ON WINNING STRATEGIES IN SYLVER COINAGE WHEN 4 HAS BEEN CHOSEN

Morgan Boyers
Dept. of Math. and Stats., Northern Arizona University, Flagstaff, Arizona
mdb527@nau.edu

Natalie Burton
Dept. of Math. and Stats., Northern Arizona University, Flagstaff, Arizona
nrb293@nau.edu

Jeff Rushall
Dept. of Math. and Stats., Northern Arizona University, Flagstaff, Arizona
jlr34@nau.edu

James Warner
Dept. of Math. and Stats., Northern Arizona University, Flagstaff, Arizona
jtw346@nau.edu

Tara Zurick
Dept. of Math. and Stats., Northern Arizona University, Flagstaff, Arizona
trz26@nau.edu

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Abstract

We verify several statements or claims made about Sylver Coinage game scenarios in which 4 has been chosen. In particular, we verify unproven statements by Conway and others, and provide an algorithm for achieving and maintaining a winning strategy in some of these games.

1. Introduction

In the game of Sylver Coinage, two players alternate choosing natural numbers. Each choice removes from play all natural numbers that can be expressed as linear combinations, over the nonnegative integers, of previous choices. The player who is forced to choose 1 loses.

The game is named in honor of J. J. Sylvester (1814-1897) and is the topic of

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Chapter 18 of *Winning Ways for your Mathematical Plays* by Berlekamp, Conway and Guy [1]. Early results on Sylver Coinage appear in both [6] and [4]. Much of the published research on Sylver Coinage has focused on games in which 4 has been played (see [1], [2], and [4]); there are several reasons for this assumption, which will be addressed in detail shortly. Some recent publications have established connections between Sylver Coinage and numerical semigroups (see [2] and [3]) to prove results associated with achieving and maintaining winning and losing positions, in addition to developing strategies for playing the game from specific positions.

Much of the available literature on Sylver Coinage focuses on determining if a given game position is an $\mathcal{N}$-position (a winning position for the next player) or a $\mathcal{P}$-position (a winning position for the previous player). For instance, if 2 and 3 are the first two moves in a game of Sylver Coinage, the only move left for the next player is the losing choice of 1. Hence, the position $\{2, 3\}$ is a $\mathcal{P}$-position. Similarly, the position $\{2\}$ is an $\mathcal{N}$-position if one assumes the player given that position plays optimally. In fact, unless otherwise stated we will assume that both players will play optimally at each stage of any game of Sylver Coinage.

In [1], [4], and [6], considerable attention is devoted to Sylver Coinage game positions in which 4 has been played. In particular, each author includes statements that, while reasonable, are not explicitly proven. Rigorous justifications of said claims do not yet appear in the literature, and attempting to verifying these statements led to the results presented in this document.

The rest of this paper is structured as follows. In Section 2 we will present background information on Sylver Coinage, and discuss important connections between Sylver Coinage and various properties of numerical semigroups. In Sections $3, 4, 5$, and 6 we will state and prove several of the aforementioned unproven claims from various sources. In particular, we will will provide a recursive algorithm for maintaining a winning position in Sylver Coinage games in which 4 has been played. In Section 7 we present some interesting open problems.

2. Background Information

Definition 1. *Sylver Coinage* is a two-player game played on the set $\mathbb{N}$ of natural numbers. The rules of Sylver Coinage are:

1. Players alternate choosing available natural numbers.
2. If $a_1, a_2, ..., a_k$ denote the first k choices in the game, the available natural numbers on the next turn are the elements of the set
$$\mathbb{N} \setminus \{n_1a_1 + n_2a_2 + ... + n_ka_k | n_i \in \mathbb{N} \cup \{0\}\}.$$
3. The player who chooses the natural number 1 loses the game.

Example 1. As an example of a game of Sylver Coinage, suppose Player 1 begins by choosing 5. Player 2 can then respond with any natural number that is not a multiple of 5, say 6. Next, Player 1 can choose any natural number that is not of the form $5x + 6y$ for some $x, y \in \mathbb{N} \cup \{0\}$. The entire game might look like this:

Available Plays	Player 1	Player 2
$\mathbb{N}$	5	
$\mathbb{N} \backslash 5\mathbb{N}$		6
$\{1, 2, 3, 4, 7, 8, 9, 13, 14, 19\}$	7	
$\{1, 2, 3, 4, 8, 9\}$		4
$\{1, 2, 3\}$	3	
$\{1, 2\}$		2
$\{1\}$	1	

Player 1 is forced to play 1 and loses.

The following facts about Sylver Coinage are well known in the literature; the curious reader can consult [2] for details.

1. If $a_1, a_2, ..., a_k$ denote the natural numbers that have been chosen in a game of Sylver Coinage and if $\gcd\{a_1, a_2, ..., a_k\} = 1$, then the set of available plays is finite.

2. Every game of Sylver Coinage involves a finite number of plays.

3. If both players play intelligently - i.e., if both players use optimal strategy - then the first player to choose $1, 2$, or 3 loses.

Definition 2. The following terms and notational conventions pertain to Sylver Coinage; see [2] and [5] for more information.

1. A *position* in a game of Sylver Coinage is a set $M = \{a_1, a_2, ..., a_k\}$ of natural numbers that have been chosen by the players.

2. The *set of legal plays of a position* $M = \{a_1, a_2, ..., a_k\}$, denoted $L(M)$, is defined as $L(M) = \mathbb{N} \setminus \{n_1a_1 + n_2a_2 + ... + n_ka_k | n_i \in \mathbb{N} \cup \{0\}\}$.

3. Given a position $M = \{a_1, a_2, ..., a_k\}$, we say that a_i is *superfluous* if $L(M) = L(M \setminus \{a_i\})$.

4. We say that a position $M = \{a_1, a_2, ..., a_k\}$ is in *canonical form* if none of the a_i are superfluous. Unless otherwise stated, we will assume that position $M = \{a_1, a_2, ..., a_k\}$ is in canonical form with $a_1 < a_2 < ... < a_k$.

5. We say that a position M is a *finite position* if $L(M)$ is a finite set - that is, if $|L(M)| < \infty$. In this case, we let $F(M)$ denote the largest element of $L(M)$.

6. Given a position M, we say that $x \in L(M)$ is an *end* if the choice of x does not eliminate any other elements of $L(M)$. We say that M is an *ender* if $F(M)$ is the only end.

7. A position is called an $\mathcal{N}$-*position* if the next player to play can win from that position. We often refer to $\mathcal{N}$-positions as *winning positions*.

8. A position is called an $\mathcal{P}$-*position* if the previous player can win from that position. We often refer to $\mathcal{P}$-positions as *losing positions*.

Note that given a position M, the set $\mathbb{N} \setminus L(M)$ is closed under addition. A fundamental result in Sylver Coinage is due to R. L. Hutchings. See [1] for more information.

Theorem 1. *In a game of Sylver Coinage, if Player* 1 *chooses any prime* $p > 3$ *for an opening move, then* $M = \{p\}$ *is a* $\mathcal{P}$*-position.*

At first glance, Hutchings' Theorem appears to imply that Sylver Coinage can be trivialized if a game begins with a choice of any prime 5 or greater. However, the proof of Hutchings' Theorem is nonconstructive, and consequently provides no insight into general strategies either player should employ in these scenarios.

It is well known that any response to the position $M = \{p\}$ results in an ender, which is due to the following results.

Proposition 1. *If* M *is an ender with* $F(M) > 1$*, then every element of* $L(M)$ *less than* $F(M)$ *eliminates* $F(M)$.

Proof. Suppose there exist elements of $L(M)$ that do not eliminate $F(M)$; let $x \in L(M)$ be the maximal such element. Since M is an ender, the choice of x must eliminate another element $y \in L(M)$ with $y > x$. It then follows that y is either an integer multiple of x or y can be written as an integer multiple of x combined with integer multiples of previous choices. In both cases, by Definition 2.3 we may write $y = kx + r_k$ for some $k \in \mathbb{N}$ and some $r_k \notin L(M)$. Since $y > x$, the choice of y would eliminate $F(M)$, and so it must be the case that $F(M) = ly + r_l$, again for some $l \in \mathbb{N}$ and some $r_l \notin L(M)$. Combining these expressions yields $F(M) = l(kx + r_k) + r_l = (lk)x + lr_k + r_l$. Since $r_k, r_l \notin L(M)$, it follows that $lr_l + r_k \notin L(M)$ as well. This in turn implies that the choice of x eliminates $F(M)$, a contradiction, ergo no such x exists. Thus, if M is an ender, every element of $L(M)$ eliminates $F(M)$. □

Proposition 2. *If* M *is an ender with* $F(M) > 1$*, then* M *is an* $\mathcal{N}$*-position.*

Proof. Assume a Sylver Coinage position M is an ender. By Proposition 2.5, every element of $L(M)$ eliminates $F(M)$. Now consider the following scenario: suppose a player on an ender M chooses $F(M)$. If their opponent has a winning response

$m \in L(M) \setminus \{F(M)\}$, then the original player should instead choose m rather than $F(M)$ and reach the same winning position. This implies that M is an $\mathcal{N}$-position. □

The argument outlined in the proof of Proposition 2.6, which allows a player to determine a winning move by considering potential moves that could give their opponent a $\mathcal{P}$-position, is called *strategy stealing*. See [1] for more examples of situations in which strategy stealing has been used to analyze Sylver Coinage positions.

We next present appropriate background information on numerical semigroups, as there are important connections between some types of numerical semigroups and certain Sylver Coinage positions. Consult [5] for more information.

Definition 3. A *numerical semigroup* is a subset S of $\mathbb{N} \cup \{0\}$ satisfying these three conditions:

1. $0 \in S$;
2. S is closed under addition;
3. $|\mathbb{N} \setminus S|$ is finite.

The smallest positive element of S is called the *multiplicity* of S, denoted $m(S)$. The largest element of $\mathbb{N} \setminus S$ is called the *Frobenius number* of S and is denoted $F(S)$. The value of $|\mathbb{N} \setminus S|$ is called the *genus* of S and is denoted $g(S)$.

Example 2. Let $S = \{0, 4, 5, 8, 9, 10\} \cup \{\mathbb{N} : n \geq 12\}$. Then S is a numerical semigroup, with Frobenius number $F(S) = 11$, genus $g(S) = 6$, and $m(S) = 4$.

From [5] we know that given any numerical semigroup S, there exists a unique finite subset $\{a_1, a_2, ..., a_k\}$ of elements in S that is minimal (with respect to containment) and such that for any $s \in S$ there exist $n_1, n_2, ..., n_k \in \mathbb{N}$ such that $s = n_1a_1 + n_2a_2 + ... + n_ka_k$. In this case, we call $\{a_1, a_2, ..., a_k\}$ a *minimal system of generators* for S, denoted $msg(S)$, and will often write $S = \langle a_1, a_2, ..., a_k\rangle$. The number of elements in the minimal generating set for a numerical semigroup S is called the *embedding dimension* of S, denoted $e(S)$. For the numerical semigroup in the last example, $S = \langle 4, 5\rangle$ and $e(S) = 2$. Finally, it is well known (see [5] for details) that if $S = \langle a_1, a_2\rangle$, then $F(S) = a_1 \cdot a_2 - a_1 - a_2$.

From the definition of $F(S)$, we know that if $x \in S$, then $F(S) - x \notin S$. It then follows that $g(S) \geq \frac{F(S)+1}{2}$ if $F(S)$ is odd, and $g(S) \geq \frac{F(S)+2}{2}$ if $F(S)$ is even.

Definition 4. We say that a numerical semigroup is *symmetric* if $F(S)$ is odd and if $g(S) = \frac{F(S)+1}{2}$. We say that a numerical semigroup is *pseudo-symmetric* if $F(S)$ is even and if $g(S) = \frac{F(S)+2}{2}$.

Example 3. The following examples pertain to Definition 2.9.

1. Let $S = \langle 5, 6, 9\rangle = \{0, 5, 6, 9, 10, 11, 12\} \cup \{\mathbb{N} : n \geq 14\}$. Note that $F(S) = 13$ and $g(S) = 7$, and hence S is symmetric.

2. Let $S = \langle 4, 7, 9\rangle = \{0, 4, 7, 8, 9\} \cup \{\mathbb{N} : n \geq 11\}$. Note that $F(S) = 10$ and $g(S) = 6$, and hence S is pseudo-symmetric.

3. Let $S = \langle 5, 6, 7\rangle = \{0, 5, 6, 7\} \cup \{\mathbb{N} : n \geq 10\}$. Note that $F(S) = 9$ and $g(S) = 6$, and hence S is neither symmetric nor pseudo-symmetric.

There are several ways to show that a given numerical semigroup is symmetric or pseudo-symmetric. They include the following, which are well known in the literature; see [3] and [5] for details.

Proposition 3. *If S is a numerical semigroup, then*

1. *If $e(S) = 2$, then S is symmetric.*

2. *If $m(S) = 4, e(S) = 3$, then S is pseudo-symmetric if and only if $S = \langle 4, k, k+2\rangle$ with k odd, $k > 3$.*

Example 4. Let $S = \{0, 4, 8, 9, 11, 12, 13\} \cup \{\mathbb{N} : n \geq 15\}$. Then S is a numerical semigroup, and in fact $S = \langle 4, 9, 11\rangle$. Note that $F(S) = 14$, $g(S) = 8$ and $m(S) = 4$. We conclude by both definition and via Proposition 2.11.2 that S is pseudo-symmetric.

There is a close connection between the elements of a given numerical semigroup and, when it is finite, the set of legal moves in a game of Sylver Coinage. This connection is discussed in detail in [3]. Here are the pertinent results from [3], presented without proof.

Proposition 4. *Let $M = \{a_1, a_2, ..., a_k\}$ be a finite position in a game of Sylver Coinage written in canonical form. Define $S(M) = (\mathbb{N} \cup \{0\}) \setminus L(M)$. Then:*

1. *$S(M)$ is a numerical semigroup with $msg(S) = \{a_1, a_2, ..., a_k\}$.*

2. *If $S(M)$ is symmetric or pseudo-symmetric, then M is an $\mathcal{N}$-position.*

3. *$F(M) = F(S)$.*

To paraphrase: given any finite position M in a game of Sylver Coinage written in canonical form, the complement of $L(M)$ in $\mathbb{N} \cup \{0\}$ corresponds to the numerical semigroup $S(M)$. Moreover, if $S(M)$ is either a symmetric or pseudo-symmetric numerical semigroup, then the position M is an $\mathcal{N}$-position. The curious reader should consult [3] for more information. In the remainder of this paper we will periodically make use of this connection between numerical semigroups and Sylver Coinage positions.

3. Verifying Sicherman's Claim on Unique Responses to $\{4, x\}$

In [6], Sicherman refers to "*...the unique winning move*" when 4 and an odd x are the first two choices in a game of Sylver Coinage. This implies that when $M = \{4, x\}$, only one response allows Player 1 to maintain their winning position. Given a position M, the existence of at least one element of $L(M)$ that is a "*... winning move*" leads to the following useful term.

Definition 5. The *set of optimal responses to a position* $M = \{a_1, a_2, ..., a_k\}$, denoted $O(M)$, is defined as $O(M) = \{\alpha \in L(M) : M \cup \{\alpha\}$ is a $\mathcal{P}$-position$\}$.

Equivalently, Sicherman is claiming that $|O(M)| = 1$. But it is not obvious why this is the case. In order to verify Sicherman's claim, first note that by Proposition 2.11 and Proposition 2.13, any position of the form $M = \{4, x\}$ with $x \geq 5$ and x odd is known to be an $\mathcal{N}$-position.

Proposition 5. *Given the $\mathcal{N}$-position $M = \{4, x\}$ with $x \geq 5$, x odd, and any even $b \in L(M)$, $M_b = \{4, x, b\}$ is also an $\mathcal{N}$-position.*

Proof. Let $M = \{4, x\}, x$ odd, and assume some even $b \in L(M)$ is chosen. Since $L(M)$ contains no multiples of 4, it must be the case that $b \equiv 2 (\text{mod } 4)$. Since every linear combination of 4 and b over $\mathbb{N} \cup \{0\}$ is even, no elements of $L(M)$ congruent to x modulo 4 are eliminated by choosing b. The minimal element of $L(M)$ congruent to $x - 2$ modulo 4 that is eliminated is $x + b$.

Because $F(M) = 4 \cdot x - 4 - x = 3x - 4$ is also congruent to $x - 2$ modulo 4, the number of elements of $L(M)$ congruent to $x - 2$ modulo 4 that are eliminated by choosing b is

$$\frac{3x - 4 - (x + b - 4)}{4} = \frac{2x - b}{4}.$$

Similarly, the number of elements of $L(M)$ congruent to 2 modulo 4 that are eliminated by choosing b is

$$\frac{2x - 4 - (b - 4)}{4} = \frac{2x - b}{4}.$$

As a result, choosing b eliminates $\frac{2x-b}{2}$ elements of $L(M)$.

It then follows that $F(M_b)$ remains odd and the equation

$$g(M_b) = \frac{F(M_b) + 1}{2}$$

holds. Thus, $S(M_b)$ is symmetric, and hence M_b is an $\mathcal{N}$-position.

□

Proposition 6. *Given the $\mathcal{N}$-position $M = \{4, x\}$ with x odd and $x \geq 5$, there is a unique $q \in L(M)$ such that $\{4, x, q\}$ is a $\mathcal{P}$-position, and q must satisfy $q \equiv x - 2$ (mod 4).*

Proof. Let $M = \{4, x\}$ with x odd. Since M is an $\mathcal{N}$-position, there exists some $q \in L(M)$ such that $\{4, x, q\}$ is a $\mathcal{P}$-position. Note that if $q \equiv x \pmod 4$, then $q < x$, and so $\{4, x, q\} = \{4, q\}$ is an $\mathcal{N}$-position. Also note that if $q \equiv 2 \pmod 4$, then the resulting position $\{4, x, q\}$ is an $\mathcal{N}$-position by Proposition 3.1. Thus, the $q \in L(M)$ that results in a $\mathcal{P}$-position $\{4, x, q\}$ must satisfy $q \equiv x - 2 \pmod 4$. Suppose two such choices exist, say q_1, q_2, with $q_1 < q_2$ and $q_1 \equiv q_2 \equiv x - 2 \pmod 4$. But then $\{4, x, q_1\} = \{4, x, q_1, q_2\}$ is a $\mathcal{P}$-position, and so $\{4, x, q_2\}$ cannot be a $\mathcal{P}$-position. The result follows. □

Example 5. Consider the $\mathcal{N}$-position $M = \{4, 7\}$. It is easy to verify that the even elements of $L(M)$ are 2, 6 or 10. By Proposition 3.2, each of these choices results in an $\mathcal{N}$-position. For instance, $\{4, 7, 10\}$ is an $\mathcal{N}$-position.

Example 6. Consider the $\mathcal{N}$-position $M = \{4, 13\}$. The only elements in $L(M)$ congruent to 13 modulo 4 are 5 and 9, and one can show that each yields an $\mathcal{N}$-position. Thus, any $q \in L(M)$ such that $\{4, 13, q\}$ is a $\mathcal{P}$-position must satisfy $q \equiv x - 2 \pmod 4$, as stated in Proposition 3.3. The possible values of q are $q = 7, 11, 15, 19, 23, 27, 31, 35$, and only one of these values of q, namely $q = 7$, yields a $\mathcal{P}$-position.

Given the $\mathcal{N}$-position $M = \{4, x\}$ with x odd, the unique value of q described in Proposition 3.3 for which $\{4, x, q\}$ is a $\mathcal{P}$-position can be found for small values of x via brute force. However, in Section 6 we will present an efficient algorithm for determining the unique q corresponding to any value of x.

We can actually say more about this situation. Namely, every odd natural number $q > 3$ is the winning response to exactly one $\mathcal{N}$-position of the form $M = \{4, x\}$ with $x \in \mathbb{N}$, which is a special case of the *Single Win Theorem* in [6].

Proposition 7. *Given any odd integer $q > 3$, there exists a unique odd $x_q \in \mathbb{N}$ with the property that $\{4, x_q\}$ is an $\mathcal{N}$-position and $\{4, x_q, q\}$ is a $\mathcal{P}$-position.*

Proof. Let $q > 3$ be odd. Then $M_q = \{4, q\}$ is an $\mathcal{N}$-position. By Proposition 4.2, there exists a unique $x_q \in L(M_q)$ such that $\{4, q, x_q\}$ is a $\mathcal{P}$-position, where none of $4, q, x_q$ are superfluous. Then $M_{x_q} = \{4, x_q\}$ is an $\mathcal{N}$-position, and q is the unique element of $L(M_{x_q})$ resulting in $\{4, x_q, q\}$ being a $\mathcal{P}$-position. □

Corollary 1. *Given any odd integer $q > 3$, there exists a unique odd $x_q \in \mathbb{N}$ such that $\{4, q\}$ and $\{4, x_q\}$ are both $\mathcal{N}$-positions and $\{4, q, x_q\}$ is a $\mathcal{P}$-position.*

We note that if $q < x_q$ in Proposition 4.3, then (q, x_q) is precisely a 4-pair as discussed in [4]. We will investigate 4-pairs in more detail in Section 6.

4. Verifying the Conway Table

In [1], Conway et al. state that each entry in Table 1 is the smallest $b = 4k + 2$ which has not appeared earlier in its row or column and, consequently, when ac is odd, $M = \{4, a, b, c\}$ "*is a $\mathcal{P}$-position*" in a game of Sylver Coinage. But the reason these minimal available values of b result in $\mathcal{P}$-positions is not justified.

a \ $c =$	7	11	15	19	23	27	31	35	39	43	47	51	55	59	63	67
5	6															
9	10	6	14													
13			6	10												
17			10	6	14	18	22	26	30							
21			18	14	6	10	26	22	34	30	38					
25			22		10	6	14	18	26		30	34	38	42	46	
29			26		18	14	6	10	22		34	30	42	38	50	46
33					22	26	10	6	14	18						
37					26	22	18	14	6	10	42	38	30	34	54	
41					30	34	38	42	10	6	14	18	22	26	58	
45					34	30	42	38	18	14	6	10	26	22	62	
49					38	42	30	34	46	22	10	6	14	18	26	
53					42	38	34	30	50	26	18	14	6	10	22	
57									38		22	26	10	6	14	18
61						46	50	54	42		26	22	18	14	6	10
65						50	46	58	54		62		34	30	10	6
69								46			50				18	14

Table 1: Values of b for which $\{4, a, b, c\}$ is claimed to be a $\mathcal{P}$-position in [1].

Example 7. Consider the position $\{4, 37, 31\}$ in a game of Sylver Coinage. Note that in the Table 1, 37 is the 9^{th} entry in the a column along the left side of the table, and 31 is the 7^{th} entry in the b row along the top of the table. The table entry in the 9^{th} row and the 7^{th} column is 18, the minimal even number congruent to 2 modulo 4 that does not appear previously in the 9^{th} row and the 7^{th} column of the table. Conway et al. claim that $\{4, 37, 31\}$ is an $\mathcal{N}$-position and that $\{4, 37, 31, 18\}$ is a $\mathcal{P}$-position.

We will now describe how this table was constructed. We begin with a result that appears in [2] as Propositions 10 and 11 and their corollaries.

Proposition 8. *Let $M = \{4, x, y\}$ be a position in a game of Sylver Coinage with xy odd. Then:*

1. *If $x \equiv 1 \pmod 4$ and $y = x + 2$, then $\{4, x, y, 6\}$ is a $\mathcal{P}$-position.*

2. *If* $x \equiv 7 \ (mod\, 8)$ *and* $y = x + 2$*, then* $\{4, x, y, 10\}$ *is a* $\mathcal{P}$*-position.*

3. *If* $x \equiv 5 \ (mod\, 8)$ *and* $y = x + 6$*, then* $\{4, x, y, 10\}$ *is a* $\mathcal{P}$*-position.*

4. *If* $x \equiv 1 \ (mod\, 8)$ *and* $y = x + 6$*, then* $\{4, x, y, 14\}$ *is a* $\mathcal{P}$*-position.*

5. *If* $x \equiv 5 \ (mod\, 8)$ *and* $y = x - 2$*, then* $\{4, x, y, 14\}$ *is a* $\mathcal{P}$*-position.*

Example 8. Consider the position $M = \{4, 17, 19\}$, the first image in Table 2. Note that M satisfies the conditions of Proposition 4.1(i), and so $\{4, 17, 19, 6\}$, the second image in Table 2, is a $\mathcal{P}$-position. One can justify why this is a $\mathcal{P}$-position by noting that each potential choice has a corresponding response that preserves this $\mathcal{P}$-position. For instance, if 13 is chosen, then 15 is the winning response (and vice versa). Similarly, if 9 is chosen, 11 is the winning response (and vice versa), and if 5 is chosen, 7 is the winning response (and vice versa).

0	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23
24	25	26	27
28	29	30	31
32	33	34	35
36	37	38	39
⋮	⋮	⋮	⋮

0	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23
24	25	26	27
28	29	30	31
32	33	34	35
36	37	38	39
⋮	⋮	⋮	⋮

Table 2: The $\mathcal{N}$-position $\{4, 17, 19\}$ and the $\mathcal{P}$-position $\{4, 17, 19, 6\}$.

The results from Proposition 4.1 verify some of the even entries in the Table 1 Conway table [1]. The fact that all entries in Table 1 correspond to $\mathcal{P}$-positions can now be proven.

Theorem 2. *Let* $M_{i,j} = \{4, a_i, c_j\}$*, where* a_i *and* c_j *denote the leading* i^{th}*-row and* j^{th}*-column table entries, respectively, in Table 3. If the* i^{th}*-row,* j^{th}*-column entry in the table is the minimal available* $b_{i,j} \in L(\{M_{i,j}\})$ *of the form* $4k+2$ *that has not yet appeared in either the* i^{th}*-row or the* j^{th}*-column of the table, then* $\{4, a_i, b_{i,j}, c_j\}$ *is a* $\mathcal{P}$*-position.*

Proof. Consider the position $\{4, a_i, c_j\}$. Suppose, say, Player 1 chooses some $b \equiv 2 \pmod 4$ with $b > b_{i,j}$. Then Player 2 can choose $b_{i,j}$, resulting in the position $\{4, a_i, b_{i,j}, c_j\}$. Any subsequent choice from Player 1 results in a position of the form $\{4, a', b', c'\}$ with $a' \leq a_i, b' \leq b_{i,j}, c' \leq c_j$, and at least one of these inequalities must be strict. Such a position is a known $\mathcal{N}$-position from the table. Thus, if $b > b_{i,j}$, it follows that $\{4, a_i, b, c_j\}$ is an $\mathcal{N}$-position, and hence $\{4, a_i, b_{i,j}, c_j\}$ is a $\mathcal{P}$-position, as desired. □

One useful consequence of Theorem 4.3 is that it allows us to identify certain positions of the form $\{4, a, c\}$ with ac odd that are $\mathcal{P}$-positions. Said differently, we can fill in some of the blank areas of the Conway table.

Corollary 2. *Given $M_{i,j}$ and $b_{i,j}$ as above, if $b_{i,j} + 4 = 2a_i$, then $\{4, a_i, c_j + 4\} = \{4, a_i, c_{j+1}\}$ is a $\mathcal{P}$-position, and if $b_{i,j} + 4 = 2c_j$, then $\{4, a_i + 4, c_j\} = \{4, a_{i+1}, c_j\}$ is a $\mathcal{P}$-position. Also, every position $\{4, q, x_q\}$ from Proposition 3.3 can be described as either $\{4, a_i, c_j + 4\}$ or $\{4, a_i + 4, c_j\}$, which satisfy $b_{i,j} + 4 = 2a_i$ or $b_{i,j} + 4 = 2c_j$, respectively.*

Example 9. In Table 1, note that if 4 and 15 have been played, the minimal available even response to 29 is 26. Note that $b_{i,j} + 4 = 2c_j$, and Corollary 4.5 claims that $\{4, 33, 15\}$ is a $\mathcal{P}$-position, which is indeed the case.

A more complete version of the Conway table appears in Table 3. The red integers correspond to results in Proposition 4.2. The blue integers are known odd responses; and the black integers follow from Theorem 4.4. If no response appears, the implication is that $\{4, a, c\}$ is a $\mathcal{P}$-position.

Example 10. The position $\{4, 15, 25\}$ has 22 as a response in Table 3, so $\{4, 15, 25\}$ is an $\mathcal{N}$-position and $\{4, 15, 22, 25\}$ is a $\mathcal{P}$-position. Similarly, the positions $\{4, 15, 49\}$ and $\{4, 15, 33, 49\}$ are $\mathcal{N}$- and $\mathcal{P}$-positions, respectively. The position $\{4, 21, 51\}$ has no response given in Table 3, so it must be a $\mathcal{P}$-position.

It is worth noting that in some positions of the form $M = \{4, a, c\}$ as described above, $|O(M)| > 1$. For instance, if $M = \{4, 41, 43\}$, then both 17 and 6 are optimal responses.

Example 11. Consider the position $\{4, 19, 21\} = \{4, a_5, c_4\}$, and assume that all previous even winning responses from Table 3 are known - i.e., all entries $b_{i,j}$, with $i \leq 5$ and $j \leq 4$ and at least one inequality being strict, are known. In the column above $b_{5,4}$ we see the entries 6 and 10, and in the row to the left of $b_{5,4}$ we see 18. The minimal available even integer is 14, and hence $b_{5,4} = 14$. That is, $\{4, 19, 21, 14\}$ is a $\mathcal{P}$-position.

5. An Algorithm for Finding Winning Moves to $\{4, x\}$ Positions

In [4], given a position $M = \{a_1, a_2, ..., a_k\}$ in Sylver Coinage, Guy's second question asks this: "*Is there an effective technique for producing good replies when such exist?*" In general, this remains an open question.

a \ $c =$	7	11	15	19	23	27	31	35	39	43	47	51	55	59	63	67
5	6		11	11	11	11	11	11	11	11	11	11	11	11	11	11
9	10	6	14		19	19	19	19	19	19	19	19	19	19	19	19
13		5	6	10	7	7	7	7	7	7	7	7	7	7	7	7
17	13	5	10	6	14	18	22	26	30		43	43	43	43	43	43
21	13	5	18	14	6	10	26	22	34	30	38		51	51	51	51
25	13	5	22	9	10	6	14	18	26	17	30	34	38	42	46	
29	13	5	26	9	18	14	6	10	22	17	34	30	42	38	50	46
33	13	5		9	22	26	10	6	14	18	15	15	15	15	15	15
37	13	5	33	9	26	22	18	14	6	10	42	38	30	34	54	25
41	13	5	33	9	30	34	38	42	10	6	14	18	22	26	58	25
45	13	5	33	9	34	30	42	38	18	14	6	10	26	22	62	25
49	13	5	33	9	38	42	30	34	46	22	10	6	14	18	26	25
53	13	5	33	9	42	38	34	30	50	26	18	14	6	10	22	25
57	13	5	33	9		23	23	23	38	17	22	26	10	6	14	18
61	13	5	33	9	57	46	50	54	42	17	26	22	18	14	6	10
65	13	5	33	9	57	50	46	58	54	17	62	21	34	30	10	6
69	13	5	33	9	57		27	46	27	17	50	21	27	27	18	14

Table 3: Even (red or black) and minimal odd (blue) values of b for which $\{4, a, b, c\}$ is a $\mathcal{P}$-position.

The results in Sections 3 and 4 imply that Table 3 can be inductively completed to determine values of b such that given distinct $a, c \in \mathbb{N}$ with ac odd, the position $\{4, a, b, c\}$ is a $\mathcal{P}$-position. This in turn provides a recursive solution to a special case of the second of Guy's 20 questions in [4]. Specifically, given any $\mathcal{N}$-position $\{4, x\}$, x odd, we can determine the unique x_q such that $\{4, x, x_q\}$ is a $\mathcal{P}$-position.

Theorem 3. *If $x \geq 5$ is an odd integer, then $O(\{4, x\})$ can be computed.*

Proof. For any $x \in \mathbb{N}$, let $M_x = \{4, x\}$, and recall that $O(M_x)$ is known if $x =$ 5, 7, 9, 11, 13, 15, 17, or 19. Proceeding inductively, given any $j \in \mathbb{N}, j > 3$, assume that $O(M_5)$, $O(M_7)$, ... , $O(M_{2j-1})$ are known, as are all Table 3 entries if $y < x$ and $y \equiv x \pmod 4$. The unique element of $O(M_{2j+1})$ must be an element of $P_{2j+1} = \{q \in \mathbb{N} : q \equiv 2j - 1 \pmod 4), 7 \leq q < 3(2j + 1)\}$.

Let e_q denote the minimal available even integer corresponding to the position $\{4, x, q\}$ in Table 3, and let x_q be as defined in Proposition 3.3. We will consider each value of q in increasing order as follows:

1. If x_q is known and $\min\{2q, 2x_q\} \leq e_q$, then x_q is a winning response to the position $\{4, x, q\}$ (since there is no winning even response) and hence $q \notin O(M_{2j+1})$.

2. If x_q is known and $\min\{2q, 2x_q\} > e_q$, then as discussed in Section 4, e_q is a winning response to the position $\{4, x, q\}$, and hence $q \notin O(M_{2j+1})$.

3. If x_q is not known, then e_q is a winning response to the position $\{4, x, q\}$ (since x_q is too large to be a winning response), and again $q \notin O(M_{2j+1})$.

This process terminates after each of the $\frac{2(2j+1)-6}{4} = j-1$ even integers in $L(M_{2j+1})$ is identified as a specific value of e_q so that $\{4, x, q, e_q\}$ is a $\mathcal{P}$-position. The next odd value of $q \in P_{2j+1}$ is then the unique element of $O(M_{2j+1})$, completing the proof. □

Example 12. To compute $O(M_{29})$, first note that since $29 \equiv 1(\text{mod } 4)$, the unique $q \in O(M_{29})$ must be an element of $\{q \in \mathbb{N} : q \equiv 3(\text{mod } 4), 7 \leq q < 87\}$. We will analyze each such q in increasing order based on both x_q and known entries in Table 3 as discussed in Theorem 5.1:

1. If x_q is known and if $\min\{2q, 2x_q\} \leq e_q$, then x_q is a winning response to $\{4, 29, q\}$. For instance, if $q = 7$, then $x_7 = 13$, $e_7 = 14$, and $\min\{2 \cdot 7, 2 \cdot 13\} \leq 14$. Thus, $\{4, 29, 7, 13\}$ is a $\mathcal{P}$-position and therefore $7 \notin O(M_{29})$.

2. If x_q is known and if $\min\{2q, 2x_q\} > e_q$, then e_q is a winning response to $\{4, x, q\}$. For instance, if $q = 15$, then $x_{15} = 33$ and $e_{15} = 26$, and $\min\{2 \cdot 15, 2 \cdot 33\} \leq 26$ is not satisfied. Thus, $\{4, 29, 15, 26\}$ is a $\mathcal{P}$-position, and therefore $15 \notin O(M_{29})$.

3. If x_q is not known, then e_q is a winning response to $\{4, 29, q\}$. For instance, if $q = 39$, then x_{39} is not known, and since $e_{39} = 22$, we know that $\{4, 29, 39, 22\}$ is a $\mathcal{P}$-position, and therefore $39 \notin O(M_{29})$.

As seen in Theorem 5.1, this process terminates after each of the $\frac{2 \cdot 29-6}{4} = 13$ even integers in $L(M_{29})$ is identified as a specific e_q so that $\{4, 29, q, e_q\}$ is a $\mathcal{P}$-position. The next value of q, namely $q = 75$, must be the unique element of $O(M_{29})$.

We readily admit that the recursive algorithm from Theorem 5.1 is not ideal for determining unique winning responses to positions of the form $\{4, x\}$ with x odd. It is, however, far more efficient than using sheer brute force.

Note that even if both players in a game of Sylver Coinage have access to this algorithm, it only benefits Player 1, as it allows them to maintain a winning position. But if Player 1 is unaware of the algorithm or is not allowed to use it, then the advantage provided to Player 1 from Hutching's Theorem is nullified if both Player 1 makes a non-optimal choice given the position $\{4, x\}, x$ odd and if Player 2 uses the algorithm.

6. Verifying the Conway Inequality

In [4], Guy describes the following scenario: let $a, b \in \mathbb{N}$ with ab odd, and assume without loss of generality that $a < b$. If $O(\{4, a\}) = \{b\}$ and $O(\{4, b\}) = \{a\}$ both hold, Guy declares "*Conway can prove that* $2 < \frac{b}{a} < 3$*....*" But no verification of this statement exists in the literature.

We can restate this claim in the context of Corollary 3.4: for every odd integer $q > 3$, there exists a unique odd $x_q \in \mathbb{N}$ such that $\{4, q\}$ and $\{4, x_q\}$ are both $\mathcal{N}$-positions, $\{4, q, x_q\}$ is a $\mathcal{P}$-position, and if without loss of generality $q < x_q$, then it must be the case that $2q < x_q < 3q$.

The first few pairs (q, x_q) that satisfy the conditions of Corollary 3.4 are $(5, 11)$, $(7, 13)$ and $(9, 19)$. We note that 4-pair $(7, 13)$ does not satisfy the bounds claimed by Conway. But all other 4-pairs do satisfy these bounds.

Theorem 4. *Let (q, x_q) be any 4-pair with $q > 7$ and $q < x_q$. Then $2q < x_q < 3q$.*

Proof. We proceed using induction. One can show via brute force that for every odd integer in the set $\{5, 9, 11, ..., 41\}$, the corresponding 4-pair (q, x_q) satisfies $2q < x_q < 3q$. Now assume the result holds for each odd integer in the the set $\{5, 9, 11, ..., 2j - 1\}$ and consider the ordered pair $(2j+1, x_{2j+1})$. Recall from Section 5 that $x_{2j+1} = O(\{4, 2j+1\})$ can be computed. If $x_{2j+1} < 2j+1$, then by the inductive hypothesis the result holds. If $2j+1 < x_{2j+1}$, then via the recursive argument in Theorem 5.1, it must be the case that $2(2j+1) < x_{2j+1}$. Since $F(\{2j+1, x_{2j+1}\}) < 3(2j+1)$, it then follows that

$$2(2j + 1) < x_{2j+1} < 3(2j + 1),$$

verifying Conway's inequality. □

Theorem 5. *The algorithm presented in Theorem 5.1 can be extended to give a general strategy on positions M where $L(M)$ is finite and four has been chosen, as well as a method for determining if such positions are $\mathcal{N}$-positions or $\mathcal{P}$-positions.*

Proof. Simply note that any finite board $M = \{4, x, ..., y\}$ can be expressed as $\{4, a, b, c\}$ where a, b, c are the minimal natural numbers such that $a, b, c \notin L(M)$, $a \equiv 1 (\text{mod } 4)$, $b \equiv 2 (\text{mod } 4)$, and $c \equiv 3 (\text{mod } 4)$. Now implement the algorithm from Theorem 5.1 and use Table 3 as needed. The result follows. □

In the following example, we will assume Player 1 can use both the Theorem 5.1 algorithm and the extended Conway table from Table 3.

Example 13. Suppose Player 1 is given the position $M = \{4, 17\}$. Using the algorithm from Theorem 5.1, Player 1 can compute $O(M) = 43$, resulting in the $\mathcal{P}$-position $\{4, 17, 43\}$. Any response to this position by Player 2 yields an $\mathcal{N}$-position.

- If Player 2 responds with an even choice, say 22, the resulting position is $\{4, 17, 43, 22\} = \{4, 17, 22\}$. This is an $\mathcal{N}$-position, since Player 1 can then use Table 3 to select 31, which results in the $\mathcal{P}$-position $\{4, 17, 22, 31\}$.

- If Player 2 responds with an odd choice, say, 19, the resulting position is $\{4, 17, 43, 19\} = \{4, 17, 19\}$. This is also an $\mathcal{N}$-position, since Player 1 can again use Table 3 to select 6, which results in the $\mathcal{P}$-position $\{4, 6, 17, 19\}$.

In both cases, any subsequent move by Player 2 yields an $\mathcal{N}$-position. Player 1 can again use Table 3 to make a choice from the remaining set of legal moves that results in a $\mathcal{P}$-position.

To summarize: the repeated use of the algorithm from Theorem 5.1, together with the corresponding extended Conway table in Table 3 (assuming the table has been completed to a needed and appropriate size) results in Sylver Coinage being solved from any position of the form $\{4, x\}, x \geq 5$ and x odd. More generally, Sylver Coinage is solved from any position of the form $\{4, a, b, c\}$ as described above.

7. Future Work

Recall that in [4], Guy mentions the inequality that we verified in Section 6 in the context of 4-pairs, expressed as

$$2 < \tfrac{x_q}{q} < 3.$$

Guy further claims that Conway can improve these upper and lower bounds on the ratio $\frac{x_q}{q}$ if the values of q are restricted to the separate congruence classes $q \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$, as well as under the assumption that the corresponding values of x_q are monotonic in each of these two congruence classes as q increases. We hope to sharpen these bounds, as well as verify the monotonicity assumption.

In [4], the 14$^{\text{th}}$ question Guy asks about Sylver Coinage is this: does $\lim_{q\to\infty} \frac{x_q}{q}$ exist? The value of $\frac{x_q}{q}$ for most values of q is approximately 2.56. We hope to shed light on the asymptotic behavior of the value of this ratio.

Recall that, by Proposition 3.3, given any odd $x > 3$, if $M = \{4, x\}$, then $|O(M)| = 1$.

Brute force searches have yielded the following:

- If $M = \{5, p\}$, then $|O(M)| \geq 2$ if $p = 23$.
- If $M = \{6, p\}$, then $|O(M)| \geq 2$ if $p = 13$.
- If $M = \{7, p\}$, then $|O(M)| \geq 2$ if $p = 11, 37$.

- If $M = \{8, p\}$, then $|O(M)| \geq 2$ if $p = 23$.
- If $M = \{9, p\}$, then $|O(M)| \geq 2$ if $p = 7, 17, 19, 23, 29, 37$.
- If $M = \{10, p\}$, then $|O(M)| \geq 2$ if $p = 11, 23, 29, 31$.
- If $M = \{11, p\}$, then $|O(M)| \geq 2$ if $p = 13, 17, 37$.
- If $M = \{12, p\}$, then $|O(M)| \geq 2$ if $p = 17, 23, 31$.
- If $M = \{13, p\}$, then $|O(M)| \geq 2$ if $p = 17$.

If Player 1 opens with a prime $p > 3$, it appears reasonable to suspect that *every* predetermined response $y > 4$ from Player 2 results in at least one position $M = \{p, y\}$ with $|O(M)| \geq 2$. We conjecture that this is the case.

Another interesting question associated with Sylver Coinage game positions is based on an observation in [6]. Namely, there are only 7 known Sylver Coinage positions $M\{x, y\}$ with $(x, y) = 1$ for which the unique winning move is $F(\{x, y\})$, the maximal element of $L(\{x, y\})$. They are:

$$\{2, 5\}, \{4, 5\}, \{5, 6\}, \{5, 9\}, \{8, 15\}, \{13, 14\}, \{13, 21\}.$$

Are there others? And do they all involve Fibonacci numbers? Our initial searches have not provided other such examples.

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