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CARMICHAEL NUMBERS WITH A PRIME NUMBER OF PRIME FACTORS

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Abstract

Under the assumption of Heath-Brown's conjecture on the first prime in an arithmetic progression, we prove that there are infinitely many Carmichael numbers n such that the number of prime factors of n is prime.

1. Introduction

Recall that a Carmichael number is a composite number n for which

$$a^n \equiv a \pmod{n}$$

for every $a \in \mathbb{Z}$.

Although the set of Carmichael numbers was proven to be infinite in 1994 in a paper by Alford, Granville, and Pomerance [2], there are still many open conjectures about Carmichael numbers. Chief among those conjectures is one about the number of prime factors of a Carmichael number.

Conjecture 1. For any fixed $R \in \mathbb{N}$ with $R \geq 3$, there exist infinitely many Carmichael numbers with exactly R prime factors.

In fact, specific conjectures [8] have been made about the number of Carmichael numbers up to x with specific numbers of prime factors[1].

Conjecture 2. (Granville-Pomerance, 1999) For any fixed $R \in \mathbb{N}$ with $R \geq 3$, let $C_R(x)$ denote the number of Carmichael numbers up to x with exactly R prime factors. Then

$$C_R(x) = x^{\frac{1}{R}+o(1)}.$$

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[1] Interestingly, this behavior does not seem to hold if R is large relative to x; Granville and Pomerance conjectured that if $R = \log^{\nu+o(1)} x$ where $0 < \nu < 1$ then there are $x^{\nu+o(1)}$ Carmichael numbers up to x with exactly R prime factors once x is sufficiently large (see Conjecture 2a of [8]).

It is also known that there exist Carmichael numbers with R prime factors where R is any number between 3 and 10,333,229,505 [1].

Unfortunately, little has been proven unconditionally about the number of Carmichael numbers with given numbers of prime factors. As such, we have generally turned to conditional results involving the assumption of rather strong conjectures. In this paper, we push those conditional results further, using a conjecture of Heath-Brown that has recently been applied to Carmichael numbers in a couple of different papers.

1.1. Conditional Results

In 1939, J. Chernick [5] noted that under the assumption of Dickson's prime k-tuples conjecture, one could prove that there are infinitely many Carmichael numbers with exactly R prime factors for any $R \geq 3$. This insight came from the fact that one could construct tuples, such as the triple $6k+1$, $12k+1$, and $18k+1$, where any occurrence of all elements of the tuple being prime simultaneously would lead to a Carmichael number with exactly that many prime factors.

In papers [14] and [15], the current author was able to weaken the requirement of Dickson's full conjecture in proving the infinitude of Carmichael numbers with bounded numbers of prime factors; however, this weakened version of the theorem no longer allows us to prove Carmichael numbers with a specific number of prime factors.

In this paper, we tackle a slightly different question; rather than asking whether we can bound the number of prime factors, we wish to determine whether one can ensure that the number of prime factors is in a specific set. To do so, however, we will need to invoke a conjecture of Heath-Brown's that has often been used in papers about Carmichael numbers:

Conjecture 3. (Heath-Brown, 1978): For any $(c, d) = 1$, the smallest prime that is congruent to $c \pmod{d}$ is $\ll d \log^2 d$.

This conjecture has been applied to Carmichael numbers several times recently, including [7], [3], and [17]; in some cases, it has been used to prove results that later went on to be proven unconditionally [10], [18], [16]. Here, we show that the assumption of this conjecture allows us to prove a statement about Carmichael numbers with prime numbers of prime factors. As is standard, we will let $\Omega(n)$ denote the number of prime factors of n.

Main Theorem 1. *If Heath-Brown's conjecture is true then there are infinitely many Carmichael numbers n for which $\Omega(n)$ is prime.*

We note that the conjecture we use here is actually quite a bit stronger than necessary; it would actually be sufficient to replace the $d \log^2 d$ with something like $de^{\sqrt{\log d}}$. However, since we are not computing anything particularly sharply in this paper, and since even this weakened conjecture is nowhere close to being proven, we have decided to work instead with the more widely recognized conjecture since it makes for cleaner exposition.

2. Introduction: Outline

Any proof involving Carmichael numbers inevitably begins with Korselt's criterion, which Korselt discovered in 1899 [9].

Korselt's Criterion. A positive composite integer n is a Carmichael number if and only if n is squarefree and $(p-1) \mid (n-1)$ for every prime $p \mid n$.

In order to prove Main Theorem 1, we begin by following the methods commonly used to construct an infinitude of Carmichael numbers as were first laid out in [2]. First, we find an L such that L has a large number of prime factors q where each $q-1$ is relatively smooth. Having done this, we then prove that there must exist a k such that there are many primes $p = dk+1$ with $d \mid L$. From here, our goal will be to find a subset of these primes $p_1, \cdots, p_r$ such that the product $n = p_1 \cdots p_r$ is 1 modulo L (and obviously 1 modulo k); since we will then have $p-1 \mid Lk \mid n-1$ for each prime $p \mid n$, this product is then Carmichael number.

Aiding us in this quest are two combinatorial results that appear in [2], although these results are simply an application of the work of van Emde Boas and Kruyswijk [6] and Meshulam [11]. For an abelian group G, let $n(G)$ be the smallest number such that any sequence of length $n(G)$ must contain a subsequence whose product is the identity in G. We can then state the following theorems, which are Theorem 1.1 and Proposition 1.2 in [2].

Theorem 1 (Alford, Granville, and Pomerance [2])**.** *If G is a finite abelian group and m is the maximal order of an element in G, then $n(G) \leq m\,(1+\log(|G|/m))$.*

Proposition 1 (Alford, Granville, and Pomerance [2])**.** *Let G be an abelian group, and let $r > t > n(G)$ be integers. Then any sequence of r elements in G must contain $\binom{r}{t} / \binom{r}{n(G)}$ distinct subsequences of length at most t and at least $t-n(G)$ whose product is the identity.*

By the pigeonhole principle, there must then be an h with $t \geq h \geq t-n(G)$ such that there are many sequences of length h whose product is the identity. If we took g distinct such sequences of length h, the product of these g sequences would then yield a sequence of length gh whose product is also the identity. We can then use Heath-Brown's conjecture to append one more prime onto these sequences, giving us sequences of length $gh+1$ whose product is the identity as well. We know that there must exist a relatively small g for which $gh+1$ is prime ($g \ll \log^2 h$ under

Heath-Brown's conjecture, $g \ll h^{4.2}$ unconditionally by [19]); thus, we can use this construction to give a Carmichael number with a prime number of prime factors.

3. Constructing an L

The first section follows roughly the same blueprint as [2] and subsequent Carmichael papers. Let $1 < \theta < 2$, and let $P(q-1)$ denote the size of the largest prime divisor of $q-1$. For some choice of y, let us define the set $\mathcal{Q}$ by

$$\mathcal{Q} = \{q \text{ prime} : \frac{y^\theta}{\log y} \leq q \leq y^\theta,\ P(q-1) \leq y\}.$$

We cite the following lemma, the proof of which appears in many places including Lemma 5.1 of [18] and Lemma 2.1 of [16].

Lemma 1. *Let $\theta \in (1,2)$. For $\mathcal{Q}$ as above, there exist constants $\gamma = \gamma_\theta$ and $Y = Y_\theta$ such that*

$$|\mathcal{Q}| \geq \gamma \frac{y^\theta}{\log(y^\theta)}$$

if $y > Y$.

From this, we let

$$L' = \prod_{q \in \mathcal{Q}} q.$$

Let $B = \frac{5}{12}$, and let $\pi(x, q, a)$ denote the number of primes $p \leq x$ with $p \equiv a \pmod{q}$. For a given x, we know that for any $d \mid L'$ with $d \leq x^B$,

$$\pi(x, d, 1) \geq \frac{\pi(x)}{2\phi(d)} \tag{1}$$

as long as d does not have a divisor in some small exceptional set $\mathcal{D}_B(x)$ (see e.g. Theorem 2.1 of [2]). Here, it is known that $|\mathcal{D}_B(x)|$ is bounded by some constant D_B determined only by B. So for each $d \in \mathcal{D}_B(x)$, we choose a prime divisor $q \mid d$, and we let $\mathcal{P}_B(x)$ be the collection of such divisors. Then we can define

$$L = \prod_{q \in \mathcal{Q}, q \notin \mathcal{P}_B(x)} q.$$

With this alteration, we can now ensure that any divisor $d \mid L$ with $d \leq x^B$ must satisfy (1).

4. Bounds for Primes in Arithmetic Progressions

Next, define

$$\mathcal{P}_k = \{p = dk + 1 : p \text{ prime}, d \mid L, (k, L) = 1\}.$$

Our goal in this section will be to prove that, for some value of k, $\mathcal{P}_k$ is relatively large. For this particular choice of k, $\mathcal{P}_k$ will then comprise the prime factors of our Carmichael numbers. This section follows the standard framework laid in Section 4 of [2].

First, we know by Montgomery and Vaughan's explicit version of the Brun-Titchmarsh theorem (i.e. Theorem 2 of [12]) that if $z > h^2$ then

$$\pi(z, h, 1) \leq \frac{4z}{\phi(h) \log z}. \tag{2}$$

By our choice of L, we can assume that the sum

$$\sum_{\substack{q \mid L \\ q \text{ prime}}} \frac{1}{q-1} \leq \frac{1}{32}. \tag{3}$$

Moreover, if $d < x^B$ then

$$(1 - B) \log x \leq \log \left(dx^{1-B}\right) \leq 2(1 - B) \log x. \tag{4}$$

Putting (1)-(4) together, we find that if $d < x^B$ and $d \mid L$, then the number of primes $p \leq dx^{1-B}$ with $p \equiv 1 \pmod d$ and $(\frac{p-1}{d}, L) = 1$ is

$$\begin{aligned}
&\geq \pi(dx^{1-B}, d, 1) - \sum_{\substack{q \mid L \\ q \text{ prime}}} \pi(dx^{1-B}, dq, 1) \\
&\geq \frac{dx^{1-B}}{4(1-B)\phi(d)\log x} - \sum_{\substack{q \mid L \\ q \text{ prime}}} \frac{4dx^{1-B}}{(1-B)\phi(dq)\log x} \\
&\geq \frac{dx^{1-B}}{4(1-B)\phi(d)\log x} - \frac{dx^{1-B}}{8(1-B)\phi(d)\log x} \\
&\geq \frac{x^{1-B}}{8(1-B)\log x}.
\end{aligned}$$

Hence, by Lemma 3.1,

$$\begin{aligned}
\sum_{k \leq x^{1-B}} |\mathcal{P}_k| &\geq \sum_{d \mid L, d \leq x^B} \#\left\{p \leq dx^{1-B} : p \equiv 1 \pmod d, \left(\frac{p-1}{d}, L\right) = 1\right\} \\
&\geq \sum_{d \mid L, d \leq x^B} \frac{x^{1-B}}{8(1-B)\log x}
\end{aligned}$$

As we are under no imperative to make x or k small, we can simply take $x^B = L$ and find that

$$\sum_{k \le x^{1-B}} |\mathcal{P}_k| \ge \frac{2^{\gamma \frac{y^\theta}{\log y}} x^{1-B}}{8(1-B)\log x}.$$

Since there are x^{1-B} choices for k in the sum above, we know by pigeonhole principle that there must exist a k_0 for which

$$|\mathcal{P}_{k_0}| \ge \frac{2^{\gamma \frac{y^\theta}{\log y}}}{8(1-B)\log x}. \tag{5}$$

5. Small Orders mod L

In this section, we will show that the order of an element mod L is very small relative to the size of the set $\mathcal{P}_{k_0}$ above. Let $\lambda(L)$ denote the Carmichael lambda function, which signifies the largest order of an element mod L. In this case, if we index the $q \in \mathcal{Q}$ as $q_1, \cdots, q_s$, we have

$$\lambda(L) = lcm\,(q_1 - 1, \cdots q_s - 1)\,.$$

Since all of the q that divide L are chosen such that $q-1$ is y-smooth, we know that $\lambda(L)$ must be free of prime factors of size greater than y. Moreover, if $r^j \mid \lambda(L)$ for some prime r then r^j must also divide $q-1$ for some $q \le \mathcal{Q}$, which means $r^j \le q \le y^\theta$. Thus, for a given prime r, let a_r be the largest power of r such that $r^{a_r} \le y^\theta$. Then

$$\lambda(L) \le \prod_{\substack{r \le y \\ r\ prime}} r^{a_r} \le \prod_{\substack{r \le y \\ r\ prime}} y^\theta \le y^{\theta\pi(y)} \le e^{2\theta y}. \tag{6}$$

This is small relative to $|\mathcal{P}_{k_0}|$. Indeed, we can bound L with

$$L \le \prod_{q \in \mathcal{Q}} q \ll \prod_{q \in \mathcal{Q}} y^\theta \le y^{\theta \frac{2y^\theta}{\log y}} = e^{2\theta y^\theta} \tag{7}$$

and hence since $x^B = L$, we apply (5) to find

$$|\mathcal{P}_{k_0}| \ge \frac{2^{\gamma \frac{y^\theta}{\log y}}}{8(1-B)\log x} \gg \frac{2^{\gamma \frac{y^\theta}{\log y}}}{y^\theta} \gg e^{\frac{\gamma y^\theta}{2\log y}}.$$

6. The Extra Prime

In Section 7, we will construct a Carmichael number n; in doing this, we must be able to ensure that the number of prime factors of n is 1 modulo a chosen variable

h. To do this, we introduce a conveniently constructed prime that we can often affix to the end of a Carmichael number coming from our construction to yield another Carmichael number.

In general, the method of construction of Carmichael numbers outlined in [2] will require finding $n = p_1 \cdots p_r$ such that $p_j - 1 \mid Lk_0$ and $Lk_0 \mid n - 1$. As such, it would be most convenient if we could find a prime $P = Lk_0 + 1$; this P would trivially satisfy $P - 1 \mid Lk_0$, and if $Lk_0 \mid n - 1$ for some n then $Lk_0 \mid Pn - 1$ as well. Unfortunately, it is difficult to guarantee that such a prime exists, so instead, we use the conjecture of Heath-Brown to find a prime that is almost as convenient. In particular, we have the following lemma.

Lemma 2. *Let L be as defined above, and k_0 be as in Section 4. Under the assumption of Conjecture 3, there exists a $k_1 \ll \log^2 L$ such that $P = Lk_0k_1 + 1$ is prime.*

The proof merely requires us to take $d = Lk_0$ and note that $k_0 < L^3$.

7. Sizes of Sets

Now, we will need to find subsets of $\mathcal{P}_{k_0}$ whose products are 1 mod Lk_0k_1. For a given h, let C_h be the set of distinct subsets of exactly h elements in $\mathcal{P}_{k_0}$ such that the product of all the elements in any of these subsets is indeed 1 mod Lk_0k_1. Let $l = n(G)$ by the notation of Proposition 1 above, where G is the group of integers mod Lk_1. We can then prove the following theorem.

Theorem 2. *There must exist an h with $l \le h \le 2l$ such that*

$$|C_h| \gg \left(\frac{l}{4}\right)^l.$$

Proof. This is an application of Proposition 1. In our case, since $k_1 \ll \log^2(Lk_0)$, we note that

$$\lambda(Lk_1) \le \lambda(L) \cdot k_1 \ll e^{2\theta y} \log^2(Lk_0),$$

where we have bounded $\lambda(L)$ by (6). So by Theorem 1, bounding Lk_0 by (7), we have

$$l \ll e^{2\theta y} \log^3(Lk_0) \ll e^{3\theta y}. \tag{8}$$

Now, following the notation in Proposition 1, the computations here will be cleanest if $t - l$ is close to l and r is roughly t^2. So let

$$r = 4l^2 \ll e^{6\theta y}$$

and

$$t = 2l.$$

Recall that, for any integers v and w with $v > w$, the classical binomial coefficient bound gives the following:

$$\left(\frac{v}{w}\right)^w \leq \binom{v}{w} \leq \left(\frac{ve}{w}\right)^w.$$

We can then invoke Proposition 1 to find

$$\begin{aligned}\sum_{h=l}^{2l} |C_h| &\gg \binom{4l^2}{2l} \Big/ \binom{4l^2}{l} \\ &\gg \left(\frac{4l^2}{2l}\right)^{2l} \Big/ \left(\frac{4el^2}{l}\right)^l \\ &\gg (2l)^{2l} \,/\, (12l)^l \\ &\gg (2l)^l \,/\, (6)^l \\ &= \left(\frac{l}{3}\right)^l\end{aligned}$$

So there must exist an h between l and $2l$ for which

$$|C_h| \gg \frac{\left(\frac{l}{3}\right)^l}{l+1} = \frac{\left(\frac{l}{4}\right)^l \left(\frac{4}{3}\right)^l}{l+1}.$$

Since

$$\left(\frac{4}{3}\right)^l \gg l+1,$$

the theorem then follows. □

From here, we can give a construction that will ultimately prove Main Theorem 1.

Theorem 3. *Choose a sufficiently large y, and choose $\theta \in (1, 2)$ as above. Then there must exist a Carmichael number n such that $e^{\frac{\gamma}{4} y^\theta \log y} \leq n \leq e^{e^{19\theta y}}$ and n has a prime number of prime factors.*

Proof. Let us take a set of primes $S \in C_h$. Then

$$n = \prod_{p \in S} p \equiv 1 \pmod{Lk_0k_1},$$

where the congruence mod Lk_1 is by definition of C_h, and all of the p in $\mathcal{P}_{k_0}$ are such that $p \equiv 1 \pmod{k_0}$.

Similarly, if we take sets $S_1, \cdots, S_j \in C_h$ then

$$n = \prod_{p \in S_1 \bigcup \cdots \bigcup S_j} p \equiv 1 \pmod{Lk_0k_1},$$

since the product of a bunch of 1's is still 1.

Note, then, that for any g that is significantly smaller than $\left(\frac{l}{4}\right)^l$ (say $g \ll \left(\frac{l}{4}\right)^{\frac{l}{2}}$), we can take a collection of g sets $S_1, \cdots, S_g$ such that

$$n = \prod_{p \in S_1 \bigcup \cdots \bigcup S_g} p \equiv 1 \pmod{Lk_0k_1},$$

where n has exactly gh distinct prime factors.

Recall also that in Section 6, we defined an "extra" prime P such that

$$P = Lk_0k_1 + 1.$$

We now wish to find a prime number of the form $gh+1$ where g is small. While this is easily shown with Conjecture 3, we instead use a weaker but unconditional result by Xylouris [19], the idea here being to invoke conjectures as few times as possible in our proof.

Theorem 4. *(Xylouris [19]) For any c and d relatively prime, there exists a prime p congruent to c mod d such that*

$$p \ll d^{5.2}.$$

In other words, for any such c and d there exists a $k \ll d^{4.2}$ such that $p = dk + c$ is prime.

In our construction above, then, for a given h, we can find a $g_0 \ll h^{4.2}$ such that $g_0h + 1$ is prime. Clearly, $h^{4.2} \ll l^5$, which is much smaller than $\left(\frac{l}{4}\right)^l$. So for $g_0 \ll h^{4.2}$, let us choose sets $S_1, \cdots, S_{g_0} \in C_h$. Then

$$n = P \cdot \prod_{p \in S_1 \bigcup \cdots \bigcup S_{g_0}} p \equiv 1 \pmod{Lk_0k_1}.$$

We know that $P - 1 = Lk_0k_1$, and for any $p \in S_j$, $p - 1 \mid Lk_0$. So for any $p \mid n$, $p - 1 \mid Lk_0k_1 \mid n - 1$. So n is a Carmichael number consisting of exactly $g_0h + 1$ prime factors, and $g_0h + 1$ is itself prime by assumption.

For the bounds on n, we know that all of the primes p in the sets S_i are such that

$$p \le Lk_0 + 1 \ll L\log^2 L \ll e^{3\theta y^\theta}.$$

by (7).

So

$$n \le P \cdot \left(e^{3\theta y^\theta}\right)^{g_0 h}.$$

Since

$$g_0 h \ll h^{5.2} \ll l^6 \ll e^{18\theta y}$$

by (8), and

$$P = Lk_0k_1 + 1 \ll L\log^5(L) \ll e^{3\theta y^\theta},$$

we see that

$$n \le e^{3\theta y^\theta}\left(e^{3\theta y^\theta e^{18\theta y}}\right) \ll e^{e^{19\theta y}}.$$

On the other hand, $n > P > L$ and L can be bounded below with

$$L = \prod_{q\in\mathcal{Q}, q\notin\mathcal{P}_B(x)} q \gg y^{\frac{\gamma y^\theta}{4\log y}} = e^{\frac{\gamma}{4}y^\theta \log y}.$$

□

Since there are an infinitude of disjoint intervals of the form $[e^{\frac{\gamma}{4}y^\theta \log y}, e^{e^{19\theta y}}]$, there must then exist an infinite sequence of y's such that each y yields a unique Carmichael number n with the required prime number of prime factors. Hence Main Theorem 1 follows.

8. Remarks

We note a couple of things here:

First, we see no imperative in this paper to try to find sharp bounds for the number of prime factors or the asymptotic density of Carmichael numbers with $\Omega(n)$ equal to some specific prime. This is because there are so many ways that this estimate could be improved. For instance, in the proof of Theorem 3, rather than simply taking one choice of g for which $gh+1$ is prime, we could find many by simply applying (1) to the progression $gh+1$. One could also invoke the Heath-Brown conjecture in a number of different places (in lieu of [19] or in lieu of the estimates for primes in arithmetic progressions in the lower bounds for $|\mathcal{P}_{k_0}|$) which would also change the bounds rather dramatically.

Second, a couple of other possible results fall out of this method fairly easily. For instance, the fact that there are infinitely many Carmichael numbers with a composite number of prime factors falls out almost trivially (and unconditionally); of course, this result seems unlikely to be a surprise. Also following rather trivially (and unconditionally) from this method is the fact that there are infinitely many Carmichael numbers where the number of prime factors is a perfect square (by taking $g = h$), perfect cube (with $g = h^2$), or, in fact, any perfect power.

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