ON THE MINIMUM CARDINALITY OF MPTQ SETS

Anupriya Shetty
Department of Post Graduate Studies and Research in Mathematics, St. Aloysius, Mangaluru, India
anupriyashetty.207ma500@nitk.edu.in

B. R. Shankar
Department of Mathematical and Computational Sciences, NITK, Surathkal, Mangaluru, Karnataka, India
shankarbr@nitk.edu.in

Received: 1/18/24, Accepted: 10/15/24, Published: 10/25/24

Abstract

Given a finite set $A \subset \mathbb{R}$, we define $A + A = \{a + a' \mid a, a' \in A\}$ and $A - A = \{a - a' \mid a, a' \in A\}$. A set A is said to be an MSTD (More Sums than Differences) set if $|A+A| > |A-A|$. We define $A.A = \{aa' \mid a, a' \in A\}$ and $A/A = \{a/a' \mid a, a' \in A,\ a' \neq 0\}$. Analogous to MSTD sets, H V Chu defines a set $A \subset \mathbb{R} \setminus \{0\}$ to be an MPTQ (More Products than Quotients) set if $|A.A| > |A/A|$. It is known by the exponentiation of MSTD sets that there exist MPTQ sets of cardinality 8. In an attempt to determine the smallest cardinality of an MPTQ set, Chu proved that an MPTQ set of real numbers must have at least 5 elements. In this work, we prove that a set of real numbers with cardinality 5 is not an MPTQ set. So we conclude an MPTQ set of real numbers must contain at least 6 elements. We have identified certain cases of sets with cardinality 6 that are not MPTQ sets. Further, we give an infinite family of MPTQ sets that are not the exponential of an MSTD set.

1. Introduction

Definition 1. Given a finite set $A \subset \mathbb{R}$, we define

$$A + A = \{a + a' \mid a, a' \in A\}; \quad A - A = \{a - a' \mid a, a' \in A\};$$
$$A.A = \{aa' \mid a, a' \in A\}; \quad A/A = \{a/a' \mid a, a' \in A,\ a' \neq 0\}.$$

A set A is said to be an *MSTD (More Sums than Differences) set* [16] if $|A + A| > |A - A|$ and an *MDTS (More Differences than Sums) set* if $|A + A| < |A - A|$.

DOI: 10.5281/zenodo.13992611

Similarly, a set $A \subset \mathbb{R} \setminus \{0\}$ is said to be an *MPTQ (More Products than Quotients) set* if $|A.A| > |A/A|$ and an *MQTP (More Quotients than Products) set* if $|A.A| < |A/A|$. In either case, we say A is *balanced* if the cardinalities are equal.

In recent years, the notion of MSTD sets has received good attention from researchers working in the field of additive number theory. For a history and overview of MSTD sets, we suggest [7, 11, 14, 17, 18], and for explicit constructions of MSTD sets, we refer the reader to [4, 6, 2, 8, 12, 13, 19]. After Nathanson's review of the concept [16], we can find various generalizations, extensions to finite groups, and other settings; see [1, 9, 10, 20] for more details. It was proved by Martin and O'Bryant [12] that as $n \to \infty$, the percentage of MSTD subsets in $\{1, 2, \ldots, n\}$ is bounded below by a positive constant, which Zhao [21] gave to several digits.

As multiplication is commutative and division is not, just like addition and subtraction, we expect that the number of quotients is at least the number of products for a given set. However, MPTQ sets do exist.

Example 1. For the set $A = \{4, 9, 16, 18, 24, 36, 162, 216, 243, 432, 972\}$, $|A.A| = 52 > 51 = |A/A|$. So A is an MPTQ set.

We can also find examples of MPTQ sets consisting of both positive and negative integers.

Example 2. The set $A = \{-1944, -864, -648, -162, -96, 12, 27, 36, 108, 144, 243\}$ is an MPTQ set.

For preliminaries, basic constructions, and probability measures for MPTQ subsets, one may refer to [5]. Though the number of MSTD subsets of $\{1, 2, \ldots, n\}$ grows quickly as n grows, Theorem 1.3 of [5] shows that as $n \to \infty$ the proportion of MPTQ subsets of $\{1, 2, \ldots, n\}$ approaches 0, unlike MSTD sets. This shows that MPTQ sets are rare compared to MSTD sets.

The notion of MPTQ sets is closely linked to that of MSTD sets through the logarithmic transformation and exponentiation of sets. Though there is extensive research happening in the area of MSTD sets and their generalizations, MPTQ sets have not received as much attention as MSTD sets. However, exploring the concept of MPTQ sets can enhance our understanding of MSTD sets and may uncover new connections in the area of additive number theory.

In [15] Nathanson asked, *What is the smallest cardinality of an MSTD set*? To answer this, Hegarty proved the following using Mathematica programming.

Theorem 1 ([8])**.** *There are no MSTD subsets of the integers of size* 7*. Up to linear transformations the only set of size* 8 *is* $\{0, 2, 3, 4, 7, 11, 12, 14\}$.

However, a computer-free proof of the result has not been produced because of the complexity involved. To answer Nathanson's question, H. V. Chu [3] presented a proof that an MSTD set must have a minimum of 7 elements without computers' help.

Question 1.5 of [5] concerns the smallest cardinality of an MPTQ set of real numbers and Chu remarks that answering this question is quite challenging because it requires more memory and computational power for computers to do multiplication and division than addition and subtraction. However, he proved the following.

Theorem 2 ([5])**.** *Let A be an MPTQ set of real numbers. The following claims are true.*

1. *If A contains only positive numbers, then* $|A| \geq 8$.
2. *If A contains negative numbers, then* $|A| \geq 5$.

A property of a set is *affine-invariant* if it remains unchanged under a dilation followed by a translation. The property of being an MSTD set is affine-invariant [16]. So for problems related to the cardinality of MSTD sets of real numbers, it is enough to consider the case of positive reals. In particular, when we work with integers it suffices to take our domain to be the interval of integers $[1, n]$. Restricting ourselves to $[1, n]$ would work even up to rational numbers as any finite set of rationals is affinely equivalent to a set of positive integers. But for questions related to the smallest cardinality of MPTQ sets we have to take into consideration negative numbers as well because the property of being an MPTQ set is not preserved under translation.

For a finite set $A \subseteq \mathbb{R} \setminus \{0\}$ and a nonzero real number r,

$$|(rA).(rA)| = |r^2(A.A)| = |A.A| \text{ and } |(rA)/(rA)| = |A/A|.$$

So A is an MPTQ set if and only if rA is an MPTQ set.

Thus, the property of being an MPTQ set is preserved under dilation. So any set with only negative real numbers can be dilated to a set of positive numbers without change in cardinality. On the other hand, Theorem 2 confirms the minimum cardinality of an MPTQ set of positive reals to be 8. Therefore, we can conclude that there is no MPTQ set of cardinality less than 8 consisting of only negative numbers. So if there exists an MPTQ set of reals with cardinality 5 it must contain at least one positive and one negative number.

In this paper, we shall prove that a set of real numbers with cardinality 5 cannot be an MPTQ set.

2. Useful Results

We write $x \longrightarrow A$ to mean the adjoining of the number x to the set A that yields the set $A \cup \{x\}$. The products and quotients generated due to $x \longrightarrow A$ will be referred to as 'new' elements.

We shall first prove the following lemma.

Lemma 1. *Let A be a finite set of real numbers which is not an MPTQ set. Let $A' = A \cup \{x\}$ where x is real such that $x \longrightarrow A$ gives at most n new products and at least m new quotients. If $n \leq m$, then A' is not an MPTQ set.*

Proof. We have,

$$|A'.A'| \quad \leq \quad |A.A| + n \qquad \text{and} \qquad |A'/A'| \quad \geq \quad |A/A| + m.$$

Now,

$$|A'/A'| - |A'.A'| \quad \geq \quad |A/A| - |A.A| + m - n.$$

As A is not an MPTQ set, we get $|A/A| - |A.A| \geq 0$. Therefore, the above inequality implies that A' is not an MPTQ set. □

Lemma 2. *If $a, b, c,$ and d are distinct positive real numbers such that $ab = cd$, then neither $ad = bc$, nor $ac = bd$, holds. That is, the product of any two of them can be equal to the product of the remaining two numbers in exactly one way.*

Proof. Let $a, b, c,$ and d be distinct positive real numbers with $ab = cd$. Suppose $ad = bc$. Then combining the two equalities we get $a = c$, and $b = d$. A similar argument shows that $ac = bd$ will also lead to a contradiction. We have completed the proof. □

Similarly, we observe the following.

Lemma 3. *It is not possible to find three distinct positive real numbers such that the product of any two of them is always the square of the third number. In fact, among the three equalities, $a^2 = bc, b^2 = ac$, and $c^2 = ab$, at most one of them can be true at a time.*

Proof. Let a, b and c be distinct positive real numbers. Suppose $a^2 = bc$ and $b^2 = ac$. Then we get $a^3 = b^3$, which implies $a = b$. This completes the proof. □

2.1. Sets of Cardinality 5 with Only One Negative Number

Lemma 4. *A set of cardinality 5 with one negative number is not an MPTQ set.*

Proof. Let $A = \{a_1, a_2, a_3, a_4\}$ be a set of positive real numbers and let $A' = A \cup \{-x\}$, where $x > 0$. Note that, the products in $A.A$ and the quotients in A/A are all positive.

- If $x^2 \notin A.A$, then $-x \longrightarrow A$ yields 5 new products,

$$-a_1x, -a_2x, -a_3x, -a_4x, x^2,$$

and 8 new quotients,

$$-\frac{x}{a_1}, -\frac{a_1}{x}, -\frac{x}{a_2}, -\frac{a_2}{x}, -\frac{x}{a_3}, -\frac{a_3}{x}, -\frac{x}{a_4}, -\frac{a_4}{x}.$$

- If $x^2 \in A.A$, then we get 4 new products,

$$-a_1x, -a_2x, -a_3x, -a_4x,$$

and at least 4 new quotients,

$$-\frac{a_1}{x}, -\frac{a_2}{x}, -\frac{a_3}{x}, -\frac{a_4}{x}.$$

Therefore, by Lemma 1 it follows that A' is not an MPTQ set. $\square$

We shall now work with sets having at least two negative numbers. Due to symmetry, it suffices to consider sets with exactly two negative and three positive numbers.

2.2. Sets of Cardinality 5 with Two Negative Numbers

Lemma 5. *A set of cardinality* 5 *with two negative numbers is not an MPTQ set.*

Proof. Let $A = \{a_1, a_2, a_3\}$ be a set of positive real numbers, and let A' denote the set obtained by adjoining two negative numbers to A. We shall divide the proof into three cases depending on the magnitude of the negative numbers appended. Under each of the three cases, we shall list the subcases discussing all possible simultaneous equalities among products that would reduce the number of newly generated quotients.

`Case 1:` $A' = A \cup \{-a_1, -a_2\}$. In this case, the numbers $-a_1, -a_2$ appended to A yield at most 5 new products,

$$-a_1^2, -a_2^2, -a_1a_2, -a_1a_3, -a_2a_3.$$

The new quotients obtained are

$$-1, -\frac{a_1}{a_2}, -\frac{a_1}{a_3}, -\frac{a_2}{a_1}, -\frac{a_2}{a_3}, -\frac{a_3}{a_1} \text{ and } -\frac{a_3}{a_2},$$

of which at least 5 are distinct as $a_1^2 = a_2a_3$ and $a_2^2 = a_1a_3$ cannot be true simultaneously by Lemma 3.

Therefore, by Lemma 1 it follows that A' is not an MPTQ set.

`Case 2:` $A' = A \cup \{-a_1, -x\}$, where $x > 0$ and $x \notin A$. In this case, $-a_1, -x \longrightarrow A$ give at most 8 new products namely,

$$-a_1^2, -a_1a_2, -a_1a_3, -a_1x, -a_2x, -a_3x, x^2, a_1x.$$

We shall list all newly produced 'negative' quotients.

Due to $-a_1 \longrightarrow A$, we get the quotients

$$-1, -\frac{a_1}{a_2}, -\frac{a_2}{a_1}, -\frac{a_1}{a_3}, -\frac{a_3}{a_1},$$

and due to $-x \longrightarrow A$, we get the quotients

$$-\frac{x}{a_1}, -\frac{a_1}{x}, -\frac{x}{a_2}, -\frac{a_2}{x}, -\frac{x}{a_3}, -\frac{a_3}{x}.$$

If all quotients listed above are distinct, then we get 11 new quotients in A'/A'. So A' will have more quotients than products. Otherwise, we can identify the following 6 subcases based on the possible equalities among the above-listed quotients with the help of Lemma 2 and Lemma 3.

`Subcase 2.1:` $x/a_1 = a_2/x,\ a_1/a_2 = a_3/a_1,\ x/a_3 = a_3/a_1$. Accordingly, we have the following equalities:

$$x^2 = a_1a_2,\ a_1^2 = a_2a_3,\ a_3^2 = a_1x,\ \text{and } a_1a_3 = a_2x.$$

In this case, we get 5 new products,

$$-a_1^2, -a_1a_2, -a_1a_3, -a_1x, -a_3x,$$

and 5 new quotients,

$$-1, -\frac{a_1}{a_2}, -\frac{a_2}{a_1}, -\frac{x}{a_1}, -\frac{a_1}{x}.$$

`Subcase 2.2:` $x/a_1 = a_3/x,\ a_1/a_2 = a_3/a_1,\ x/a_2 = a_2/a_1$. These result in the equalities:

$$x^2 = a_1a_3,\ a_1^2 = a_2a_3,\ a_2^2 = a_1x,\ \text{and } a_1a_2 = a_3x,\ \text{respectively.}$$

In this case, we get 5 new products,

$$-a_1^2, -a_1a_2, -a_1a_3, -a_1x, -a_2x,$$

and 5 new quotients,

$$-1, -\frac{a_1}{a_2}, -\frac{a_2}{a_1}, -\frac{x}{a_1}, -\frac{a_1}{x}.$$

`Subcase 2.3:` $x/a_1 = a_2/x,\ x/a_1 = a_1/a_3,\ x/a_3 = a_1/a_2$. The resulting equalities are

$$x^2 = a_1a_2,\ a_1^2 = a_3x,\ \text{and } a_1a_3 = a_2x,\ \text{respectively.}$$

In this case, we get 4 new products,

$$-a_1^2, -a_1a_2, -a_1a_3, -a_1x,$$

and 5 new quotients,

$$-1, -\frac{a_1}{a_2}, -\frac{a_2}{a_1}, -\frac{a_3}{a_1}, -\frac{a_1}{a_3}.$$

`Subcase 2.4:` $x/a_1 = a_3/x,\ x/a_1 = a_1/a_2,\ x/a_2 = a_1/a_3$. The resulting equalities are

$$x^2 = a_1a_3,\ a_1^2 = a_2x,\ \text{and } a_1a_2 = a_3x.$$

In this case, we get 4 new products,

$$-a_1^2, -a_1a_2, -a_1a_3, -a_1x,$$

and 5 new quotients,

$$-1, -\frac{a_1}{a_2}, -\frac{a_2}{a_1}, -\frac{a_3}{a_1}, -\frac{a_1}{a_3}.$$

`Subcase 2.5:` $x/a_2 = a_3/x,\ x/a_1 = a_1/a_2,\ x/a_3 = a_3/a_1$. The resulting equalities with products are

$$x^2 = a_2a_3,\ a_1^2 = a_2x,\ a_3^2 = a_1x,\ \text{and } a_1a_2 = a_3x,\ \text{respectively.}$$

In this case, we get 4 new products,

$$-a_1^2, -a_1a_2, -a_1a_3, -a_1x,$$

and 5 new quotients,

$$-1, -\frac{a_1}{a_2}, -\frac{a_2}{a_1}, -\frac{a_3}{a_1}, -\frac{a_1}{a_3}.$$

`Subcase 2.6:` $x/a_3 = a_2/x,\ x/a_1 = a_1/a_3,\ x/a_2 = a_2/a_1$. The resulting equalities are:

$$x^2 = a_2a_3,\ a_1^2 = a_3x,\ a_2^2 = a_1x,\ \text{and } a_1a_3 = a_2x,\ \text{respectively.}$$

In this case, we get 4 new products,

$$-a_1^2, -a_1a_2, -a_1a_3, -a_1x,$$

and 5 new quotients,

$$-1, -\frac{a_1}{a_2}, -\frac{a_2}{a_1}, -\frac{a_3}{a_1}, -\frac{a_1}{a_3}.$$

Thus, in any case, the number of new products obtained will not exceed the number of new quotients. Therefore, by Lemma 1 it follows that A' is not an MPTQ set.

Remark 1. If we interchange the roles of a_2 and a_3 in Subcase 2.1 we get the Subcase 2.2. Similarly, we can get Subcase 2.4 from Subcase 2.3 and Subcase 2.6 from Subcase 2.5.

`Case 3:` $A' = A \cup \{-x, -y\}$, where $x, y > 0$ and $x, y \notin A$. With $-x, -y \longrightarrow A$ we get at most 9 new products,

$$-a_1x,\ -a_2x,\ -a_3x,\ -a_1y,\ -a_2y,\ -a_3y,\ x^2,\ y^2,\ xy,$$

and at most 14 new quotients,

$$-\frac{a_1}{x},\ -\frac{x}{a_1},\ -\frac{a_2}{x},\ -\frac{x}{a_2},\ -\frac{a_3}{x},\ -\frac{x}{a_3},$$
$$-\frac{a_1}{y},\ -\frac{y}{a_1},\ -\frac{a_2}{y},\ -\frac{y}{a_2},\ -\frac{a_3}{y},\ -\frac{y}{a_3},\ \frac{x}{y},\ \frac{y}{x}.$$

If all the quotients listed above are distinct, then A' cannot be an MPTQ set.

Now we shall discuss various cases that arise due to the possible equalities among the above listed quotients.

`Subcase 3.1:` $x/a_1 = a_2/x,\ a_1/x = y/a_1,\ x/a_2 = y/a_1,\ y/a_2 = a_3/y$. These will result in the following equalities:

$$x^2 = a_1a_2,\ a_1^2 = xy,\ a_1x = a_2y,\ \text{and}\ \ y^2 = a_2a_3,\ \text{respectively.}$$

In this case, we get 5 new products,

$$-a_1x, -a_2x, -a_3x, -a_1y, -a_3y,$$

and 6 new quotients,

$$-\frac{a_1}{x}, -\frac{x}{a_1}, -\frac{a_3}{x}, -\frac{x}{a_3}, -\frac{a_2}{y}, -\frac{y}{a_2}.$$

`Subcase 3.2:` $x/a_1 = a_2/x,\ a_1/x = y/a_1,\ x/a_2 = y/a_1,\ x/a_1 = y/a_3,\ y/a_1 = a_3/y$. These will result in the following equalities: $x^2 = a_1a_2,\ a_1^2 = xy,\ a_1x = a_2y,\ a_3x = a_1y,$ and $y^2 = a_1a_3,$ respectively. In this case, we get 4 new products,

$$-a_1x, -a_2x, -a_3x, -a_3y,$$

and 4 new quotients,

$$-\frac{a_1}{x}, -\frac{x}{a_1}, -\frac{a_3}{x}, -\frac{x}{a_3}.$$

`Subcase 3.3:` $x/a_1 = a_2/x,\ a_1/x = y/a_1,\ x/a_2 = y/a_1,\ x/a_3 = y/a_2$. These will result in the following equalities:

$$x^2 = a_1a_2,\ a_1^2 = xy,\ a_1x = a_2y,\ \text{and}\ a_2x = a_3y,\ \text{respectively.}$$

In this case, we get 5 new products,

$$-a_1x, -a_2x, -a_3x, -a_1y, y^2,$$

and 6 new quotients,

$$-\frac{a_1}{x}, -\frac{x}{a_1}, -\frac{a_3}{x}, -\frac{x}{a_3}, -\frac{a_3}{y}, -\frac{y}{a_3}.$$

`Subcase 3.4:` $x/a_1 = a_2/x,\ a_2/x = y/a_2,\ x/a_2 = y/a_3,\ x/a_3 = y/a_2,\ y/a_2 = a_3/y$. These will result in the equations, $x^2 = a_1a_2,\ a_2^2 = xy,\ a_2y = a_3x,\ a_2x = a_3y$, and $y^2 = a_2a_3$, respectively. In this case, we get 4 new products,

$$-a_1x, -a_2x, -a_3x, -a_3y,$$

and 4 new quotients,

$$-\frac{a_1}{x}, -\frac{x}{a_1}, -\frac{a_3}{x}, -\frac{x}{a_3}.$$

`Subcase 3.5:` $x/a_1 = a_2/x,\ a_3/x = y/a_3,\ x/a_3 = y/a_2,\ y/a_2 = a_3/y$. These will result in the following equalities:

$$x^2 = a_1a_2,\ a_3^2 = xy,\ a_3y = a_2x, \text{ and } y^2 = a_2a_3, \text{ respectively.}$$

In this case, we get 5 new products,

$$-a_1x, -a_2x, -a_3x, -a_1y, -a_2y,$$

and 6 new quotients,

$$-\frac{a_1}{x}, -\frac{x}{a_1}, -\frac{a_3}{x}, -\frac{x}{a_3},\ -\frac{a_1}{y},\ -\frac{y}{a_1}.$$

`Subcase 3.6:` $x/a_1 = a_2/x,\ a_3/x = y/a_3,\ x/a_3 = y/a_1,\ y/a_1 = a_3/y$. These will result in the following equalities:

$$x^2 = a_1a_2,\ a_3^2 = xy,\ a_3y = a_1x, \text{ and } y^2 = a_1a_3, \text{ respectively.}$$

In this case, we get 5 new products,

$$-a_1x, -a_2x, -a_3x, -a_1y, -a_2y,$$

and 6 new quotients,

$$-\frac{a_1}{x}, -\frac{x}{a_1}, -\frac{a_3}{x}, -\frac{x}{a_3},\ -\frac{a_2}{y},\ -\frac{y}{a_2}.$$

`Subcase 3.7:` $x/a_1 = a_2/x,\ a_2/x = y/a_3,\ x/a_1 = y/a_3,\ y/a_1 = a_3/y,\ x/a_2 = y/a_1$. These will result in the following equalities: $x^2 = a_1a_2,\ a_2a_3 = xy,\ a_3x = a_1y,\ y^2 = a_1a_3$, and $a_1x = a_2y$, respectively. In this case, we get 4 new products,

$$-a_1x, -a_2x, -a_3x, -a_3y,$$

and 4 new quotients,

$$-\frac{a_1}{x}, -\frac{x}{a_1}, -\frac{a_3}{x}, -\frac{x}{a_3}.$$

`Subcase 3.8:` $x/a_1 = a_2/x,\ a_1/x = y/a_3,\ x/a_2 = y/a_3,\ y/a_2 = a_3/y,\ x/a_1 = y/a_2$. These will result in the following equalities: $x^2 = a_1a_2,\ a_1a_3 = xy,\ a_3x = a_2y,\ y^2 = a_2a_3,$ and $a_1y = a_2x,$ respectively. In this case, we get 4 new products,

$$-a_1x, -a_2x, -a_3x, -a_3y,$$

and 4 new quotients,

$$-\frac{a_1}{x}, -\frac{x}{a_1}, -\frac{a_3}{x}, -\frac{x}{a_3}.$$

Thus, in all the cases the number of new products due to $-x$ and $-y$ is at most the number of new quotients. Hence A' cannot be an MPTQ set. □

Remark 2. The cases when $x^2 = a_1a_3$ or $x^2 = a_2a_3$ are not discussed in Case 3 as the roles of a_1, a_2, a_3 are interchangeable. Similarly, the cases with $y^2 = a_ia_j$, where $i \neq j, 1 \leq i, j \leq 3$ can be discussed just by replacing x with y in each of the above subcases.

2.3. Main Theorem

Theorem 3. *There is no MPTQ set of real numbers with cardinality* 5.

Proof. We have noted earlier, that if there exists an MPTQ set of real numbers with cardinality 5 then it must contain at least one positive and one negative number. Further, the detailed analysis carried out in Subsections 2.1 and 2.2, shows that the number of new products due to the negative numbers adjoined is at most the number of newly generated quotients. So applying the Lemma 1 we concluded A' is not an MPTQ set in either case. We have completed the proof. □

Remark 3. One can easily construct balanced sets of the form discussed in Subsection 5.

Example 3. The set $A = \{-16, -4, 4, 8, 16\}$ is a balanced set with,

$$A.A = \{-256, -128, -64, -32, -16, 16, 32, 64, 128, 256\},$$

and

$$A/A = \{-4, -2, -1, -0.5, -0.25, 0.25, 0.5, 1, 2, 4\}.$$

Remark 4. The equalities listed under subcases in Subsection 5 provide us with a method of constructing balanced sets of real numbers with cardinality 5 consisting of both positive and negative numbers with distinct magnitudes.

Example 4. Using the Subcase 3.4, we get a balanced set,
$A = \{-54, -24, 16, 36, 81\}$ with,

$$A.A = \{-4374, -1944, -864, -384, 256, 576, 1296, 2916, 6561\},$$

and,

$$A/A = \{-3.375, -1.5, -0.6667, -0.29623, 0.1975, 0.4444, 1, 2.25, 5.0625\}.$$

We shall now consider certain cases of sets with a cardinality of 6 and show that they do not form MPTQ sets.

3. Sets of Cardinality 6

As there is no MPTQ set of positive real numbers with cardinality 6 and the property of being an MPTQ set is preserved under dilation, it follows that there is no MPTQ set of cardinality 6 consisting of only negative real numbers. The case when A' contains exactly one negative number can be easily handled as in Subsection 2.1 above. Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ be a set of positive real numbers and let $A' = A \cup \{-x\}$, where $x > 0$.

- If $x^2 \notin A.A$, then $-x \longrightarrow A$ yields 6 new products and 10 new quotients.
- If $x^2 \in A.A$, then we get 5 new products and at least 5 new quotients.

Therefore, by Lemma 1 it follows that A' is not an MPTQ set.

This will also help us conclude that A' with four negative numbers cannot be an MPTQ set. So we need to analyze the cases when A' contains two or three negative numbers.

We shall first prove the following.

Proposition 1. *A finite set A of positive real numbers is an MPTQ set if and only if $A \cup (-A)$ is an MPTQ set.*

Proof. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of positive real numbers. Let $B = A \cup (-A)$. Due to the adjoining of the elements of $-A$ to A, the new products obtained are $-a_i a_j$, where $1 \le i, j \le n$, which is precisely the set $-(A.A)$ and the newly obtained quotients are $-a_i/a_j$, where $1 \le i, j \le n$, which constitutes the set $-(A/A)$.

We know that $|-(A.A)| = |A.A|$, and $|-(A/A)| = |A/A|$. Therefore, we get

$$|B.B| = 2|A.A|, \text{ and } |B/B| = 2|A/A|.$$

Hence A is an MPTQ set if and only if B is an MPTQ set. □

We shall now discuss the following two cases for a set A' of cardinality 6 containing two or three negative numbers such that $A' \subseteq A \cup (-A)$, where A is the set of all positive numbers in A'.

Case 1: $A' = A \cup \{-a_1, -a_2\}$, where $A = \{a_1, a_2, a_3, a_4\}$ is a set of positive real numbers. Being a set with 4 positive real numbers, A cannot be an MPTQ set. Due to $-a_1, -a_2 \longrightarrow A$ the newly produced products are

$$-a_1^2, -a_2^2, -a_1a_2, -a_1a_3, -a_1a_4, -a_2a_3, -a_2a_4,$$

and the newly produced quotients are

$$-1, -\frac{a_1}{a_2}, -\frac{a_1}{a_3}, -\frac{a_1}{a_4}, -\frac{a_2}{a_1}, -\frac{a_3}{a_1}, -\frac{a_4}{a_1} - \frac{a_2}{a_3}, -\frac{a_3}{a_2}, -\frac{a_2}{a_4}, \text{ and } -\frac{a_4}{a_2}.$$

If all quotients listed here are distinct, then A' cannot be an MPTQ set.

We shall identify the following six subcases depending on the possible equalities among the quotients.

Subcase 1.1: $a_1/a_2 = a_3/a_1$, and $a_1/a_2 = a_4/a_3$. In this case, we get 5 new products,

$$-a_1^2, -a_2^2, -a_1a_2, -a_1a_3, -a_1a_4,$$

and 7 new quotients,

$$-1, -\frac{a_1}{a_2}, -\frac{a_1}{a_3}, -\frac{a_1}{a_4}, -\frac{a_4}{a_1}, -\frac{a_2}{a_4}, \text{ and } -\frac{a_4}{a_2}.$$

Subcase 1.2: $a_1/a_2 = a_3/a_1$, and $a_1/a_2 = a_2/a_4$. Here we get 5 new products,

$$-a_1^2, -a_1a_2, -a_1a_3, -a_1a_4, -a_2a_4,$$

and 7 new quotients,

$$-1, -\frac{a_1}{a_2}, -\frac{a_1}{a_3}, -\frac{a_1}{a_4}, -\frac{a_4}{a_1}, -\frac{a_2}{a_3}, \text{ and } -\frac{a_3}{a_2}.$$

Subcase 1.3: $a_1/a_2 = a_4/a_3$, and $a_1/a_2 = a_2/a_4$. In this case, we get 5 new products,

$$-a_1^2, -a_1a_2, -a_1a_3, -a_1a_4, -a_2a_3,$$

and 7 new quotients,

$$-1, -\frac{a_1}{a_2}, -\frac{a_1}{a_3}, -\frac{a_1}{a_4}, -\frac{a_4}{a_1}, -\frac{a_2}{a_1}, \text{ and } -\frac{a_3}{a_1}.$$

Subcase 1.4: $a_1/a_2 = a_4/a_1$, and $a_1/a_2 = a_3/a_4$.

`Subcase 1.5:` $a_1/a_2 = a_4/a_1$, and $a_1/a_2 = a_2/a_3$.

`Subcase 1.6:` $a_1/a_2 = a_3/a_4$, and $a_1/a_2 = a_2/a_3$.

The Subcases 1.4, 1.5, and 1.6, can be obtained just by interchanging the roles of a_3 and a_4 in each of the Subcases 1.1, 1.2, and 1.3, respectively. So each of them yields exactly 5 new products and 7 new quotients. Therefore, we conclude that A' is not an MPTQ set in this case.

`Case 2:` $A' = A \cup (-A)$, where $A = \{a_1, a_2, a_3\}$ is a set of positive real numbers. By Proposition 1 it follows that A' is not an MPTQ set. In the above two cases, we have shown that a set A' of cardinality 6, obtained by appending negative numbers to a set A of positive real numbers such that $A' \subseteq A \cup (-A)$, cannot be an MPTQ set. So, if there exists an MPTQ set A' of reals with cardinality 6, then it must contain at least 2 negative numbers and $A' \nsubseteq A \cup (-A)$, where A denotes the set of positive numbers in A'.

4. An Infinite Family of MPTQ Sets

The idea used in Proposition 1 has helped us formulate the following result.

Theorem 4. *A set A of integers is an MPTQ set if and only if $A \cup pA$ is an MPTQ set for any prime p which is coprime to the numbers in A.*

Proof. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of integers. Consider a prime number p with $(p, a_i) = 1$ for each $i, 1 \leq i \leq n$. Let $B = A \cup pA$.

Due to the adjoining of the elements of pA to A, the newly obtained products are as follows:

- pa_ia_j, $1 \leq i, j \leq n$, which form the set $pA.A$
- $p^2a_ia_j$, $1 \leq i, j \leq n$, which constitute the set $pA.pA$.

Similarly, the newly obtained quotients are:

- pa_i/a_j, $1 \leq i, j \leq n$, which constitute the set pA/A
- a_i/pa_j, $1 \leq i, j \leq n$, which yield the set A/pA.

We know that $|pA.A| = |pA.pA| = |A.A|$, and $|pA/A| = |A/pA| = |A/A|$.
Therefore, we get

$$|B.B| = 3|A.A|, \text{ and } |B/B| = 3|A/A|.$$

Hence A is an MPTQ set if and only if B is an MPTQ set. □

Remark 5. If A is an MPTQ set of integers, then so is pA. So using Theorem 4 we can say that there exist MPTQ sets whose union is also an MPTQ set. We know that for any positive real number $r \neq 1$, the r-log transformation of an MPTQ set results in an MSTD set. So taking the log transformation of A and pA we can conclude that there exist MSTD sets whose union is also an MSTD set.

Theorem 4 also allows us to generate infinitely many MPTQ sets starting with one. If A is an MPTQ set of integers with cardinality n, then we can generate an infinite family $\{A_k\}_{k=1}^{\infty}$ of MPTQ sets, each of cardinality kn, using prime numbers that are coprime to the elements in A, as demonstrated in the following algorithm.

1. Choose an MPTQ set A of integers with cardinality n.
2. Let $A_1 = A$.
3. Let p_1 be the first prime (with respect to the natural ordering of numbers) that is coprime to every element in A.
4. Let $A_2 = A \cup p_1 A$. Then A_2 is an MPTQ set of cardinality $2n$.
5. If p_k denotes the k^{th} prime number which is coprime to every number in A, then $A_k = A \cup p_1 A \ldots \cup p_{k-1} A$ is an MPTQ set of cardinality kn.

Thus, from an MPTQ set A of cardinality n, we have generated an infinite family $\{A_k\}_{k=1}^{\infty}$ of MPTQ sets, where each A_k is a union of k MPTQ sets and $|A_k| = kn$.

5. Conclusion

Using the method of case analysis we have proved that there is no MPTQ set of real numbers with cardinality 5. Therefore, an MPTQ set of real numbers must have at least 6 elements. We know that the minimum cardinality of an MSTD set of real numbers is 8. Then the natural question one can ask is: *does there exist an MPTQ set of real numbers with* 6 *or* 7 *elements?*

Using a method similar to the one used for a set of cardinality 5, we have shown that a set A' of cardinality 6 with one or four negative numbers is not an MPTQ set. Also, a set A' of cardinality 6 consisting of two or three negative numbers with $A' \subseteq A \cup (-A)$, where A is the set of positive numbers in A', cannot be an MPTQ set. But handling the cases when A' contains two or three negative numbers such that $A' \nsubseteq A \cup (-A)$ requires careful consideration due to the tedious nature of the subcases involved. So using the above technique in such cases does not look feasible. One may have to adopt a new approach to deal with such sets of cardinality 6.

In the literature, the only technique used to generate MPTQ sets is the concept of exponentiation of MSTD sets. But in this work, we were able to present a method of generating an infinite family of MPTQ sets without using the exponentiation of MSTD sets.

Acknowledgement. The authors would like to thank the anonymous referee and the editor for a careful reading of this article.

References

[1] M. Asada, S. Manski, S. J. Miller, and H. Suh, Fringe pairs in generalized MSTD sets, *Int. J. Number Theory* **13** (2017), 2653-2675.

[2] H. V. Chu, N. McNew, S. J. Miller, V. Xu, and S. Zhang, When sets can and cannot have sum-dominant subsets, *J. Integer Seq.* **21** (18) (2018), Art. 18.8.2, 16 pp.

[3] H. V. Chu, When sets are not sum-dominant, *J. Integer Seq.* **22** (3) (2019), Art. 19.3.7, 16 pp.

[4] H. V. Chu, N. Luntzlara, S. J. Miller, and L. Shao, Generalizations of a curious family of MSTD sets hidden by interior blocks, *Integers* **20** (A) (2020), #A5.

[5] H. V. Chu, On sets with more products than quotients, *Rocky Mt. J. Math.* **50** (2020), 499-512.

[6] H. V. Chu, The union of two arithmetic progressions with the same common difference is not sum-dominant, *Integers* **20** (2020), #A87.

[7] G. A. Freiman and V. P. Pigarev, The relation between the invariants R and T (Russian), In: Number Theoretic Studies in the Markov Spectrum and in the Structural Theory of Set Addition, Kalinin. Gos. Univ., Moscow, 1973, 172-174.

[8] P. V. Hegarty, Some explicit constructions of sets with more sums than differences, *Acta Arith.* **130** (2007), 61-77.

[9] G. Iyer, O. Lazarev, S. J. Miller, and L. Zhang, Generalized more sums than differences sets, *J. Number Theory* **132** (2012), 1054-1073.

[10] G. Iyer, O. Lazarev, S. J. Miller, and L. Zhang, Finding and counting MSTD sets, In: Combinatorial and Additive number theory— CANT 2011 and 2012, Springer Proc. Math. Stat., vol. 101, Springer, New York, 2014, pp. 79-98.

[11] J. Marica, On a conjecture of Conway, *Canad. Math. Bull.* **12** (1969), 233-234.

[12] G. Martin and K. O'Bryant, Many sets have more sums than differences, in *Additive Combinatorics*, **43**, CRM Proc. Lecture Notes, Amer. Math. Soc., 2007, pp. 287-305.

[13] S. J. Miller and D. Scheinerman, Explicit constructions of infinite families of MSTD sets, in *Additive Number Theory*, Springer, New York, 2010, 229-248.

[14] M. B. Nathanson, *Additive Number Theory The Classical Bases*, Springer Science & Business Media, 1996.

[15] M. B. Nathanson, Problems in Additive Number Theory I, in *Additive Combinatorics*, **43**, CRM Proc. Lecture Notes, Amer. Math. Soc., Providence, RI, 2007, 263-270.

[16] M. B. Nathanson, Sets with more sums than differences, *Integers* **7** (1) (2007), #A5.

[17] I. Z. Ruzsa, On the cardinality of $A+A$ and A-A, In *Combinatorics, (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. II, Coll. Math. Soc. J. Bolyai.,* **18**, 1978, pp. 933-938.

[18] I. Z. Ruzsa, On the number of sums and differences. *Acta Math. Sci. Hungar.* **59** (1992), 439-447.

[19] Y. Zhao, Constructing MSTD sets using bidirectional ballot sequences, *J. Number Theory* **130** (2010), 1212-1220.

[20] Y. Zhao, Counting MSTD sets in finite abelian groups, *J. Number Theory* **130** (2010), 2308-2322.

[21] Y. Zhao, Sets characterized by missing sums and differences, *J. Number Theory* **131** (2011), 2107-2134.