible value of n is 60 and $d_1 = 1, d_2 = 2, d_3 = 3, d_4 = 4, d_5 = 5, d_6 = 6, d_7 = 10, d_8 = 12, d_9 = 15, d_{10} = 20, d_{11} = 30, d_{12} = 60$. This implies $\sum_{d_i|n} \frac{1}{d_i^2} = \frac{91}{60} > \frac{3}{2}$.

Hence $\tau(n) \geq 12$. □

Proposition 4. *If n is an even s-Zumkeller number, then 3 divides n.*

Proof. Suppose 3 does not divide n. This implies that multiples of 3 cannot be divisors of n. Let $\mathbb{N}$ be the set of natural numbers. Then,

$$\begin{aligned}\sum_{d|n}\frac{1}{d^2} < \sum_{m\in\mathbb{N}}\frac{1}{m^2} - \sum_{m\in\mathbb{N}}\frac{1}{(3m)^2} &= \sum_{m\in\mathbb{N}}\frac{1}{m^2} - \frac{1}{9}\sum_{m\in\mathbb{N}}\frac{1}{m^2}\\ &= \frac{8}{9}\sum_{m\in\mathbb{N}}\frac{1}{m^2}\\ &= \frac{8}{9}\times\frac{\pi^2}{6} \text{ (by Basel problem [2])}\\ &= 1.4621636149762...\\ &< \frac{3}{2},\end{aligned}$$

which contradicts (1). Thus, 3 divides n. □

As a consequence of the above result, we get the following corollary.

Corollary 1. *If n is an even s-Zumkeller number, then 6 divides n.*

Remark 1. If n is an odd s-Zumkeller number and 3 divides n, then $\frac{\sigma_2(n)-n^2}{2} \geq (\frac{n}{3})^2$ and this implies $\sum_{\substack{d|n\\2\nmid n}}\frac{1}{d^2} \geq \frac{11}{9} = 1.\overline{2}$. Now, $\sum_{\substack{d|n\\2\nmid n}}\frac{1}{d^2} < \sum_{m\in\mathbb{N}}\frac{1}{m^2} - \sum_{m\in\mathbb{N}}\frac{1}{(2m)^2}$. This implies $\sum_{\substack{d|n\\2\nmid n}}\frac{1}{d^2} < \frac{3}{4}\sum_{m\in\mathbb{N}}\frac{1}{m^2} = \frac{\pi^2}{8}$ (by Basel problem [2]). So, there may exist a positive integer n, such that $1.\overline{2} \leq \sum_{\substack{d|n\\2\nmid n}}\frac{1}{d^2} < \frac{\pi^2}{8}$. This is a necessary but not sufficient condition for n to be an s-Zumkeller number. This implies that odd s-Zumkeller numbers may exist.

Proposition 5. *If n is an s-Zumkeller number and the prime factorization of n is $p_1^{k_1}p_2^{k_2}...p_m^{k_m}$, where $p_1 < p_2 < ... < p_m$, then*

$$\frac{2}{p_1^2} + 1 < \prod_{i=1}^{m}\frac{p_i^2}{p_i^2-1}.$$

Proof. Let n be an s-Zumkeller number. Then

$$\frac{\sigma_2(n) - n^2}{2} \geq \left(\frac{n}{p_1}\right)^2.$$

This implies

$$\frac{2}{p_1^2} + 1 \leq \frac{\sigma_2(n)}{n^2}. \tag{2}$$

Again, by Lemma 1, we have

$$\frac{\sigma_2(n)}{n^2} < \prod_{i=1}^{m} \frac{p_i^2}{p_i^2 - 1}. \tag{3}$$

Hence, by (2) and (3),

$$\frac{2}{p_1^2} + 1 < \prod_{i=1}^{m} \frac{p_i^2}{p_i^2 - 1}.$$ □

Proposition 6. *(i) If n is an even s-Zumkeller number, then n must contain at least 3 distinct prime divisors.*
(ii) If n is an odd s-Zumkeller number and 3 divides n, then n must contain at least 8 distinct prime divisors.

Proof. (i) Let n be an even s-Zumkeller number. Then, by Proposition 5,

$$\frac{3}{2} < \prod_{i=1}^{m} \frac{p_i^2}{p_i^2 - 1}. \tag{4}$$

If $m \leq 2$, then

$$\prod_{i=1}^{m} \frac{p_i^2}{p_i^2 - 1} \leq \frac{2^2}{2^2 - 1} \times \frac{3^2}{3^2 - 1} = \frac{4}{3} \times \frac{9}{8} = \frac{3}{2},$$

which contradicts (4). Therefore, $m \geq 3$. Hence, n must contain at least 3 distinct prime divisors.
(ii) Let n be an odd s-Zumkeller number and let 3 divide n. Then, by Proposition 5,

$$\frac{11}{9} < \prod_{i=1}^{m} \frac{p_i^2}{p_i^2 - 1}. \tag{5}$$

If $m \leq 7$, then

$$\begin{aligned}\prod_{i=1}^{m} \frac{p_i^2}{p_i^2 - 1} &\leq \frac{3^2}{3^2 - 1} \times \frac{5^2}{5^2 - 1} \times \frac{7^2}{7^2 - 1} \times \frac{11^2}{11^2 - 1} \times \frac{13^2}{13^2 - 1} \times \frac{17^2}{17^2 - 1} \times \frac{19^2}{19^2 - 1} \\ &= \frac{9}{8} \times \frac{25}{24} \times \frac{49}{48} \times \frac{121}{120} \times \frac{169}{168} \times \frac{289}{288} \times \frac{361}{360} < \frac{11}{9},\end{aligned}$$

which contradicts (5). Therefore, $m \geq 8$. Hence, n must contain at least 8 distinct prime divisors. □

Proposition 7. *If n is an odd s-Zumkeller number, then n is a perfect square.*

Proof. Let n be an odd s-Zumkeller number. Then, $\sigma_2(n) - n^2$ must be even. Since n is odd, $\sigma_2(n)$ must be odd. If the prime factorization of n is $\prod_{i=1}^{m} p_i^{k_i}$, then $\sigma_2(n) = \prod_{i=1}^{m}(\sum_{j=0}^{k_i} p_i^{2j})$. Since $\sigma_2(n)$ is odd, all $k_i's$ must be even. Hence, n is a perfect square. □

Proposition 8. *If n is an odd s-Zumkeller number divisible by 3, then* $n \geq 12442607161209225$.

Proof. Let n be an odd s-Zumkeller number divisible by 3. Then, by Proposition 6 (ii), n must contain at least 8 distinct prime divisors. So, $n \geq 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23$. Again, by Proposition 7, n must be a perfect square. Therefore, $n \geq (3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23)^2 = 12442607161209225$. □

Proposition 9. *If n is an even s-Zumkeller number, then $H_2(n) \leq \frac{2\tau(n)}{3}$.*

Proof. Let n be an even s-Zumkeller number. By Proposition 1(iii), we have $\sigma_2(n) \geq \frac{3}{2}n^2$. Again, $H_2(n) = \frac{n^2\tau(n)}{\sigma_2(n)}$. Therefore, $H_2(n) = \frac{n^2\tau(n)}{\sigma_2(n)} \leq \frac{2n^2\tau(n)}{3n^2} = \frac{2\tau(n)}{3}$. □

Proposition 10. *For any positive integer α, let $n = 2^{\alpha}(2^{\alpha+1}-1)$ be an s-Zumkeller number. Then $H_2(n) < \frac{2^{2\alpha+3}}{3}$.*

Proof. By Proposition 9, we have $H_2(n) \leq \frac{2\tau(n)}{3} = \frac{2(\alpha+1)\tau(2^{\alpha+1}-1)}{3}$. For all $k \in \mathbb{N}$, we have $2^k \geq k+1$. Therefore, $2^{\alpha+1} \geq \alpha+1+1$ and this implies $2^{\alpha+1}-1 \geq \alpha+1$. Also, for each positive integer n, $\tau(n) \leq n$. Therefore, $\tau(2^{\alpha+1}-1) \leq 2^{\alpha+1}-1$. Hence, $H_2(n) \leq \frac{2(2^{\alpha+1}-1)^2}{3} < \frac{2(2^{\alpha+1})^2}{3} = \frac{2^{2\alpha+3}}{3}$. □

3. Unitary s-Zumkeller Numbers

In this section, we first present some preliminary arithmetic functions and results that are necessary for our work. We then define unitary s-Zumkeller numbers and provide suitable examples. Additionally, we discuss some properties of these numbers.

The function $\sigma_2^*(n)$ denotes the sum of the squares of the unitary divisors of a positive integer n and the function $\tau^*(n)$ denotes the number of unitary divisors of n.

Lemma 2 ([3])**.** *For any positive integer $n = \sum_{i=1}^{m} p_i^{k_i}$, we have $\tau^*(n) = 2^m$.*

Lemma 3 ([11])**.** *(i) If the prime factorization of n is $p_1^{k_1}p_2^{k_2}...p_m^{k_m}$, then*

$$\sigma_2^*(n) = (1+p_1^{2k_1})(1+p_2^{2k_2})...(1+p_m^{2k_m}) = \prod_{i=1}^{m}(1+p_i^{2k_i}).$$

(ii) If $n = n_1 n_2 ... n_m$, where $n_1 < n_2 < ... < n_m$ are pairwise relatively prime, then

$$\sigma_2^*(n) = \prod_{i=1}^{m} (1 + n_i^2).$$

Remark 2. Let $n = p_1^{k_1} p_2^{k_2} ... p_m^{k_m}$, where p_i's are distinct primes. Then $n = q_1 q_2 ... q_m$. (where $q_i = p_i^{k_i}$, for $i = 1, 2, \cdots, m$.). This implies $n = n_1 n_2 ... n_m$ (where $n_i = q_j$, for some $i = 1, 2, 3, ..., m$ and $j = 1, 2, 3, ..., m$), such that $n_1 < n_2 < ... < n_m$. Then, $\frac{n}{n_1}$ is the greatest proper unitary divisor of n.

We now define unitary s-Zumkeller numbers as follows.

Definition 2. A positive integer n is called a *unitary s-Zumkeller number* if the set D of proper positive unitary divisors of n can be partitioned as $\{A, B\}$, such that

$$\sum_{d \in A} d^2 = \sum_{d \in B} d^2 = \frac{\sigma_2^*(n) - n^2}{2}.$$

Example 2. The integers 60, 140, 420, 660, 1224, 1820 are unitary s-Zumkeller numbers.

3.1. Main Results

Proposition 11. *If n is a unitary s-Zumkeller number and $n = n_1 n_2 ... n_m$, where n_i's are pairwise relatively prime, such that $n_1 < n_2 < ... < n_m$, then $\sigma_2^*(n) \geq n^2(\frac{2}{n_1^2} + 1)$ and hence,*

(i)

$$\frac{2}{n_1^2} + 1 \leq \sum_{\substack{d|n \\ (d, \frac{n}{d})=1}} \frac{1}{d^2};$$

(ii)

$$\frac{2}{n_1^2} + 1 \leq \prod_{i=1}^{m} \frac{(1 + n_i^2)}{n_i^2}.$$

Proof. (i) Let $n = n_1 n_2 ... n_m$ be a unitary s-Zumkeller number, where $(n_i, n_j) = 1$, for all $i \neq j$ and $n_1 < n_2 < ... < n_m$. Since $\frac{n}{n_1}$ is the greatest proper unitary divisor of n, by the definition of unitary s-Zumkeller numbers, we have

$$\frac{\sigma_2^*(n) - n^2}{2} \geq \left(\frac{n}{n_1}\right)^2, \text{ and this implies } \sum_{\substack{d|n \\ (d, \frac{n}{d})=1}} \frac{1}{d^2} \geq \frac{2}{n_1^2} + 1.$$

(ii) From (i), it follows that

$$\sigma_2^*(n) \geq n^2 \left(\frac{2}{n_1^2} + 1 \right).$$

Also, by Lemma 3(ii),

$$\sigma_2^*(n) = \prod_{i=1}^{m} (1 + n_i^2).$$

Therefore,

$$\prod_{i=1}^{m} \frac{(1 + n_i^2)}{n_i^2} \geq \frac{2}{n_1^2} + 1. \qquad \square$$

Proposition 12. *Let n be a unitary s-Zumkeller number. If the least nontrivial unitary divisor of n is 2 or 3 or 4, then $\tau^*(n)$ is greater than 2^6 or 2^2 or 2^2, respectively.*

Proof. Let $n = n_1 n_2 ... n_m$ be a unitary s-Zumkeller number, where $n_1 < n_2 < ... < n_m$ are pairwise relatively prime. By Proposition 11(ii),

$$\frac{2}{n_1^2} + 1 \leq \prod_{i=1}^{m} \frac{(1 + n_i^2)}{n_i^2}.$$

We may consider three cases.
`Case1.` If $n_1 = 2$, then

$$\frac{3}{2} \leq \prod_{i=1}^{m} \frac{(1 + n_i^2)}{n_i^2}. \tag{6}$$

However, for $m \leq 6$, we have

$$\begin{aligned} \prod_{i=1}^{m} \frac{(1 + n_i^2)}{n_i^2} &\leq \frac{2^2 + 1}{2^2} \times \frac{3^2 + 1}{3^2} \times \frac{5^2 + 1}{5^2} \times \frac{7^2 + 1}{7^2} \times \frac{11^2 + 1}{11^2} \times \frac{13^2 + 1}{13^2} \\ &= \frac{5}{4} \times \frac{10}{9} \times \frac{26}{25} \times \frac{50}{49} \times \frac{122}{121} \times \frac{170}{169} < \frac{3}{2}, \end{aligned}$$

which contradicts (6). Therefore, $m > 6$ and hence, by Lemma 2, we have $\tau^*(n) > 2^6$. Thus, if 2 is the smallest nontrivial unitary divisor of n, then $\tau^*(n) > 2^6$.
`Case2.` If $n_1 = 3$, then

$$\frac{11}{9} \leq \prod_{i=1}^{m} \frac{(1 + n_i^2)}{n_i^2}. \tag{7}$$

However, for $m \leq 2$, we have

$$\prod_{i=1}^{m} \frac{(1+n_i^2)}{n_i^2} \leq \frac{3^2+1}{3^2} \times \frac{4^2+1}{4^2} = \frac{10}{9} \times \frac{17}{16} < \frac{11}{9},$$

which contradicts (7). Therefore, $m > 2$ and hence, by Lemma 2, we have $\tau^*(n) > 2^2$. Thus, if 3 is the smallest nontrivial unitary divisor of n, then $\tau^*(n) > 2^2$.
`Case3.` If $n_1 = 4$, then

$$\frac{18}{16} \leq \prod_{i=1}^{m} \frac{(1+n_i^2)}{n_i^2}. \tag{8}$$

However, for $m \leq 2$, we have

$$\prod_{i=1}^{m} \frac{(1+n_i^2)}{n_i^2} \leq \frac{4^2+1}{4^2} \times \frac{5^2+1}{5^2} = \frac{17}{16} \times \frac{26}{25} < \frac{18}{16},$$

which contradicts (8). Therefore, $m > 2$ and hence, by Lemma 2, we have $\tau^*(n) > 2^2$. Thus, if 4 is the smallest nontrivial unitary divisor of n, then $\tau^*(n) > 2^2$. □

Proposition 13. *There does not exist any odd unitary s-Zumkeller number.*

Proof. Let n be an odd positive integer. Then, by Lemma 2, we have that $\tau^*(n)$ is even. Since n is odd, all the unitary divisors of n are odd. Therefore, $\sigma_2^*(n)$ being the sum of the squares of an even number of odd positive integers, is even. Suppose n is a unitary s-Zumkeller number. Then $\sigma_2^*(n) - n^2$ is even. Since n is odd, $\sigma_2^*(n)$ is odd, which is a contradiction. Hence, there does not exist any odd unitary s-Zumkeller number. □

4. Conclusion

In this paper, we have generalized Zumkeller numbers to include s-Zumkeller numbers and unitary s-Zumkeller numbers, and have established several properties of these numbers. We have demonstrated that if an odd s-Zumkeller number is divisible by 3, it must be greater than 12,442,607,161,209,225. Additionally, we have shown the non-existence of odd unitary s-Zumkeller numbers. Future work could explore the introduction and investigation of graph labeling concepts using the newly defined s-Zumkeller numbers and unitary s-Zumkeller numbers.

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