NO NEW GOORMAGHTIGH PRIMES UP TO 10^{700}

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Abstract
The Goormaghtigh conjecture states that the only two numbers which have two non-trivial representations as repunits are 31 and 8191. We call such a prime number a *Goormaghtigh prime*. We show that there are no other Goormaghtigh primes less than 10^{700}.

1. Introduction

Recall that a repunit is a positive integer whose digital representation in some base b consists only of 1s. Goormaghtigh [5] observed that the only numbers up to 10^5 with two non-trivial representations as repunits are 31 (bases 2 and 5) and 8191 (bases 2 and 90). The conjecture that these are the only two such numbers has become known as the "Goormaghtigh Conjecture." More precisely,

Conjecture 1. The only solutions to

$$N = \frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1} \tag{1}$$

with integers $y > x \geq 2$, $n > m > 2$ are $N = 31$ and $N = 8191$.

We exclude the case $m = 2$ in Conjecture 1 because every integer N is a length-2 repunit in base $N - 1$. It is currently unknown whether the equation has finitely many solutions. Some results are known for small exponents and the case $\gcd(m - 1, n - 1) > 1$, which we will use below. Note that 31 and 8191 are both primes. We propose the terms *Goormaghtigh primes* for such repunits N that are primes and *Goormaghtigh numbers* for any N with two representations. Prime repunits have been also studied as Brazilian primes; see [8]. A Goormaghtigh prime can also be characterized as a Brazilian prime to two different bases. Bateman and Stemmler [1], using computations of Horn, noted that the only N less than 1.275×10^{10}

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satisfying (1) with x, y and N all prime is 31. In this paper, we look at a condition weaker than in the Goormaghtigh conjecture, but stronger than in the Bateman-Stemmler question, by searching for Goormaghtigh primes. We use recent results of Bennett, Garbuz, and Martens [2] as a key ingredient to reduce the computation.

2. There Are Only Two Goormaghtigh Primes Less than 10^{700}

Our approach is to use lower bounds on m, n, and y in (1) to significantly limit the number of cases less than 10^{700} that we need to check.

The following is part of Theorem 4 from [2].

Theorem 1. *The only solutions to Equation* (1) *with* $\gcd(m-1, n-1) > 1$ *and* $m \leq 50$ *have* $N = 31$ *or* $N = 8191$.

For Goormaghtigh primes, we must have m and n odd primes, and thus $m \geq 53$. Theorem 2 from Bennett, Gherga, and Kreso [3] rules out further examples with $n = 3$ or $n = 5$ when $\gcd(m-1, n-1) > 1$, which is true in the prime case. Theorem 3 from [2] rules out further examples with $y \leq 10^5$. In order to show that there are no Goormaghtigh primes below 10^{700}, we employ the following algorithm. Let $f_k(x) = \frac{x^k-1}{x-1}$.

1. We perform a precomputation similar to the one in [4]. For a list of small primes $\{p_i\}$, we compute the values of $f_q(b)$ for all $2 \leq b < p_i$ and $1 \leq q < p_i$.

2. Generate a list of possible values of m. We know m must be prime. From Theorem 1, we have $m \geq 53$, and since $x > 2$ in the range considered (no other known Mersenne primes are Goormaghtigh primes), $m < \log 10^{700}/\log 3 \approx 1467$.

3. For each m in the above list, we examine all x with $x > 2$ and $x < 10^{700/m}$. If $x \not\equiv 1 \pmod{p_i}$, then $f_m(x) \equiv f_{m'}(x) \pmod{p_i}$ for $m \equiv m' \pmod{p_i - 1}$. We retrieve this value from the precomputation. Call this value a_i.

4. We examine each prime n such that $m > n \geq 7$; we exclude the case $3(m-1) = (n-1)$ by Theorem 6 of [2]. For $n' \equiv n \pmod{p_i - 1}$, we see if a_i is a possible value of $f'_n(y)$. To handle the case $y \equiv 0 \pmod{p_i}$, we check if $a_i \equiv 1 \pmod{p_i}$. Finally, if $y \equiv 1$, we have $f_n(y) \equiv n$, so we check if $a_i \equiv n$. If none of these conditions hold, we have shown there is no solution mod p_i, and thus over the integers. If there is a solution, we repeat with a different p_i.

The computation up to 10^{500} took 129 minutes on a single core of an Intel SP Platinum 8280 CPU running at 2.7 GHz. The computation up to 10^{700} took approximately 480 core-days.

Note that we have, in fact, shown a slightly stronger result, that there are no new Goormaghtigh numbers up to 10^{700} where both exponents are prime.

3. A Conditional Result

Recall the abc conjecture.

Conjecture 2. The *abc conjecture* of Oesterlé [7] and Masser [6] states that if a, b, and c are relatively prime integers such that $a+b=c$, then for any $\epsilon > 0$, only finitely many (a,b,c) fail to satisfy the inequality

$$c < \mathrm{rad}(abc)^{1+\epsilon}.$$

Carl Pomerance suggested an argument that gives the following.

Theorem 2. *Assuming the abc conjecture, there are only finitely many Goormaghtigh numbers where neither representation is of length three or four.*

Proof. We see that $y^{n+1} > x^m$, so we will use that $y^{(n+1)/m} > x$. Rewrite (1) as

$$(x^m-1)(y-1) = (y^n-1)(x-1).$$

Rearranging,

$$x^m(y-1) + (x-y) = y^n(x-1).$$

Let

$$g = \gcd(x^m(y-1), x-y, y^n(x-1)).$$

Let $a = x^m(y-1)/g$, $b = (x-y)/g$ and $c = y^n(x-1)/g$. Then we have $a+b=c$, and

$$\mathrm{rad}(abc) \leq x(y-1)(y-x)y(x-1)/g < y^{3+(n+1)/m}(x-1)/g.$$

On the other hand, we have $c = y^n(x-1)/g$. Let $\epsilon = 1/4$. Then it suffices to show that for $n > m \geq 5$, $\mathrm{rad}(abc)^{1+\epsilon}/c \leq 1$, or

$$\left(y^{3+(n+1)/m}(x-1)/g\right)^{5/4} / (y^n(x-1)/g) \leq 1.$$

This is equivalent to

$$y^{15/4+(5/4)(n+1)/m-n}((x-1)/g)^{1/4} \leq 1.$$

The exponent on y is no greater than $-1/2$ (which is achieved when $m=5$, $n=6$). We have by necessity that $((x-1)/g)^{1/4} \leq (x-1)^{1/4} < (x-1)^{1/2}$, so we have established the conditions for the abc conjecture. Therefore there are only finitely many Goormaghtigh numbers with $m < 5$. □

Combining Theorem 2 with Theorem 2 from [3], which eliminates the length-3 case for primes, gives the following.

Corollary 1. *Assuming the abc conjecture, there are only finitely many Goormaghtigh primes.*

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