ON REMOTENESS FUNCTIONS OF EXACT SLOW k-NIM WITH $k+1$ PILES

Vladimir Gurvich
National Research University Higher School of Economics (HSE), Moscow, Russia
and
RUTCOR, Rutgers, the State University of New Jersey, Piscataway, New Jersey
vgurvich@hse.ru, vladimir.gurvich@gmail.com

Denis Martynov
National Research University Higher School of Economics (HSE), Moscow, Russia
taiwong2464@gmail.com

Vladislav Maximchuk
National Research University Higher School of Economics (HSE), Moscow, Russia
vladislavmaximchuk3495@gmail.com

Michael Vyalyi
National Research University Higher School of Economics, Moscow, Russia;
Institute of Physics and Technology, Dolgorpudnyi, Russia;
and
Federal Research Center "Computer Science and Control" of the Russian Academy of Sciences, Moscow, Russia
vyalyi@gmail.com

Received: 4/15/23, Revised: 6/14/24, Accepted: 9/12/24, Published: 10/9/24

Abstract

Given integers n and k such that $0 < k \leq n$ and n piles of tokens, two players alternate turns. In each move they are allowed to choose any k non-empty piles and remove exactly one token from each pile. The player who has to move but cannot is the loser. Cases $k = 1$ and $k = n$ are trivial. For $k = 2$ the game was solved for $n \leq 6$. For $n \leq 4$ the Sprague-Grundy (SG) function was efficiently computed, for both the normal and misère versions. For $n = 5, 6$ a polynomial algorithm computing P-positions for the normal version was obtained. Here we consider case $1 < k = n - 1$ in the normal version and compute the Smith remoteness function, whose even values are taken in the P-positions. An optimal move is always defined by the following simple rule:

- if all piles are odd, keep a largest one and reduce all others;
- if there exist even piles, keep a smallest one of them and reduce all others.

DOI: 10.5281/zenodo.13909093

This strategy is optimal for both players, moreover, it allows a player to win as fast as possible from an N-position and to resist as long as possible from a P-position.

1. Introduction

We assume that the reader is familiar with basic concepts of impartial game theory (for example, see [1, 2] for an introduction).

1.1. Exact Slow NIM

The game Exact Slow NIM was introduced in [7] as follows. Given two integers n and k such that $0 < k \leq n$, and n piles containing $x_1, \ldots, x_n$ tokens, respectively, two players alternate turns. In each move they are allowed to reduce any k piles by exactly one token. The player who has to move but cannot is the loser. In [7], this game was denoted $\text{NIM}^1_=(n, k)$. Here we will simplify this notation to $\text{NIM}(n, k)$. We start with three simple properties of this game.

1. In accordance with the rules, every move reduces the total number of tokens by exactly k. Hence, the value[1] $x_1 + \cdots + x_n \bmod k$ is an invariant of the game $\text{NIM}(n, k)$. Hence, $\text{NIM}(n, k)$ is split into k subgames, between which there are no moves.

2. A position $x = (x_1, \ldots, x_n)$ will be called *even* or *odd* if all its entries are even or odd, respectively. Every even position of the game $\text{NIM}(n, k)$ is a P-position, because each move of the first player can be repeated by the opponent. Respectively, every position x with exactly k odd entries is an N-position, because there is a move from x to a P-position.

Remark 1. Both above claims can be obviously extended from the game $\text{NIM}(n, k)$ to slow NIM on arbitrary hypergraphs. Given a hypergraph $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$ on the ground set $[n] = \{1, \ldots, n\}$, and a nonnegative integer vector $x = (x_i : i \in [n])$, the *hypergraph slow game* $\textit{NIM}_{\mathcal{H}}$ is defined as follows. In each move, a player chooses a hyperedge $H \in \mathcal{H}$ such that $x_i > 0$ for every $i \in H$ and reduces by 1 each x_i. The game $\text{NIM}(n, k)$ corresponds to the hypergraph slow $\text{NIM}_{\mathcal{H}}$ with $\mathcal{H} = \{H : |H| = k\}$. In $\text{NIM}_{\mathcal{H}}$, every even x is a P-position and hence, x is an N-position whenever there exists an $H \in \mathcal{H}$ such that x_i is odd if and only if $i \in H$.

3. In $\text{NIM}(n, k)$, the moves from a position are defined up to a permutation of its entries. If y is obtained from x by a permutation of entries, then two subgames, starting from x and y, are isomorphic. In particular, either x and y are both P-positions or N-positions. Thus, we can restrict ourselves to positions whose entries are non-decreasing. Such positions (and vectors) will be called *ordered*, for brevity. Note that after a move a re-ordering may be required.

[1] We use $a \bmod k$ for a residue of a modulo k and $a \equiv b \pmod{k}$ for the modulo k comparison.

The game NIM(n,k) is trivial if $k = 1$ or $k = n$. In these two cases it always ends after $x_1 + \cdots + x_n$ and $\min(x_1, \ldots, x_n)$ moves, respectively. Hence, nothing depends on the players' skills. Other cases are more complicated.

The game was solved for $k = 2$ and $n \leq 6$. In [8], an explicit formula for the SG function was found for $n \leq 4$.[2] This formula allows us to compute the SG function in linear time. Then, in [5], P-positions were found for $n \leq 6$. For the subgame with even $x_1 + \cdots + x_n$ a simple formula for the P-positions was obtained. It allows us, for a given position, to verify in linear time, if it is a P-position and, if not, to find a move to a P-position. The subgame with odd $x_1 + \cdots + x_n$ is more complicated. Still, a more sophisticated formula for its P-positions was found, which provides a linear time recognition algorithm.

1.2. Case $n = k + 1$

In this paper we solve the game in case $n = k + 1$. In every position an optimal move is defined by the following simple rule:

- if all piles are odd, keep a largest one and reduce all others;
- if there exist even piles, keep a smallest one of them and reduce all others.

We will call this strategy the *M-rule* and the corresponding moves the *M-moves*. Obviously, in every position there exists a unique M-move, up to a re-numeration of piles. We will show that the M-rule solves the game; moreover, it allows a player to win as fast as possible in an N-position and to resist as long as possible in a P-position.

Given a position $x = (x_1, \ldots, x_n)$, assume that both players follow the M-rule and denote by $\mathcal{M}(x)$ the number of moves from x to a terminal position. We will prove that $\mathcal{M} = \mathcal{R}$, where $\mathcal{R}$ is the so-called *remoteness function* considered in the next subsection.

In some positions the M-move may be a unique winning move. For example, in positions (1,1,2) and (1,1,3) of NIM(3,2) the M-moves reduce the two smallest piles to zero and win immediately, while moves to (0,1,1) and (0,1,2) are losing.

It is easily seen that there exists no M-move to an odd position. Furthermore, the latter may be a P- or N-position. For example, in NIM(3,2), position (1,1,1) is an N-position, while (3,3,3) is a P-position, since $\mathcal{M}(3,3,3) = 4$. The corresponding sequence of M-moves is: $(3,3,3) \to (2,2,3) \to (1,2,2) \to (0,1,2) \to (0,0,1)$.

1.3. Smith's Remoteness Function

In 1966 Smith [19] introduced a *remoteness* function $\mathcal{R}$ by the following algorithm.

[2] It was done for both, the normal and misère versions of the game. In the present paper we restrict ourselves to the normal version; the misère version seems much more complicated (see [10]).

Consider an impartial game given by a finite acyclic directed graph (digraph) G. Set $\mathcal{R} = 0$ for all terminal positions of G and $\mathcal{R}(x) = 1$ if and only if there is a move from x to a terminal position. Delete all labeled positions from G and repeat the above procedure increasing $\mathcal{R}$ by 2, that is, assign 2 and 3 instead of 0 and 1, and so on.

Remark 2. This algorithm was considered as early as in 1901 by Bouton [4], but only for special graphs corresponding to the game NIM. In 1944 this algorithm was extended to arbitrary digraphs by von Neumann and Morgenstern in [17]. In graph theory, the set of vertices with even $\mathcal{R}$ is called a (unique) *kernel* of the corresponding (acyclic) digraph, while in the theory of impartial games, this set is referred to as the set of P-positions.

The function $\mathcal{R}$ has the following, stronger, property: x is a P-position if $\mathcal{R}(x)$ is even (respectively, odd). Furthermore, the player making a move from x cannot win, yet, can resist for at least $\mathcal{R}(x)$ moves but not longer (respectively, wins in at most $\mathcal{R}(x)$ moves, but not faster) if the opponent plays optimally. In both cases the player reduces $\mathcal{R}$ by 1 in one move, which is always possible unless x is a terminal position, in which case $\mathcal{R}(x) = 0$.

1.4. Sprague-Grundy and Smith's Theories

Given n impartial games $\Gamma_1, \ldots, \Gamma_n$, in each move a player chooses one of them and makes a move in it. The player who has to move but cannot is the loser. The obtained game $\Gamma = \Gamma_1 \vee \cdots \vee \Gamma_n$ is called the *disjunctive sum.* For example, NIM with n piles is the disjunctive sum of n one-pile NIMs. The SG function $\mathcal{G}$ of Γ is uniquely determined by the SG functions of the n components by the formula $\mathcal{G}(\Gamma) = \mathcal{G}(\Gamma_1) \oplus \cdots \oplus \mathcal{G}(\Gamma_n)$ where $\oplus$ is the so-called NIM-sum. These results were obtained by Bouton [4] for the special case of NIM and then extended to arbitrary impartial games by Sprague [20] and Grundy [6].

Now suppose that a move consists of playing in each of the n component games, rather than in one of them. Again, the player who has to move but cannot is the loser. The obtained game $\Gamma = \Gamma_1 \wedge \cdots \wedge \Gamma_n$ is called the *conjunctive sum.* The remoteness function $\mathcal{R}$ of Γ is uniquely determined by the remoteness functions of the n components by the formula $\mathcal{R}(\Gamma) = \min(\mathcal{R}(\Gamma_1), \ldots, \mathcal{R}(\Gamma_n))$. This was proven in 1966 by Smith [19].

The above results show the importance of the SG and remoteness functions. We mention several further steps. A polynomial algorithm computing the SG function of NIM easily follows from the theorem mentioned above. It was suggested by Bouton [4]. In 1910 Moore [16] generalized this result. He introduced and solved a game called k-NIM, where in each move a player can reduce at least 1 and at most k piles. Moore found the P-positions. The SG function is known only in the case $n = k + 1$ (see Jenkins and Mayberry [15]). For Moore's game, there is a simple formula for $\mathcal{R}(x)$ when x is a P-position; in contrast, $\mathcal{R}(x)$ is NP-hard to compute when x is an N-position (see [3]).

Wythoff [21] introduced a modified version of NIM with two piles of tokens, where in each move a player can either reduce one pile by any number of tokens, or both piles by the same number of tokens. Wythoff found the P-positions of this game, while an explicit formula for the SG function and the algorithmic complexity of its computing remain unknown, despite numerous efforts (see, for example, [18] for a survey). In contrast, the remoteness function $\mathcal{R}(x)$ can be computed in polynomial time for Wythoff's game and several of its generalizations (see [3]). In this paper we obtain such results for the remoteness function of the game NIM$(k+1,k)$.

It is also worth mentioning that computations reveal a chaotic behavior of the SG function of NIM$(5,2)$ and NIM$(6,2)$, while the remoteness functions of NIM$(4,2)$ and NIM$(5,2)$ follow some simple rules (see [9]).

1.5. On m-Critical Positions of the Game NIM$(k+1,k)$

In the set of positions x with fixed value $\mathcal{R}(x)$ we distinguish minimal with respect to the entrywise order $\leq$ on vectors. Given a nonnegative integer m, a position $x = (x_1, \dots x_n)$ of the game NIM(n,k) is called *m-critical* if $\mathcal{R}(x) = m$ and $x > y$ fails for any position y with $\mathcal{R}(y) = m$. By this definition, for any position y there exists a unique m such that $y \geq x$ holds for some m-critical x, and $y \geq x$ fails for any $(m+1)$-critical x.

We will characterize m-critical positions of the game NIM$(k+1,k)$ as follows.

Theorem 1. *A position $x = (x_1, \dots, x_{k+1})$ of the game NIM$(k+1,k)$ is m-critical, that is, $\mathcal{R}(x) = m$ and $\mathcal{R}(y) \neq m$ for all $y < x$, if and only if one of the following two cases holds:*

(A) $x_1 + \dots + x_{k+1} = km$;

$\max(x_1, \dots, x_{k+1}) \leq m$;

if m is even then x is even (that is, all $k+1$ entries of x are even);

if m is odd then exactly one of the $k+1$ entries of x is even.

(B) $x_1 + \dots + x_{k+1} = km + k - 1$;

$\max(x_1, \dots, x_{k+1}) < m$;

m is even, x is odd (that is, all $k+1$ entries of x are odd).

The proof is based on Lemma 4 (the lemma and the proof are given in Section 2). The following statement easily results from the above theorem.

Corollary 1. *(i) Let x be an m-critical position with an even m. Then, x is either odd (Case (B)) or even (Case (A)). In the latter case, at least one of even entries of x is strictly less than m. A move from x leads to an $(m-1)$-critical position, where $m-1$ is odd, provided this move reduces all (maximal) entries of x that are equal to m. In particular, an M-move has such a property.*

(ii) For an m-critical position x with an odd m, there exists exactly one move leading to an $(m-1)$-critical position. This move keeps the (unique) even entry and reduces all other; furthermore, this move is a unique M-move.

Proof. (i) It is enough to note that all $n = k+1$ entries cannot be equal to m for $m > 0$, because in this case their sum would be $(k+1)m$ rather than km, as required by Theorem 1. All other statements are immediate from the definitions and Theorem 1.

(ii) If the unique even entry x_i of x is maximal, then $x_i < m$ since m is odd. Hence, by Theorem 1, the M-move is unique and leads to an $(m-1)$-critical position. All other statements are immediate from the definitions and Theorem 1. □

1.6. Main Results

Our main results are the following two properties of the game NIM$(k+1,k)$.

Theorem 2. *The equality $\mathcal{M}(x) = \mathcal{R}(x)$ holds for any position x.*

The proof is based on Claims 1, 2 and Lemmas 1, 2, 3, 4; see Section 2.

Theorem 3. *The function $\mathcal{R}$ can be computed in polynomial time, even if k is a part of the input and integers are presented in binary.*

The proof is based on Lemmas 4 and 5; see Section 3.

1.7. On P-Positions of the Game NIM$(k+1,k)$

Recall that P-positions of an impartial game can be characterized in two ways, in other words, the following three statements are equivalent: x is a P-position, $\mathcal{G}(x) = 0$, and $\mathcal{R}(x)$ is even. Hence, x is a P-position of NIM$(k+1,k)$ if and only if the number $\mathcal{M}(x) = \mathcal{R}(x)$ is even. Using the above theorems, we will obtain a polynomial algorithm verifying if x is a P-position, and if not, computing a move from x to a P-position.

Remark 3. Interestingly, we do not need this algorithm for playing NIM$(k+1,k)$ optimally. Instead, in every position (N- or P-, does not matter) we just follow the M-rule and whatever will be, will be. This is a peculiar situation for the impartial game theory, which somewhat downgrades the role of the P-positions.

For NIM$(4,3)$ some pretty complicated formulas for P-positions were confirmed by computations; see Appendix. Yet, we were not able to prove these formulas. Furthermore, there are three cases: $x_1 + x_2 + x_3 + x_4 \equiv 0, 1$, or $2 \pmod 3$. The game NIM$(k+1,k)$ is split into k disjoint subgames, which makes the general case analysis even more difficult. Interestingly, the m-critical positions, unlike the P-positions, are defined by the same formulas for all k subgames.

Theorem 1 implies that every x satisfying (B) is an odd P-position. Interestingly, the inverse holds too.

Proposition 1. *In the game NIM$(k+1,k)$, a position x is an odd P-position if and only if it satisfies (B) of Theorem 1.*

The proof is based on Lemmas 1 and 4, it is also given in Section 2. Recall that, by definition, an M-move cannot lead to an odd position. In other words, if both players follow the M-rule, then a play can start in an odd position but cannot enter one. How odd positions are partitioned into P- and N-positions is shown by Proposition 1. For example, we have $\mathcal{M}(3,3,3) = 4$ and $\mathcal{M}(3,5,5) = 6$; the corresponding M-sequences are respectively:

$$(3,3,3) \to (2,2,3) \to (1,2,2) \to (0,1,2) \to (0,0,1) \text{ and}$$
$$(3,5,5) \to (2,4,5) \to (2,3,4) \to (2,2,3) \to (1,2,2) \to (0,1,2) \to (0,0,1).$$

Hence, $(3,3,3)$ and $(3,5,5)$ are odd P-positions of NIM$(3,2)$, by Theorem 2. In contrast, $(1,1,1)$ is an odd N-position – the unique move from it leads to the terminal $(0,0,1)$.

Recall that in case (B) we have: $x_1 + \cdots + x_{k+1} = km + k - 1$, m is even, and position x is odd. Since $k = 2$ for NIM$(3,2)$, we obtain $x_1+x_2+x_3 = 2m+1$. Hence, since $\mathcal{R}(x) = m$, it follows that $\mathcal{R}(3,3,3) = 4$ and $\mathcal{R}(3,5,6) = 6$ are both even, while $\mathcal{R}(1,1,1) = 1$ is odd. Thus, positions (3,3,3) and (3,5,5) of NIM$(3,2)$ satisfy (B), while (1,1,1) does not: $m = 1$ is odd in this case (actually, $\max(1,1,1) < 1$ also fails).

2. Proofs of Theorems 1 and 2

For $k = 1$, the game and both theorems become trivial. Assume that $k > 1$. Throughout the proofs we use the following assumptions and notation. A move in NIM$(k+1,k)$ preserves exactly one entry. Its index indicates the move. Thus, a move i from a position x leads to the position $x - d^{(i)}$, where $d^{(i)}_j = 1 - \delta_{ij}$ and δ_{ij} is the Kronecker symbol. We denote the unit entry vectors by $e^{(i)}$, that is, $e^{(i)}_j = \delta_{ij}$. Let $e(x)$ be the minimal index such that $x_{e(x)}$ is the smallest even integer among x_i if there are even x_i. Otherwise, if all x_i are odd, then $e(x)$ is the minimal index such that $x_{e(x)}$ is a maximal entry in x. Thus, the M-move at x is $x \to x' = x - d^{(e(x))}$. Note that x' is not necessarily ordered even if x is ordered. For ordered x and $i < e(x)$ we have $x_i < x_{e(x)}$ and x_i is odd.

Based on the description of critical positions given in Theorem 1, we introduce a new function $\mathcal{B}(x)$ as follows. Set $\Sigma(z) = \sum_{i=1}^{k+1} z_i$ and $\max(z) = \max_i z_i$.

A vector $z = (z_1, \ldots, z_{k+1})$ is called *basic* of type (A0), (A1), or (B), if it satisfies the following conditions (A0), (A1), or (B), respectively:

$$
\begin{array}{ll}
\text{(A)} & \Sigma(z) = kb(z), \\
 & \max(z) \le b(z), \\
\text{(A0)} & \text{(A) holds, } b(z) \text{ is even, and all } z_i \text{ are even,} \\
\text{(A1)} & \text{(A) holds, } b(z) \text{ is odd, and exactly one } z_i \text{ is even,} \\
\text{(B)} & \Sigma(z) = kb(z) + k - 1, \\
 & \max(z) < b(z), \\
 & b(z) \text{ is even, and all } z_i \text{ are odd.}
\end{array}
$$

We say that a basic vector z is *m-basic*, if $b(z) = m$. Using basic vectors, we define

$$\mathcal{B}(x) = \max(b(z) : x \ge z, \text{ where } z \text{ is basic}). \tag{1}$$

For a basic vector z such that $b(z) = \mathcal{B}(x)$, we say that z *supports* x (or x is supported by z). Note that a vector can be supported by more that one basic vector (see Example 2 below).

The following claim is an immediate corollary of the definition.

Claim 1. *If $x' \le x$ then $\mathcal{B}(x') \le \mathcal{B}(x)$.*

In particular, the zero vector is basic of type (A0) with $b(0) = 0$ and $x \ge 0$ for every position x. Thus, $\mathcal{B}(x) \ge 0$ for every position. Nevertheless, we allow negative entries in basic vectors. This convention simplifies the proofs.

We do not assume that a basic vector is ordered. It simplifies the analysis, since there is no need to re-order entries after a move. Nevertheless, without loss of generality we can restrict ourselves to positions and supporting vectors whose entries are non-decreasing. A permutation of entries of a position results in an isomorphic game. For supporting vectors we have the following claim.

Claim 2. *If $z \le x$ for a basic z and an ordered x, then there exists a basic vector z^* such that $z^* \le x$, $b(z^*) = b(z)$, and z^* is ordered.*

Proof. The definition of a basic vector is invariant under permutations of entries. If x is ordered and $z_i > z_j$ for $i < j$, then $x_j \ge x_i \ge z_i$ and $x_i \ge z_i > z_j$. Thus, the transposition of z_i and z_j gives a basic position $\tilde{z}$ such that $\tilde{z} \le x$ and $b(\tilde{z}) = b(z)$. Repeating such transpositions, we come to an ordered basic z^* such that $z^* \le x$. $\square$

For our main arguments we need the following facts about the change of $\mathcal{B}(\cdot)$ after a move.

Lemma 1. *If x is supported by a basic vector z of type* (B)*, then $x = z$ and $\mathcal{B}(x') = \mathcal{B}(x) - 1$ for each move $x \to x' = x - d^{(j)}$.*

Proof. We prove $x = z$ by contradiction. Let $z < x$ and $z_p < x_p$ for some p. For $\tilde{z} = z + e^{(p)}$ we have

$$\Sigma(\tilde{z}) = 1 + \Sigma(z) = 1 + kb(z) + k - 1 = k(b(z) + 1).$$

Note that $\tilde{z}_p$ is the unique even entry in $\tilde{z}$. Also, $\tilde{z}_i \leq b(z) + 1$ for all $1 \leq i \leq k + 1$. So $\tilde{z}$ is a basic vector of type (A1) and $\tilde{z} \leq x$. Therefore $\mathcal{B}(x) \geq b(z) + 1 = \mathcal{B}(x) + 1$, a contradiction.

Thus $x = z$. Since z is a basic vector of type (B), all entries are odd. Let $x \to x' = x - d^{(j)}$ be a move. Then $\mathcal{B}(x') < \mathcal{B}(x) = b(z)$, since $\Sigma(x) = kb(z) + k - 1$, $\Sigma(x') = kb(z) - 1 < kb(z)$.

Now choose $q \neq j$. Define z' as follows:

$$z'_i = \begin{cases} x_j < b(z), & \text{if } i = j, \\ x_q - 1, & \text{if } i = q, \\ x_i - 2, & \text{otherwise.} \end{cases}$$

From the definition we get $z' \leq x' \leq x$ and

$$\Sigma(z') = -1 - 2(k - 1) + \Sigma(x) = 1 - 2k + kb(z) + k - 1 = k\big(b(z) - 1\big).$$

Note that $b(z) - 1$ is odd, since $b(z)$ is even. The only even entry in z' is z'_q and $z'_i < b(z)$ by construction. Thus, z' is basic of type (A1) and we conclude that $\mathcal{B}(x) > \mathcal{B}(x') \geq b(z) - 1 = \mathcal{B}(x) - 1$. Therefore $\mathcal{B}(x') = \mathcal{B}(x) - 1$. □

Lemma 2. *If x is supported by a basic vector z of type* (A0)*, then $\mathcal{B}(x') \leq \mathcal{B}(x) - 1$ and $\mathcal{B}(x')$ is odd for each move $x \to x' = x - d^{(j)}$. Moreover, $\mathcal{B}(x') = \mathcal{B}(x) - 1$ for moves $x \to x' = x - d^{(j)}$ such that either* (1) *$z_j < b(z)$, or* (2) *$z_j = b(z)$ and $z_i < \min(b(z), x_i)$ for some i.*

Example 1. The function $\mathcal{B}$ can be reduced by 3 or more in this case. Let $x = (2, 4, 6, 6)$ and $k = 3$. Since $\Sigma(x) = 18 = 3 \cdot 6$, x is 6-basic. Let $x' = x - d^{(4)} = (1, 3, 5, 6)$. Due to Lemma 2, $\mathcal{B}(x')$ is odd. Since $\Sigma(x') = 15 = 3 \cdot 5$, $\mathcal{B}(x') = 5$ implies that x' supports itself, i.e., it is 5-basic. But $\max(x') = 6 > 5$, a contradiction to the definition of basic vectors. Thus $\mathcal{B}(x') < 5$. On the other hand, $(0, 3, 3, 3) \leq x'$ and $(0, 3, 3, 3)$ is 3-basic. Therefore $\mathcal{B}(x') = 3$.

Proof of Lemma 2. Recall that z is of type (A0). Hence, $\Sigma(z) = k \cdot 2s > 0$ (otherwise z is terminal). Let $x \to x' = x - d^{(j)}$ be a move and z' is a basic vector supporting x'.

Let us prove that z' is not of type (B). Suppose, for contradiction, that z' is basic of type (B). Then $z' = x'$ by Lemma 1. So $\Sigma(x') = kb + k - 1$, where $b = \mathcal{B}(x')$ is even, and $\Sigma(x) = k(b+1) + k - 1 < k(b+2)$. Note that $\Sigma(z) \leq \Sigma(x)$ and z is a basic vector of type (A0), thus $b(z)$ is even. So we get $\mathcal{B}(x) = b(z) \leq b$. Therefore $\mathcal{B}(x) = b = \mathcal{B}(x')$, since $\mathcal{B}(x) \geq \mathcal{B}(x') = b$ by Claim 1. Take a vector $\tilde{z} = z' + e^{(i)}$, $i \neq j$. It is a basic vector of type (A1). Indeed, the single even entry in $\tilde{z}$ is $\tilde{z}_i$, $\Sigma(\tilde{z}) = kb + k - 1 + 1 = k(b+1)$, and from $z'_\ell < b$ we conclude that $\tilde{z}_\ell \leq b + 1$ for all $1 \leq \ell \leq k+1$. Moreover, $\tilde{z} \leq x' + e^{(i)} \leq x$, since $i \neq j$. Thus, $\mathcal{B}(x) \geq b + 1$, a contradiction.

We have proven that z' is either of type (A0) or (A1) and claim that in both cases $\mathcal{B}(x') < \mathcal{B}(x)$ holds. Assume to the contrary that $\mathcal{B}(x') = \mathcal{B}(x)$, which implies that z' is basic of type (A0). Then, $\tilde{z} = z' + d^{(j)} \leq x' + d^{(j)} = x$ is basic of type (A1), because the sum of entries is $k(b(z')+1)$; the parity and maximum conditions holds, since entries are increased by at most 1 and $b(\tilde{z}) = b(z') + 1$. We come to a contradiction: $\mathcal{B}(x') = \mathcal{B}(x) \geq b(z') + 1 = \mathcal{B}(x') + 1$, thus $\mathcal{B}(x') \leq 2s - 1$.

Recall that $\Sigma(z) = k \cdot 2s > 0$. Let $z'' = z - d^{(j)}$ and note that $z'' \leq x' = x - d^{(j)}$. We now prove two conditions under which $\mathcal{B}(x') = \mathcal{B}(x) - 1$ (hence odd), and then show that in all other cases $\mathcal{B}(x')$ is odd as well.

`Case (i):` If $z_j < 2s = b(z)$, then z'' is a basic vector of type (A1): $\Sigma(z'') = k(2s-1)$, the maximum condition and the parity condition hold. For the latter, note that all entries of z are even, thus, the single even entry in z'' is z_j. So $2s = \mathcal{B}(x) > \mathcal{B}(x') \geq 2s - 1$. We conclude that $\mathcal{B}(x') = 2s - 1 = \mathcal{B}(x) - 1$.

`Case (ii):` Now assume that $z_j = 2s = b(z)$, and without loss of generality, x and z are ordered. Some entries in z are less than $2s$, as is explained in the proof of Corollary 1. So

$$z_1 \leq z_2 \leq \cdots \leq z_r < 2s = z_{r+1} = \cdots = z_j = \cdots = z_{k+1}, \quad r < j.$$

There are two subcases.

`Subcase (a):` Let $z_i < x_i$ for some $i \leq r$, that is $z_i < \min\{x_i, b(z)\}$. In this case $z'' = z - d^{(i)} \leq x' = x - d^{(j)}$, $\Sigma(z'') = k \cdot (2s-1)$, and $\max(z'') \leq 2s - 1$. The parity condition also holds, since the single even entry in z'' is z''_i. Thus z'' is a basic vector of type (A1). It implies $2s - 1 \geq \mathcal{B}(x') \geq b(z'') = 2s - 1$. Therefore, the move $d^{(j)}$ leads to a position with $\mathcal{B}(x') = 2s - 1 = \mathcal{B}(x) - 1$.

`Subcase (b):` Let $z_i = x_i$ for all $i \leq r$. In this case, we show that $\mathcal{B}(x')$ has to be odd. Assume to the contrary that $\mathcal{B}(x') = 2p$. We have proved that $\mathcal{B}(x') < \mathcal{B}(x)$, thus $2p < 2s$. By assumption, z' supports x', thus $z' \leq x'$ and $b(z') = \mathcal{B}(x') = 2p$ is even, so z' is of type (A0). Thus $\Sigma(z') = k \cdot 2p$ and all z'_i are even. For all $i \leq r$, we have $z'_i < x'_i$, since z'_i is even and $x'_i = x_i - 1 = z_i - 1$ is odd (since z is basic of type (A0)). For $i > r$, we have $z'_i \leq 2p \leq 2s - 2 = z_i - 2 \leq x_i - 2 \leq x'_i - 1$. Therefore $z'' = z' + d^{(i)} \leq x'$ for all i, z'' is basic of type (A1), and $b(z'') = 2p + 1 > \mathcal{B}(x')$, a contradiction. □

Lemma 3. *If x is supported by a basic vector z of type* (A1)*, then $\mathcal{B}(x') \geq \mathcal{B}(x)-2$ for all moves $x \to x'$ and $\mathcal{B}(x') = \mathcal{B}(x) - 1$ for the M-move.*

Example 2. The equality $\mathcal{B}(x') = \mathcal{B}(x)$ is possible. Take $k = 4$, $x = (5, 5, 7, 8, 9)$, and $x' = x - d^{(2)} = (4, 5, 6, 7, 8)$. The vector $(3, 5, 6, 7, 7)$ is 7-basic and $(3, 5, 6, 7, 7) \leq x' \leq x$. Thus, $\mathcal{B}(x) \geq \mathcal{B}(x') \geq 7$ (here we take into account Claim 1). From $\Sigma(x) = 34 = 4 \cdot 8 + 2$ we conclude $\mathcal{B}(x) \leq 8$. Due to Lemma 1, x is not supported by basic vectors of type (B). If $z \leq x$ and z is even, then $\Sigma(z) \leq 4+4+6+8+8 = 30 < 4\cdot 8$. Thus $\mathcal{B}(x) < 8$. We conclude that $7 = \mathcal{B}(x) = \mathcal{B}(x')$. Note that x is also supported by other 7-basic vectors, say, $(3, 4, 7, 7, 7)$ or $(2, 5, 7, 7, 7)$. But they do not support x'.

Proof of Lemma 3. Recall that z is a basic vector of type (A1). Thus $\Sigma(z) = k \cdot (2s + 1)$ and there is exactly one even entry z_j in z.

Let $r \neq j$. We are going to prove that $\mathcal{B}(x') \geq 2s - 1 = \mathcal{B}(x) - 2$ for a move $x \to x' = x - d^{(r)}$. It trivially holds if $s = 0$, so we assume that $s \geq 1$. It is enough to indicate a basic vector z' such that $b(z') = 2s - 1$ and $z' \leq x'$. There are two possible cases. (a) There exists $p \neq j$ such that $z_p < 2s + 1$. Define $z'_j = z_j - 1$, $z'_p = z_p - 1$, $z'_i = z_i - 2$ for $i \notin \{j, p\}$. (b) For all $p \neq j$, the equality $z_p = 2s + 1$ holds. In this case $z_j = 0$. Define $z'_j = 0$ and $z'_i = z_i - 2$ for $i \neq j$.

In each case, a total of $2k$ tokens are removed, so $\Sigma(z') = k \cdot (2s - 1)$. There is either no switches, or one switch each from odd to even and from even to odd, so the parity condition holds. We assume that $s \geq 1$, thus $\max(z') \leq 2s - 1$ in both cases, because $z_j < 2s + 1$, so $z'_j < 2s$, and $z'_p < 2s$ in the first case. Thus, z' is basic and $z' \leq x'$ by construction in case (a) and the non-negativity of x'_j due to $s \geq 1$ in case (b). Therefore $\mathcal{B}(x') \geq b(z') = 2s - 1$.

For a move $x \to x' = x - d^{(j)}$, we are going to prove that

$$\mathcal{B}(x) = 2s + 1 \geq \mathcal{B}(x') \geq 2s = \mathcal{B}(x) - 1. \tag{2}$$

The first inequality follows from Claim 1. The vector $z' = z - d^{(j)}$ is a basic vector of type (A0) such that $z' \leq x'$. Indeed, all entries in z' are even, $\Sigma(z') = k \cdot 2s$, and $\max(z') \leq 2s$ since z_j is even, so $z_j < 2s + 1$. It implies the second inequality.

Consider now the M-move $x \to x' = x - d^{(e(x))}$. Let $\tilde{z}$ be a basic vector supporting x. W.l.o.g. we assume that x and $\tilde{z}$ are ordered. It appears that x is also supported by a basic vector z such that $z_{e(x)}$ is even. Let r be the index of the unique even entry in $\tilde{z}$. Compare $e(x)$ and r.

`Case (i):` If $e(x) = r$ then $z = \tilde{z}$.

`Case (ii):` Assume that $e(x) < r \leq k + 1$. In this case $\tilde{z}_{e(x)}$ is odd. So, if $x_{e(x)}$ is even, then $\tilde{z}_{e(x)} < x_{e(x)}$. If $x_{e(x)}$ is odd, then x has only odd entries and $x_{e(x)}$ is a maximal element. So, by definition of $e(x)$, $x_r = x_{e(x)}$ since x is ordered. Then $\tilde{z}_{e(x)} \leq \tilde{z}_r < x_r = x_{e(x)}$, where the inequality follows from parity considerations. Thus in both cases, $\tilde{z}_{e(x)} < x_{e(x)}$. Therefore $z \leq x$ for $z = \tilde{z} + e^{(e(x))} - e^{(r)}$. The vector z is also basic. The single even entry in z is $z_{e(x)}$. The sum of entries in z is $kb(\tilde{z})$. Since $\tilde{z}_{e(x)}$ is odd, $\tilde{z}_r$ is even and $\tilde{z}$ is ordered, we have $\tilde{z}_{e(x)} < \tilde{z}_r$ and $z_{e(x)} = 1 + \tilde{z}_{e(x)} \leq \tilde{z}_r < b(\tilde{z})$. Thus $\max(z) \leq b(\tilde{z})$. Therefore z is the required basic vector.

`Case (iii):` Finally, let $e(x) > r$. In this case $\tilde{z}_r$ is even by construction and x_r is odd by definition of $e(x)$. Hence $\tilde{z}_r < x_r$ and $z \leq x$ for $z = \tilde{z} + e^{(r)} - e^{(e(x))}$. Similarly to the previous case, basic vector conditions on the sum of entries and on parities of entries hold. The condition on $\max(z)$ also holds, since $\tilde{z}_{e(x)}$ is odd and $\tilde{z}_r < \tilde{z}_{e(x)} \leq b(\tilde{z})$. Therefore z is the required basic vector.

By Equation (2), to prove $\mathcal{B}(x') = \mathcal{B}(x) - 1 = 2s$ it remains to exclude the case $\mathcal{B}(x') = 2s + 1$. Suppose for contradiction that $\mathcal{B}(x') = 2s + 1$ and a basic vector $z' \leq x'$ supports x'. Since $b(z') = 2s + 1$, vector z' is basic of type (A1). Let r be the index of the single even entry in z'. We now create several basic vectors based on z' to show that x is odd, $z'_{e(x)} = x_{e(x)}$, $z'_r = x_r - 1$, $z'_i = x_i - 2$ for $i \neq e(x), r$, and that $\max(x) = x_{e(x)} \leq 2s + 1$. This implies that

$$\Sigma(x) = 1 + 2(k-1) + \Sigma(z') = 1 + 2(k-1) + k(2s+1) = k(2s+2) + k - 1,$$

and that x is a basic vector of type (B), which contradicts to the assumption that x is supported by a basic vector of type (A1).

Let $z'' = z' + d^{(r)}$. Note that $\Sigma(z'') = \Sigma(z') + k = (2s+2)k$, all entries of z'' are even, and $\max(z'') \leq \max(z') + 1 = 2s + 2$, thus z'' is basic of type (A0). Since $b(z'') = 2s + 2$, $z'' \nleq x$, so $z''_\ell > x_\ell$ for some ℓ. Since $z'_\ell \leq x'_\ell \leq x_\ell$, we conclude that $\ell \neq r$ and that $z'_\ell = x'_\ell = x_\ell$. Therefore, $\ell = e(x)$ and $x_{e(x)} = z'_{e(x)}$ is odd. By definition of $e(x)$, it implies that x is odd and that $x_i \leq x_{e(x)} = z'_{e(x)} \leq 2s + 1$ for all $1 \leq i \leq k + 1$. All together, due to parity considerations (z'_r is even, z'_i is odd for $i \neq r$), we have the following inequalities:

$$z'_\ell = x'_\ell \leq 2s{+}1,\ z'_r \leq 2s,\ z'_r \leq x'_r = x_r{-}1, \text{ and } z'_i < x'_i = x_i{-}1 \text{ for } i \neq r, e(x). \quad (3)$$

Now let us consider $z''' = z'' + 2e^{(r)} - 2e^{(e(x))}$ and $z^{(4)} = z'' + 2e^{(i)} - 2e^{(e(x))}$ for any $i \neq r, e(x)$. Clearly, $\Sigma(z''') = \Sigma(z^{(4)}) = \Sigma(z'')$, $b(z''') = b(z^{(4)}) = b(z'') = 2s{+}2$, and from Equation (3), $\max(z''') \leq 2s + 2$ and $\max(z^{(4)}) \leq 2s + 2$. Therefore, z''' and $z^{(4)}$ are basic vectors. Thus, $z''' \nleq x$ and $z^{(4)} \nleq x$. The former implies that $z'''_r = z''_r + 2 = z'_r + 2 > x_r$, and thus by Equation (3), $x_r - 2 < z'_r \leq x_r - 1$, that is $z'_r = x_r - 1$. Similarly, from $z^{(4)} \nleq x$, we obtain that $z^{(4)}_i = z'_i + 3 > x_i$. Using Equation (3), we conclude that $x_i - 2 \leq z'_i < x_i - 1$, that is, $z'_i = x_i - 2$ for $i \neq r, e(x)$, which is what we needed to show. □

Lemma 4. $\mathcal{B}(x) = \mathcal{R}(x)$ *for all positions* x.

Proof. To prove $\mathcal{B}(x) = \mathcal{R}(x)$, we verify that both functions satisfy the same recurrence. We can equivalently redefine $\mathcal{R}(x)$ as follows. Let $N^+(x)$ be the set of positions x' such that there exists a move $x \to x'$. Then, the remoteness function is determined by equations

$$\mathcal{R}(x) = \begin{cases} 0, & \text{if } \mathcal{R}(N^+(x)) = \emptyset, \\ 1 + \min(\mathcal{R}(N^+(x)) \cap 2\mathbb{Z}_{\geq 0}), & \text{if } \mathcal{R}(N^+(x)) \cap 2\mathbb{Z}_{\geq 0} \neq \emptyset, \\ 1 + \max \mathcal{R}(N^+(x)), & \text{if } \emptyset \neq \mathcal{R}(N^+(x)) \subseteq 1 + 2\mathbb{Z}_{\geq 0}. \end{cases} \quad (4)$$

Here $\mathcal{R}(S) = \{r : r = \mathcal{R}(x),\ x \in S\}$ is the image of S under the function $\mathcal{R}$.

We want to prove that the same recurrence holds for $\mathcal{B}(x)$, which would imply that these two functions coincide.

For the first line in Equation (4), note that a terminal position in NIM$(k+1, k)$ has at least two zero entries. If $b(z) > 0$ for a basic vector z, then the number of positive entries in z is at least k, since $\Sigma(z) \geq kb(z)$ and $z_i \leq b(z)$. Hence, $\mathcal{B}(x) = 0$ for a terminal position x.

In the opposite direction, if x is a non-terminal position, then at most one of its entries is 0. Thus, for an ordered x, we have $x \geq z = (0, 1, \dots, 1)$ and z is basic of type (A1). Hence, $\mathcal{B}(x) > 0$.

Let x be a non-terminal position and z be a non-zero basic vector that supports x. If $\Sigma(z) = k \cdot 2s + k - 1$, then $\mathcal{B}(x) = 2s > 0$ and z is basic of type (B). By Lemma 1, $\mathcal{B}(x') = \mathcal{B}(x) - 1$ for each move $x \to x'$. Thus, the third line in Equation (4) holds.

If $\Sigma(z) = k \cdot 2s$, then $\mathcal{B}(x) = 2s > 0$ and z is basic of type (A0). By Lemma 2, $\mathcal{B}(x')$ is odd for each move $x \to x'$. Since $2s > 0$, some entries are less than $2s$, as is explained in the proof of Corollary 1. Choose j such that $z_j = \min_i z_i$. Conditions $\Sigma(z) = k \cdot 2s > 0$ and $\max(z) \leq 2s$ imply $z_i > 0$ for $i \neq j$ (at most one entry in z is 0). So, a move $x \to x' = x - d^{(j)}$ is legal and, by Lemma 2, $\mathcal{B}(x') = \mathcal{B}(x) - 1$. Thus, the third line in Equation (4) holds too.

If $\Sigma(z) = k \cdot (2s+1)$, then $\mathcal{B}(x) = 2s+1$ and z is basic of type (A1). By Lemma 3 and Claim 1, $\mathcal{B}(x) \geq \mathcal{B}(x') \geq \mathcal{B}(x) - 2$ for each move $x \to x'$, and $\mathcal{B}(x') = \mathcal{B}(x) - 1$ for some move. Thus, the second line in Equation (4) holds. □

Now we are ready to prove Theorem 1, Theorem 2 and Proposition 1 stated in Section 1. For reader's convenience, we recall the statements.

Theorem 1. *A position $x = (x_1, \dots, x_{k+1})$ of the game NIM$(k+1, k)$ is m-critical, that is, $\mathcal{R}(x) = m$ and $\mathcal{R}(y) \neq m$ for all $y < x$, if and only if one of the following two cases holds:*

(A) $x_1 + \dots + x_{k+1} = km$;

$\max(x_1, \dots, x_{k+1}) \leq m$;

if m is even then x is even (that is, all $k+1$ entries of x are even);

if m is odd then exactly one of the $k+1$ entries of x is even.

(B) $x_1 + \cdots + x_{k+1} = km + k - 1$;

$\max(x_1, \ldots, x_{k+1}) < m$;

m is even, x is odd (that is, all $k+1$ entries of x are odd).

Proof. The theorem follows immediately from Lemma 4, since $\mathcal{B}(z) = b(z)$ for every basic position z, in other words, a basic position supports itself. □

Theorem 2. *The equality $\mathcal{M}(x) = \mathcal{R}(x)$ holds for any position x.*

Proof. By Lemma 4, it is enough to prove that an M-move $x \to x' = x - d^{(e(x))}$ reduces $\mathcal{B}(x)$ by 1. Let z be a non-zero vector supporting x. W.l.o.g. we assume that both x and z are ordered. In particular, $x_i < x_{e(x)}$ for all $i < e(x)$ by definition of $e(x)$.

If $\Sigma(z) = k \cdot 2s + k - 1$, then, by Lemma 1, $\mathcal{B}(x') = \mathcal{B}(x) - 1$ for each move $x \to x'$. So the M-move in x also reduces $\mathcal{B}(x)$ by 1.

If $\Sigma(z) = k \cdot (2s + 1)$, then $\mathcal{B}(x') = \mathcal{B}(x) - 1$ by Lemma 3.

If $\Sigma(z) = k \cdot 2s$, then we need to verify one of two conditions in Lemma 2: either $z_{e(x)} < 2s$ or $z_{e(x)} = 2s$, but $z_i < \min(2s, x_i)$ for some i. If $z_{e(x)} = 2s$, then $e(x) > 1$, since some entries in a basic vector are less than $2s$ and we assume that z is ordered. Thus $z_1 < 2s$, and z_1 is even, since z is a basic vector of type (A0). But $e(x) > 1$ implies that $x_1 < x_{e(x)}$ is odd by the definition of $e(x)$ (x is assumed to be ordered). Thus, $z_1 < \min(2s, x_1)$ and $\mathcal{B}(x - d^{(e(x))}) = \mathcal{B}(x) - 1$, by Lemma 2. □

Proposition 1. *In the game NIM$(k+1, k)$, a position x is an odd P-position if and only if it satisfies (B) of Theorem 1.*

Proof. Let x be an odd P-position and z be a basic vector supporting x. It follows from Lemma 4 that $\mathcal{R}(x) = \mathcal{B}(x)$ is even. If z is of type (B), then $x = z$ by Lemma 1.

If z is of type (A0), then entries of z are even. Hence, $x_i - z_i$ are odd for all $1 \le i \le k+1$. Thus, $\tilde{z} = z + d^{(j)} \le x$, $\Sigma(\tilde{z}) = \Sigma(z) + k$ for arbitrary $1 \le j \le k+1$, and exactly one entry of $\tilde{z}$ is even, while $\max(\tilde{z}) \le b(z) + 1$. Therefore, $\tilde{z}$ is basic of type (A1) and $b(z) = \mathcal{B}(x) \ge b(\tilde{z}) = b(z) + 1$. We come to a contradiction and conclude that this case is impossible. □

3. Structural and Algorithmic Complexity

It follows from Lemma 4 that x is a P-position in NIM$(k+1, k)$ if and only if $\mathcal{R}(x) = \mathcal{B}(x)$ is even.

For a fixed k, conditions (A) and (B) in Theorem 1 are easily expressed in Presburger arithmetic (for example, see [14]): they include inequalities and congruences modulo fixed integers. Therefore the predicate '$\mathcal{R}(x)$ is even' is also expressed in Presburger arithmetic. It implies that the set of P-positions is semilinear [14], i.e. can be expressed as a finite union of solutions of systems of linear inequalities combined with equations modulo some integer (fixed for the set). It was conjectured in [13] that the set of P-positions of any multidimensional subtraction game with nonnegative vectors of differences is semilinear. The conjecture was supported by several sporadic examples from [5,8]. Games NIM$(k+1, k)$ provide an infinite family of nontrivial subtraction games having this property.

Thus, for a fixed k, there exists a very simple linear time algorithm recognizing P-positions in NIM$(k+1, k)$. Theorem 3 is stronger. Recall it.

Theorem 3. *The function $\mathcal{R}$ can be computed in polynomial time, even if k is a part of the input and integers are presented in binary.*

In order to prove Theorem 3, note that, by Lemma 4, it is enough to compute $\mathcal{B}(x) = \mathcal{R}(x)$ or to find a basic vector supporting x. We assume in this section that x is ordered.

Note that, by Lemma 1, basic vectors of type (B) support themselves only. Verifying the condition (B) can be done in polynomial time as well as computing $b(z)$ for a basic vector z of type (B).

For any remaining position, computing $\mathcal{B}(x)$ is simpler if $\mathcal{B}(x)$ is even. To deal with this case we introduce one more auxiliary function, namely

$$\mathcal{E}(x) = \max(b(z) : x \geq z,\ z \text{ is basic of type (A0)}). \tag{5}$$

Thus $\mathcal{E}(x) = \mathcal{B}(x)$ if and only if $\mathcal{B}(x)$ is even provided x is not basic of type (B).

To compute $\mathcal{E}(x)$ it is sufficient to take into account only basic vectors in a restricted form. Namely, let an ordered $(k+1)$-dimensional vector x and integers t and b satisfy the conditions

$$\begin{aligned}
&2 \leq t \leq k+2, && \\
&x_t \geq b \geq 2 \cdot \lfloor x_{t-1}/2 \rfloor, && \text{if } t \leq k+1, \\
&b = 2 \cdot \lfloor x_{k+1}/2 \rfloor, && \text{if } t = k+2, \\
&\ell_1 = bk - \sum_{i=2}^{t-1} 2 \cdot \lfloor x_i/2 \rfloor - b(k+2-t) \leq x_1. &&
\end{aligned}$$

Then a $(k+1)$-dimensional vector $\ell(x, b, t)$ is defined as follows:

$$\ell(x,b,t)_i = \begin{cases} b, & t \leq i \leq k+1, \\ 2 \cdot \lfloor x_i/2 \rfloor, & 2 \leq i < t, \\ \ell_1 = bk - \sum_{i=2}^{k+1} \ell(x,b,t)_i\,, & i = 1. \end{cases} \tag{6}$$

Note that the first line is meaningful for $t \le k+1$. By definition $\Sigma\big(\ell(x,b,t)\big) = bk$, and, for even b, all entries of $\ell(x,b,t)$ are even. Moreover, $\ell(x,b,t)_i \le b$ and $\ell(x,b,t) \le x$, since x is ordered. Thus, $\ell(x,b,t)$ is basic of type (A0) for even b. Let $L(x)$ be the set of all vectors $\ell(x,b,t)$ for all even b and all $2 \le t \le k+2$.

Example 3. Let $x = (2,4,6,6)$. So $k = 3$ and possible vectors $\ell(x,b,t)$ are shown in the table:

t	b	$\ell(x,b,t)$
2	2, 3, 4	$(0,2,2,2)$, $(0,3,3,3)$, $(0,4,4,4)$
3	4, 5, 6	$(0,4,4,4)$, $(1,4,5,5)$, $(2,4,6,6)$
4	6	$(2,4,6,6)$
5	6	$(2,4,6,6)$.

For the last line, b should satisfy the conditions $b = 2\cdot\lfloor x_{k+1}/2\rfloor = 6$ and $3b-16 \le 2$, the latter being equivalent to $b \le 6$. So $b = 6$. Removing vectors corresponding to odd values of b, we get $L(x) = \{(0,2,2,2),\ (0,4,4,4),\ (2,4,6,6)\}$.

Lemma 5. *For all x there exists $z^*(x) \in L(x)$ such that $z^*(x)$ is basic of type (A0), $z^*(x) \le x$, and $b(z^*(x)) = \mathcal{E}(x)$.*

Proof. Recall the definition of the reverse lexicographical order on $(k+1)$-dimensional integer vectors: $x <_{\text{lex}} y$ if $x_i = y_i$ for all $i > r$ and $x_r < y_r$. Among basic vectors z of type (A0) such that $z \le x$ and $b(z) = \mathcal{E}(x)$, choose the maximal one with tespect to the reverse lexicographical order. Denote it by $z^*(x)$. We are going to prove that $z^*(x) \in L(x)$.

At first, note that $z^*(x)$ is ordered. The arguments we use are similar to the proof of Claim 2. If $z^*(x)_i > z^*(x)_{i+1}$, then $z^*(x)_{i+1} < x_i$ and $z^*(x)_i \le x_i \le x_{i+1}$, since x is assumed to be ordered. Swapping $z^*(x)_i$ and $z^*(x)_{i+1}$ gives a vector $\tilde{z}$ such that $\tilde{z} \le x$, $\tilde{z}$ is basic of type (A0), $b(\tilde{z}) = \mathcal{E}(x)$, and $z^*(x) <_{\text{lex}} \tilde{z}$, a contradiction to the choice of $z^*(x)$.

For brevity, we set $b = b(z^*(x))$. From $\Sigma\big(z^*(x)\big) = kb$ we conclude that either all entries of $z^*(x)$ are strictly positive, or exactly one entry is 0 and the rest of them are b. In the second case, $z^*(x) = \ell(x,b,2)$ (there are no i such that $2 \le i < 2$) and the lemma follows. So, in the sequel we assume the first case.

For the sake of contradiction, suppose that $z^*(x) \notin L(x)$. Let t^* be the smallest index such that $z^*(x)_{t^*} = b$ (if $z^*(x)_i < b$ for all i then $t^* = k+2$) and i^* be the largest index such that $x_{i^*} - z^*(x)_{i^*} \ge 2$ and $1 \le i^* < t^*$ (if $x_i - z^*(x)_i \le 1$ for all $1 \le i < t^*$ then $i^* = 0$). Note that $t^* > 1$ since $\Sigma(z^*) = kb$, and $i^* > 1$, otherwise $z^*(x) = \ell(x,b,t^*) \in L(x)$. Define z as follows:

$$z_i = \begin{cases} z^*(x)_i - 2, & i = i^* - 1, \\ z^*(x)_i + 2, & i = i^*, \\ z^*(x)_i & \text{otherwise.} \end{cases}$$

All entries of z are even, and $\Sigma(z) = \Sigma\big(z^*(x)\big) = kb$, and $z_{i^*} \le \min(x_{i^*}, b)$. So z is basic of type (A0), $z \le x$, and $z^*(x) <_{\text{lex}} z$, a contradiction to the choice of $z^*(x)$. □

Proof of Theorem 3. It follows from Lemma 5 that, to compute $\mathcal{E}(x)$, only vectors from $L(x)$ need to be considered. Therefore we express $\mathcal{E}(x)$ in the following form:

$$\begin{aligned}
&\mathcal{E}(x) = \max_{2\le t\le k+2} b_t(x),\\
&b_t(x) = \max\{b : b \text{ is even}, \ \ell(x,b,t) \text{ does exist}\}.
\end{aligned}$$

If for some pairs x, t, conditions in the definition of $b_t(x)$ are inconsistent, then we set $b_t(x) = -\infty$. It follows from Equation (6) that $b_t(x) = 2s^*$, where s^* is the maximum in the optimization problem

$$\begin{aligned}
&s \to \max, \ s \text{ is a non-negative integer},\\
&2s \le x_t \ (\text{if } t < k+2),\\
&2s \ge 2\cdot\lfloor x_{t-1}/2\rfloor,\\
&x_1 \ge \ell(x,2s,t)_1 = 2sk - \sum_{i=2}^{t-1} 2\cdot\lfloor x_i/2\rfloor - 2s(k-t+2).
\end{aligned} \tag{7}$$

The third inequality in Equation (7) is equivalent to

$$2s(t-2) \le \sum_{i=2}^{t-1} 2\cdot\lfloor x_i/2\rfloor + x_1,$$

so it trivially holds for $t = 2$.

Anyway, we get a system of at most three linear inequalities in one integer variable. In polynomial time, one can check the satisfiability of the system and can find the maximum value s^* provided the system is satisfiable. Thus $b_t(x)$ and $\mathcal{E}(x)$ are computable in polynomial time.

To complete the proof we explain how to compute $\mathcal{B}(x)$ efficiently using an oracle computing $\mathcal{E}(x)$. As it said, computing $\mathcal{B}(x)$ for basic positions of type (B) can be done in polynomial time as well as detecting these positions. For the remaining positions, the supporting vectors are of type either (A0) or (A1), by Lemma 1. Note that $\mathcal{E}(x) \le \mathcal{B}(x)$ and equality holds exactly for x such that $\mathcal{B}(x)$ is even. Let $x \to x' = x - d^{(e(x))}$ be an M-move. If $\mathcal{B}(x)$ is odd then $\mathcal{B}(x')$ is even, by Theorem 2 and the fact that $\mathcal{B}(x) = \mathcal{B}(x') + 1 = \mathcal{E}(x') + 1$. Therefore, $\mathcal{E}(x') = \mathcal{B}(x') = \mathcal{B}(x) - 1 > \mathcal{E}(x) - 1$. Similarly, if $\mathcal{B}(x)$ is even then we have $\mathcal{E}(x) = \mathcal{B}(x) = \mathcal{B}(x') + 1 > \mathcal{E}(x') + 1$.

It gives us a rule to compute $\mathcal{B}(x)$ from $\mathcal{E}(x)$:

$$\mathcal{B}(x) = \begin{cases} \mathcal{E}(x), & \text{if } \mathcal{E}(x) > \mathcal{E}(x') + 1,\\ \mathcal{E}(x') + 1, & \text{if } \mathcal{E}(x) < \mathcal{E}(x') + 1, \end{cases}$$

where $x \to x'$ is the M-move. □

4. Plans for Future Research

The concept of m-critical positions is valid for arbitrary impartial games. However, characterizing these positions remains an open problem already for NIM(n, k).

Conjecture 1. For any integers k, n, m such that $0 < k < n$ and $0 \leq m$, the inequalities

$$km \leq (x_1 + \cdots + x_n) < k(m+1) \quad \text{and} \quad \max(x_1, \ldots, x_n) \leq m$$

hold for any m-critical position $x = (x_1, \ldots, x_n)$ of the game NIM(n, k).

The case $k = 1$ is trivial and $k = n - 1$ is studied in the present paper. Yet, characterizing m-critical positions for arbitrary k, n, and m seems difficult. Consider, for example, $k = 3$, $n = 5$, and $m = 8$. Computations show that $(1, 3, 7, 7, 7), (1, 5, 5, 7, 7), (3, 3, 5, 7, 7)$, and $(5, 5, 5, 5, 5)$ are m-critical positions of the game NIM$(5, 3)$, while $(3, 5, 5, 5, 7)$ is not; instead, $(3, 5, 5, 6, 7)$ is.

The function $\mathcal{M}$ and the M-rule can be generalized as follows. As before, consider integers n and k such that $0 < k < n$ and add an integer $\ell \geq 2$, replacing the previous value $\ell = 2$. Introduce the GM-rule $x \to x'$ as follows.

Let $\mu = \mu(x)$ denote the number of entries of x that are multiples of ℓ. If $\mu \geq n - k$, keep the smallest $n - k$ of these entries; otherwise, if $\mu < n - k$, keep all entries that are multiples of ℓ, as well as the $n - k - \mu$ largest among the other entries of x. In both cases reduce the remaining k entries by 1.

The GM-rule defines a unique *GM-move*. These concepts generalize the M-rule and M-moves considered above, for which $k = n - 1$ and $\ell = 2$. The GM-rule uniquely defines the GM function $\mathcal{M} = \mathcal{M}(n, k, \ell, x)$. An efficient algorithm computing $\mathcal{M}(n, k, \ell, x)$ in polynomial time in the variables n, k, ℓ and $\sum_{i=1}^{n} \log(|x_i|+1)$ was suggested in [11] for $k = n-1$ and in [12] for any k between 1 and $n-1$. However, in general, it remains an open question, to determine how the function $\mathcal{M}(n, k, \ell, x)$ is related to the remoteness functions of impartial games. Some partial results and conjectures in this direction were recently suggested.

In [9], for $n = 4, 5$, and $k = 2$, computations confirm that the functions $\mathcal{M}$ and $\mathcal{R}$ are still closely related: their difference is given by simple explicit formulas. In [10], for $n \leq 30$ and $k = n - 1$, computations confirm that the functions $\mathcal{M}$ and $\mathcal{R}$ are also related, in a pretty complicated way, for the misère version of the game.

Acknowledgements. The authors are thankful to an anonymous reviewer for valuable remarks and suggestions. This research was prepared within the framework of the HSE University Basic Research Program.

References

[1] M.H. Albert, R.J. Nowakowski, and D. Wolfe, *Lessons in Play: an Introduction to Combinatorial Game Theory*, second ed., A. K. Peters Ltd., Wellesley, MA, 2007.

[2] E.R. Berlekamp, J.H. Conway, and R.K. Guy, *Winning Ways for Your Mathematical Plays*, second ed., vol. 1-4, A.K. Peters, Natick, MA, 2001-2004.

[3] E. Boros, V. Gurvich, K. Makino, and M. Vyalyi, Computing remoteness functions of Moore, Wythoff, and Euclid's games, preprint, `arXiv:2311.02685`.

[4] C.L. Bouton, Nim, a game with a complete mathematical theory, *Ann. of Math. (2)* **3** (1901-1902), 35-39.

[5] N. Chikin, V. Gurvich, K. Knop, M. Paterson, and M. Vyalyi, More about Exact Slow k-nim, *Integers* **21** (2021), #G4.

[6] P.M. Grundy, Mathematics of games, *Eureka* **2** (1939), 6-8.

[7] V. Gurvich and N.B. Ho, Slow k-Nim, *RUTCOR Research Report* **3** (2015). Available online also at `arXiv:1508.05777`.

[8] V. Gurvich, S. Heubach, N.B. Ho, and N. Chikin, Slow k-Nim, *Integers* **20** (2020), #G3.

[9] V. Gurvich, V. Maximchuk, and M. Naumova, On remoteness functions of Exact Slow NIM, $\text{NIM}^1_=(4,2)$ and $\text{NIM}^1_=(5,2)$, *SSRN* (2023) 8 pp., `http://dx.doi.org/10.2139/ssrn.4647107`

[10] V. Gurvich, V. Maximchuk, G. Miheenkov, and M. Naumova, On remoteness functions of Exact Slow k-NIM with $k+1$ piles in normal and misère versions, preprint, `arXiv:2311.13511`.

[11] V. Gurvich and M. Naumova, GM-rule and its applications to impartial games, preprint, `arXiv:2311.03257`.

[12] V. Gurvich and M. Naumova, Screw discrete dynamical systems and their applications to Exact Slow NIM, *Discrete Appl. Math.* **358** (2024), 382–394.

[13] V. Gurvich and M. Vyalyi, On computational hardness of multidimensional subtraction games, *Algorithms* **14** (3) (2021), 71.

[14] C. Haase, A survival guide to Presburger arithmetic, *ACM SIGLOG News* **5** (2018), 67-82.

[15] T.A. Jenkyns and J.P. Mayberry, The skeletion of an impartial game and the Nim-function of Moore's Nim_k, *Internat. J. Game Theory* **9** (1) (1980), 51-63.

[16] E. H. Moore, A generalization of the game called Nim, *Ann. of Math. (2)* **11** (3) (1910), 93-94.

[17] J. von Neumann, O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, NJ, 1944.

[18] G. Nivasch, More on the Sprague-Grundy function for Wythoff's game: on the location of the q-values, in *Games of No Chance 3*, MSRI Publications 56, Cambridge Univ. Press, Berkeley, Ca., 2009.

[19] C.A.B. Smith, Graphs and composite games, *J. Combin. Theory* **1** (1966), 51-81.

[20] R. Sprague, Über mathematische Kampfspiele, *Tohoku Math. J.* **41** (1936), 438-444.

[21] W.A. Wythoff, A modification of the game of Nim, *Niew Arch. Wiskd.* **7** (1907), 199–202.

Appendix: P- and N-positions of NIM(4,3)

Here we use the notation $[A]$ for the indicator function: $[A] = 1$ if A is true and $[A] = 0$ otherwise.

Case 0. $\boldsymbol{x_1 + x_2 + x_3 + x_4 = 0 \pmod 3}$

- If x_1 and x_2 are both odd, then x is an N-position.
- Otherwise, if $(x_3 - x_2 - x_1) \geq 0$, then x is a P-position if and only if $(x_1 + x_2)$ is even.
- Otherwise, if $(x_3 - x_2 - x_1) = -1$, then x is a P-position if and only if $(x_1 + x_2)$ is odd.
- Otherwise, if $(x_1 + x_3 - x_2)$ is even, then x is a P-position if and only if $(x_1 + x_2)$ is even.
- Otherwise, x is a P-position if and only if $(x_2 + p + q)$ is even, where

$$p = [(x_1 + x_3 - x_2) \equiv 3 \pmod 4],$$
$$q = [(x_1 + x_2 + x_3 - 2x_4 - 3) = 12k,\ k \in \mathbb{Z}_{\geq 0}].$$

Case 1. $\boldsymbol{x_1 + x_2 + x_3 + x_4 = 1 \pmod 3}$

- If x_1 and x_2 are both odd, then x is an N-position.
- Otherwise, if $(x_3 - x_2 - x_1) \geq 0$, or $(x_3 - x_2 - x_1)$ is even, or $(x_3 - x_2 - x_1) = -3$, then x is a P-position if and only if $(x_1 + x_2)$ is even.
- Otherwise, if $(x_3 - x_2 - x_1) = -1$ or $(x_3 - x_2 - x_1) = -5$, then x is a P-position if and only if $(x_1 + x_2)$ is odd.
- Otherwise, x is a P-position if and only if $(x_2 + p + q)$ is odd, where

$$p = [(x_1 + x_3 - x_2) \equiv 1 \pmod 4],$$
$$q = [(x_1 + x_2 + x_3 - 2x_4 - 7) = 12k,\ k \in \mathbb{Z}_{\geq 0}].$$

Case 2. $\boldsymbol{x_1 + x_2 + x_3 + x_4 = 2 \pmod 3}$

- Let x_1 and x_2 be both odd:
  - If $(x_3 - x_2 - x_1) \geq 0$ or $(x_3 - x_2 - x_1) \in \{-1, -3, -4, -7\}$, then x is an N-position.

- Otherwise, x is a P-position if and only if p is odd, where

$$p = \begin{bmatrix} (x_1 + x_2 + x_3 - 2x_4 - c) = 12k, \ k \in \mathbb{Z}_{\geq 0}, \\ c = 2 + 3\big((x_1 + x_2) \bmod 4\big) \end{bmatrix}.$$

- Let x_1 be odd and x_2 even.
  - If $(x_3 - x_2 - x_1) \geq 0$ or $(x_3 - x_2 - x_1) \in \{-2, -3, -6\}$, then x is an N-position.
  - Otherwise, if $(x_3 - x_2 - x_1) = -1$, then x is an P-position.
  - Otherwise, x is a P-position if and only if $(p + q)$ is odd, where

$$p = [(x_1 + x_3 - x_2) \equiv 1 \pmod 4],$$
$$q = \begin{bmatrix} (x_1 + x_2 + x_3 - 2x_4 - c) = 12k, & k \in \mathbb{Z}_{\geq 0}, \\ c = 5, & \text{if } (x_1 + x_2) \text{ is odd,} \\ c = (8 - 3((x_1 + x_2) \bmod 4)), & \text{otherwise.} \end{bmatrix}$$

- Let x_1 be even.
  - If $(x_3 - x_2 - x_1) \geq 0$ or $(x_3 - x_2 - x_1) \in \{-2, -3, -6\}$, then x is a P-position if and only if x_2 is even.
  - Otherwise if $(x_3 - x_2 - x_1) = -1$, then x is a P-position if and only if x_2 is odd.
  - Otherwise x is a P-position if and only if $(p + q + x_2)$ is even, where

$$p = [(x_1 + x_3 - x_2) \equiv 3 \pmod 4],$$
$$q = \begin{bmatrix} (x_1 + x_2 + x_3 - 2x_4 - c) = 12k, & k \in \mathbb{Z}_{\geq 0}, \\ c = 5, & \text{if } (x_1 + x_2) \text{ is odd,} \\ c = 2 + 3\big((x_1 + x_2) \bmod 4\big), & \text{otherwise.} \end{bmatrix}$$