OLYMPIC GAMES: THREE IMPARTIAL GAMES WITH INFINITE OCTAL CODES

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Abstract

We compute the $\mathcal{P}$-positions of three octal games with infinite octal codes. The octal codes are defined using Beatty sequences constructed from the golden mean, the silver mean, and bronze mean, respectively.

1. Introduction

We investigate a family of three octal games, each with an infinite octal code. What sort of behavior should we expect from their nim-sequence? Berlekamp, Conway, and Guy [5], [10] conjectured that any octal game with a finite octal code has an ultimately periodic nim-sequence. The octal games which have received the most attention over the years are the ones with at most 3 octal digits. Among the 79 non-trivial octal games with such a code, only 14 have been solved [8], [9]. Achim Flammenkamp [8] has computed millions of nim-values for the remaining 63 octal games and, to date, no patterns have been found in their nim-value sequences.

Less is known about octal games with infinite octal codes. The work that has been done has dealt with octal games that have periodic codes [3], [8].

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1.1. Combinatorial Game Theory

A *Taking-and-Breaking* game is an impartial game played on heaps of tokens. On her move, a player may take a prescribed number of tokens from any one heap, and then, possibly, split the remaining tokens in that heap into several heaps. The number of tokens which may be removed and the number of heaps one may split the remaining tokens into are given by the rules of the game. The most famous example of such a game is the game of NIM.

The types of taking-and-breaking games we consider in this article are called octal games. Richard Guy and Cedric Smith pioneered the study of these games in the seminal paper [11]. An *octal game* is characterized by its octal code

$$\mathbf{d}_0.\mathbf{d}_1\mathbf{d}_2\mathbf{d}_3\ldots,$$

where $0 \leq \mathbf{d}_i \leq 7$. The number $\mathbf{d}_0$ can only be 0 or 4. If $i > 0$, then

$$\mathbf{d}_i = e_0 2^0 + e_1 2^1 + e_2 2^2,$$

where each e_i can be 0 or 1. For a given $\mathbf{d}_i$, a player may

- reduce a heap of size i to 0 if $e_0 = 1$;
- remove i tokens from a heap of size $k > i$ if $e_1 = 1$; and
- remove i tokens from a heap of size $k > i$ and split the remaining $k - i$ tokens into two nonempty heaps, if $e_2 = 1$.

An example of such a game is the game of Kayles. This game can be played on a row of bowling pins. A move consists of knocking out either one pin or two adjacent pins, until all the pins are knocked over. Kayles is equivalent to the octal game with code $\mathbf{0.77}$.

To see why this is so, first note that $7 = 1 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2$. Hence, according to the rules of the octal game $\mathbf{0.77}$, a player may remove a heap with one or two tokens in it (since the coefficient of $2^{\mathbf{0}}$ is 1). Additionally, a player may remove one or two tokens from a heap and leave tokens in the heap (the coefficient of $2^{\mathbf{1}}$ is also 1). Finally, if she wishes, a player may remove one or two tokens from a heap, leave tokens in that heap, and then split the remaining tokens into two nonempty heaps (because the coefficient of $2^{\mathbf{2}}$ is 1).

We also note that NIM is an octal game. Its code is infinite and is $\mathbf{0.333}\ldots$.

A position reachable from a given game G in one move is called a *follower* of G. The *minimum excluded element*, or mex, of a set S of non-negative integers is the smallest non-negative integer not present in S. Using these ideas, we can define the *nim-value* of an impartial game G by $\mathscr{G}(G) = \text{mex}\{\mathscr{G}(H) \mid H \text{ is a follower of } G\}$.

If $\mathscr{G}(G) = 0$, then G is a second-player win and we call G a $\mathcal{P}$-position. If $\mathscr{G}(G) \neq 0$, then G is a first-player win and we call G an $\mathcal{N}$-position.

The *nim-sum* of two non-negative integers, m and n, written as $m \oplus n$, can be computed by adding m's and n's binary representation without carrying. A game G is the *disjunctive sum* of two games H and K, that is, $G = H + K$, if and only if, on a players turn, she may choose one of H or K and then make a valid move on that summand. Note, it follows from the theory of impartial games that if $G = H + K$, then $\mathscr{G}(G) = \mathscr{G}(H) \oplus \mathscr{G}(K)$.

Due to the fact that all taking-and-breaking games are examples of disjunctive games, it suffices to know how to play one of these games on a single heap. We may then compute what is called the $\mathscr{G}$-sequence of a given taking-and-breaking game G, $\mathscr{G}(0), \mathscr{G}(1), \mathscr{G}(2), \ldots$, where $\mathscr{G}(n)$ denotes the nim-value of G played on a heap of size n. A more thorough presentation of the basic concepts of Combinatorial Game Theory can be found in [1], [6], or [12].

1.2. Beatty Sequences

A *Beatty sequence* [4] is a sequence of non-negative integers which arises by applying the floor function to the positive multiples of a given positive irrational number, α. That is, given $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the Beatty sequence generated by α is $\{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \ldots\}$. For example, if $\alpha = (1 + \sqrt{5})/2$, then we get $\{1, 3, 4, 6, \ldots\}$. It can be shown that the complement of a Beatty sequence (the complement taken in the set of non-negative integers) is itself a Beatty sequence with generating irrational $\alpha/(\alpha - 1)$. Moreover, it is true that a given Beatty sequence and its complementary sequence form a partition of the set of non-negative integers.

It is customary to refer to the starting Beatty sequence as the *α-sequence* and it is denoted by $\mathcal{B}_\alpha$. Its complementary Beatty sequence is called the *β-sequence* and is denoted $\mathcal{B}_\beta$, where $\beta = \alpha/(\alpha - 1)$.

The earliest known appearance of a Beatty sequence in combinatorial game theory is in the well-known game of Wythoff [13]. Interestingly, the $\mathcal{P}$-positions of Wythoff can be described in terms of a Beatty sequence and its complement. A discussion of these sequences can also be found in [2].

1.3. Metallic Means

For a given positive integer n, the number $\frac{n+\sqrt{n^2+4}}{2}$ is referred to as the n^{th} *metallic mean.* If $n = 1$, then we get $\phi = \frac{1+\sqrt{5}}{2}$, the golden ratio. Similarly, $n = 2$ and $n = 3$ give the silver and bronze ratio, respectively. It is interesting to note that the continued fraction expansion for $\frac{n+\sqrt{n^2+4}}{2}$ is given by

$$n + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{n + \ddots}}}}.$$

There is no agreed upon naming convention for metallic means for $n \geq 4$, but, as they will not be considered in this article, this is of no consequence to us. Additional details surrounding the concept of metallic mean can be found in [7].

2. The Gold Game

Before we get started, recall that the *Fibonacci numbers* can be expressed in the following way:

$$F_0 = 0, F_1 = 1, \text{ and } F_{n+1} = F_n + F_{n-1}, \text{ for } n \geq 1.$$

First we will prove a theorem which asserts that every positive natural number can be expressed as a sum of odd-indexed Fibonacci numbers, subject to specific conditions. This theorem, and others like it in subsequent sections, is inspired by the presentations of the theorems of Zeckendorf and Ostrowski given in [2] (see Chapter 3).

Theorem 1. *Let $t \in \mathbb{N}^+$. Then*

$$t = \sum_{k=0}^{n} a_{2k+1} F_{2k+1},$$

where

(1) $F_{2n+1} \leq t$, but $F_{2n+3} > t$,

(2) $0 \leq a_1 \leq 1$ and $0 \leq a_{2k+1} \leq 2$ for $k \geq 1$,

(3) $2F_3 + F_1$ never appears in any representation, and

(4) $2F_{2k+1} + 2F_{2k-1}$ never appears in any representation.

Proof. Let $t \in \mathbb{N}^+$. Choose the largest odd-indexed Fibonacci number F_{2m+1} such that $t \geq F_{2m+1}$, but $F_{2k+3} > t$. Continue this process by choosing the largest odd-indexed Fibonacci number $F_{2\ell+1}$ that is less than or equal to $t - F_{2m+1}$, etc.

Condition (1) is clear by construction. For condition (2), note that $F_3 = 2F_1$. Also, because

$$2F_{2k+1} < F_{2k+3} < 3F_{2k+1}, \text{ for } k \geq 1,$$

we must have $0 \leq a_{2k+1} \leq 2$ for $k \geq 1$. Condition (3) follows as

$$F_5 = 2F_3 + F_1.$$

Finally, condition (4) follows from an easy calculation which shows that

$$2F_{2k+1} + 2F_{2k-1} > F_{2k+3}.$$

$\square$

Let $\varphi = \frac{1+\sqrt{5}}{2}$. Then, $1+\varphi = \frac{3+\sqrt{5}}{2}$. We need the following proposition to move forward.

Proposition 1. *Let $k \geq 0$. Then*

(1) $F_{2k+1} \cdot \varphi = F_{2k+2} + \left(\frac{\sqrt{5}-1}{2}\right)^{2k+1}$ *and*

(2) $F_{2k+1} \cdot (1+\varphi) = F_{2k+3} + \left(\frac{\sqrt{5}-1}{2}\right)^{2k+1}$.

Proof. Statement (1) follows from Binet's formula and simple algebra. Statement (2) follows from statement (1) and the Fibonacci recurrence. □

Next, we will use Theorem 1 to express a given positive integer t in terms of odd-indexed Fibonacci numbers and then we will consider the product

$$t \cdot \varphi = \left(\sum_{k=0}^{n} a_{2k+1} F_{2k+1}\right) \cdot \varphi.$$

By Proposition 1, the last expression can be written as

$$\sum_{k=0}^{n} a_{2k+1} F_{2k+2} + \sum_{k=0}^{n} a_{2k+1} \left(\frac{\sqrt{5}-1}{2}\right)^{2k+1}.$$

We refer to the second summand in the above sum as the **error** of the representation. We will show that it is positive and strictly less than 1. Hence, it will disappear after applying the floor function to the expression $t \cdot \varphi$.

Our next goal is to show that

$$0 \leq \sum_{k=0}^{n} a_{2k+1} \left(\frac{\sqrt{5}-1}{2}\right)^{2k+1} < 1$$

for a given valid odd-indexed Fibonacci representation, thus establishing the following theorem.

Theorem 2. *Let $n \geq 0$. Then*

(1)
$$\left\lfloor \left(\sum_{k=0}^{n} a_{2k+1} F_{2k+1}\right) \cdot \varphi \right\rfloor = \sum_{k=0}^{n} a_{2k+1} F_{2k+2}, \text{ and}$$

(2)
$$\left\lfloor \left(\sum_{k=0}^{n} a_{2k+1} F_{2k+1}\right) \cdot (1+\varphi) \right\rfloor = \sum_{k=0}^{n} a_{2k+1} F_{2k+3}.$$

Proof. In what follows, we will see that the largest errors come from representations of the form

$$F_{2n+1} + F_{2n-1} + \cdots + F_5 + F_3 + F_1.$$

Note that

$$\begin{aligned}\left(\sum_{k=0}^{n} F_{2k+1}\right) \cdot \varphi &= \left(\sum_{k=0}^{n} F_{2k+2}\right) + \sum_{k=0}^{n} \left(\tfrac{\sqrt{5}-1}{2}\right)^{2k+1} \\ &= \left(\sum_{k=0}^{n} F_{2k+2}\right) + \left(1 - (3-\sqrt{5})\left(\tfrac{\sqrt{5}-1}{2}\right)^{2n}\right)\end{aligned}$$

and

$$0 < \left(1 - (3-\sqrt{5})\left(\frac{\sqrt{5}-1}{2}\right)^{2n}\right) < 1$$

for all $n \geq 0$.

We will show that the error of any representation $t = \sum_{k=0}^{n} a_{2k+1}F_{2k+1}$ is at most the error associated with the sum of the form $\sum_{k=0}^{\ell} F_{2k+1}$ which is the largest representation of this type less than or equal to t. The proof of this fact will proceed by induction. In table 1 we provide the representations of the first 21 positive integers to establish our basis steps.

From this table, we can see that the error of the representation of $t = 2$ is less than the error for $t = 1$. Similarly, the errors for $t = 4, 5, 6, 7$ are less than for $t = 3$. We also see that the errors for $t = 9, 10, \ldots 20$ are all less than the error for $t = 8$. (It is interesting to note that the bolded rows occur at the even-indexed Fibonacci number positions.)

Before we begin the induction step, we recall one identity involving Fibonacci numbers:

$$F_{2\ell+1} + F_{2\ell-1} + \cdots + F_3 + F_1 = F_{2\ell+2}.$$

Now let $t = \sum\limits_{k=0}^{n} a_{2k+1}F_{2k+1}$ be a given representation, where $t \geq 21$. If $a_{2n+1} = 1$, we consider $t - F_{2n+1} = \sum\limits_{k=0}^{n-1} a_{2k+1}F_{2k+1}$. Either

(*i*) $F_{2\ell-1} \leq t - F_{2n+1} \leq F_{2\ell}$, or

(*ii*) $F_{2\ell} \leq t - F_{2n+1} \leq F_{2\ell+1}$, for some ℓ with $1 \leq \ell \leq n$.

If $\ell = 1$, then we have $F_1 \leq t - F_{2n+1} \leq F_2$ or $F_2 \leq t - F_{2n+1} \leq F_3$. If $F_1 \leq t - F_{2n+1} \leq F_2$, then $t = F_{2n+1} + F_1$. Hence,

$$\begin{aligned}\mathbf{error}(F_{2n+1} + F_1) &= \mathbf{error}(F_{2n+1}) + \mathbf{error}(F_1) \\ &\leq \mathbf{error}(F_{2n-1} + \cdots + F_3 + F_1)\end{aligned}$$

$\mathbf{t}$	$\sum \mathbf{a_{2k+1} F_{2k+1}}$	$(\sum \mathbf{a_{2k+1} F_{2k+1}}) \cdot \varphi$	$\sum \mathbf{a_{2k+1} F_{2k+2}}$	**Error**
1	$\mathbf{F_1}$	1.61803398874989	1	**0.61803398874989**
2	F_3	3.23606797749979	3	0.23606797749979
3	$\mathbf{F_3 + F_1}$	4.85410196624968	4	**0.85410196624968**
4	$2F_3$	6.47213595499958	6	0.47213595499958
5	F_5	8.09016994374947	8	0.09016994374947
6	$F_5 + F_1$	9.70820393249937	9	0.70820393249937
7	$F_5 + F_3$	11.3262379212493	11	0.3262379212493
8	$\mathbf{F_5 + F_3 + F_1}$	12.9442719099992	12	**0.9442719099992**
9	$F_5 + 2F_3$	14.5623058987491	14	0.5623058987491
10	$2F_5$	16.1803398874989	16	0.1803398874989
11	$2F_5 + F_1$	17.7983738762488	17	0.7983738762488
12	$2F_5 + F_3$	19.4164078649987	19	0.4164078649987
13	F_7	21.0344418537486	21	0.0344418537486
14	$F_7 + F_1$	22.6524758424985	22	0.6524758424985
15	$F_7 + F_3$	24.2705098312484	24	0.2705098312484
16	$F_7 + F_3 + F_1$	25.8885438199983	25	0.8885438199983
17	$F_7 + 2F_3$	27.5065778087482	27	0.5065778087482
18	$F_7 + F_5$	29.1246117974981	29	0.1246117974981
19	$F_7 + F_5 + F_1$	30.7426457862480	30	0.7426457862480
20	$F_7 + F_5 + F_3$	32.3606797749979	32	0.3606797749979
21	$\mathbf{F_7 + F_5 + F_3 + F_1}$	33.9787137637478	33	**0.9787137637478**

Table 1: Odd-indexed Fibonacci representations of the first 21 positive integers

by Proposition 1. If $F_2 \le t - F_{2n+1} \le F_3$, then $t = F_{2n+1} + F_2 = F_{2n+1} + F_1$ or $t = F_{2n+1} + F_3$. Only the latter case is new. Note that

$$\begin{aligned} \mathbf{error}(F_{2n+1} + F_3) &= \mathbf{error}(F_{2n+1}) + \mathbf{error}(F_3) \\ &\le \mathbf{error}(F_{2n-1} + \cdots + F_3 + F_1) \end{aligned}$$

as before, by Proposition 1.

If $\ell > 1$ and we are in case (i), then

$$\mathbf{error}(t - F_{2n+1}) \le \mathbf{error}(F_{2\ell-3} + \cdots + F_3 + F_1).$$

Hence,

$$\begin{aligned} \mathbf{error}(\sum_{k=0}^{n} a_{2k+1} F_{2k+1}) &\le \mathbf{error}(F_{2n+1} + F_{2\ell-3} + \cdots + F_3 + F_1) \\ &\le \mathbf{error}(F_{2n-1} + F_{2n-3} + \cdots + F_3 + F_1). \end{aligned}$$

If we are in case (ii), then

$$\mathbf{error}(t - F_{2n+1}) \quad \leq \quad \mathbf{error}(F_{2\ell-1} + \cdots + F_3 + F_1).$$

Hence,

$$\begin{aligned} \mathbf{error}(\sum_{k=0}^{n} a_{2k+1}F_{2k+1}) \quad &\leq \quad \mathbf{error}(F_{2n+1} + F_{2\ell-1} + \cdots + F_3 + F_1) \\ &\leq \quad \mathbf{error}(F_{2n-1} + \cdots + F_3 + F_1). \end{aligned}$$

Finally, we consider the case where $a_{2n+1} = 2$. In this case, $F_{2n+1} \leq t - F_{2n+1}$. Thus,

$$\begin{aligned} \mathbf{error}(\sum_{k=0}^{n} a_{2k+1}F_{2k+1}) \quad &\leq \quad \mathbf{error}(F_{2n+1}) + \mathbf{error}(F_{2n-1} + \cdots + F_3 + F_1) \\ &\leq \quad \mathbf{error}(F_{2n+1} + \cdots + F_3 + F_1). \end{aligned}$$

□

We can now introduce the octal game that we call the gold game. This game is given by the octal code

$$\mathbf{0.d_1d_2d_3}\ldots,$$

where

$$\mathbf{d}_i = \begin{cases} 1, & \text{if } i \in \mathcal{B}_\varphi \\ 2, & \text{if } i \in \mathcal{B}_{1+\varphi}. \end{cases}$$

Thus, the beginning of this infinite octal code looks like:

$$\mathbf{0.12112121121121211212112112121121121211212}\ldots.$$

We will denote the heap game given by this octal code as the *gold game*, $\mathscr{G}$. A plot of the first 10,000 nim-values is shown in figure 1.

Our next aim is to prove the following theorem.

Theorem 3. *The $\mathcal{P}$ positions of the gold game occur at the following heap sizes:*

$$0, F_3, F_5, F_7, \ldots, F_{2k+1}, \ldots.$$

Proof. It is easy to see from the game rules that any nonempty heap $\mathscr{G}_k$ with size $k \in \mathcal{B}_\varphi$ has nonzero nim-value. Next, note that no heap $\mathscr{G}_k$ with $k \in \mathcal{B}_{1+\varphi}$ has a move to the empty heap. Now let $t \in \mathcal{B}_{1+\varphi}$ with $t \notin \{F_3, F_5, \ldots, F_{2k+1}, \ldots\}$. Then

$$t = \sum_{k=0}^{n} a_{2k+1}F_{2k+3} = \left\lfloor \left(\sum_{k=0}^{n} a_{2k+1}F_{2k+1} \right) \cdot (1+\varphi) \right\rfloor,$$

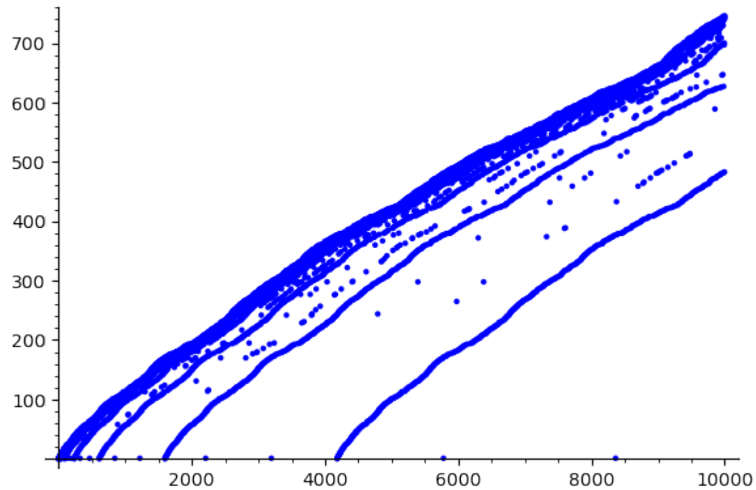

Figure 1: Nim-values for the gold game up to heap size 10,000

by Theorem 2. Notice that if

$$j = \begin{cases} F_{2n+3} + \sum_{k=0}^{n-1} a_{2k+1} F_{2k+3}, & \text{if } a_{2n+1} = 2 \\ \sum_{k=0}^{n-1} a_{2k+1} F_{2k+3}, & \text{if } a_{2n+1} = 1 \end{cases}$$

then $t - j \in \mathcal{B}_{1+\varphi}$ is an option. Thus, every non-Fibonacci $t \in \mathcal{B}_{1+\varphi}$ has a move to a Fibonacci number in $\mathcal{B}_{1+\varphi}$. The last thing that we need to show is that no move exists from $F_{2m+1} \in \mathcal{B}_{1+\varphi}$ to $F_{2\ell+1} \in \mathcal{B}_{1+\varphi}$.

If there was such a move from F_{2m+1} to $F_{2\ell+1}$, then $F_{2m+1} - F_{2\ell+1} \in \mathcal{B}_{1+\varphi}$. However,

$$\begin{aligned} F_{2m+1} - F_{2\ell+1} &= (F_{2m+1} - F_{2m-1}) + (F_{2m-1} - F_{2m-3}) + \cdots + (F_{2\ell+3} - F_{2\ell+1}) \\ &= F_{2m} + F_{2m-2} + \cdots + F_{2\ell+2} \\ &= \left\lfloor \left(\sum_{k=\ell}^{m-1} F_{2k+1} \right) \cdot \varphi \right\rfloor. \end{aligned}$$

Thus, $F_{2m+1} - F_{2\ell+1} \in \mathcal{B}_{\varphi}$ and hence there is no legal way to move from $\mathscr{G}_{F_{2m+1}}$ to $\mathscr{G}_{F_{2\ell+1}}$. Therefore, the $\mathcal{P}$-positions of $\mathscr{G}$ are precisely the heaps of size

$$0, F_3, F_5, F_7, \ldots, F_{2k+1}, \ldots.$$

$\square$

3. The Silver Game

We first recall what the *Pell numbers* and the *half-companion Pell numbers* are. The Pell numbers are given by

$$P_0 = 0, P_1 = 1, \text{ and } P_{n+1} = 2P_n + P_{n-1}, \text{ for } n \geq 1.$$

Thus, the first few are

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985.$$

The half-companion Pell numbers are given by

$$H_0 = 1, H_1 = 1, \text{ and } H_{n+1} = 2H_n + H_{n-1}, \text{ for } n \geq 1.$$

The first few are

$$1, 1, 3, 7, 17, 41, 99, 239, 577, 1393.$$

Note that $H_n = P_n + P_{n-1}$. It is also straightforward to check that, except for $P_1 = H_0 = H_1 = 1$, the two sets of numbers share no common elements. As it turns out, the Pell numbers and the half-companion Pell numbers are related to the silver mean, $1 + \sqrt{2}$, in a manner similar to how the Fibonacci numbers and the golden mean are related (see the proof of Proposition 2 for further clarification).

We will prove a theorem which asserts that every positive natural number can be expressed as a sum of the mixed set of numbers comprised of

$$\{H_0, 2P_1, H_2, 2P_3, H_4, 2P_5, H_6, \dots\},$$

subject to several conditions.

Theorem 4. *Let $t \in \mathbb{N}^+$. Then*

$$t = \sum_{k=0}^{n} a_k R_k,$$

where

(1) $R_k = 2P_k$, if k is odd, and $R_k = H_k$, if k is even,

(2) $R_n \leq t$, but $R_{n+1} > t$,

(3) $0 \leq a_0 \leq 1$ and if $a_1 = 1$, then $a_0 = 0$,

(4) $0 \leq a_{2k} \leq 3$ and $0 \leq a_{2k+1} \leq 1$,

(5) for $k \geq 1$, if $a_{2k} = 3$, then $a_{2k-1} = 0$ and $a_{2k+1} = 0$,

(6) if $a_{2k} = 3$, then $a_{2k-2} < 3$ and $a_{2k+2} < 3$.

Proof. As with the odd-indexed Fibonacci representation, a representation from this mixed set of numbers can be constructed in a greedy fashion. To see that condition (3) is true, we first note that $2P_1 = 2$ and $H_2 = 3$. Thus, $0 \leq a_0 \leq 1$. We also cannot have $a_0 = 1$ and $a_1 = 1$ as $H_2 = 3$.

For condition (4), note that $4H_{2k} > 2P_{2k+1}$, so $a_{2k} \leq 3$. Also, observe that $4P_{2k+1} > H_{2k+2}$. Hence, $a_{2k+1} \leq 1$.

For condition (5), first suppose that $a_{2k} = 3$ and $a_{2k-1} = 1$. If this were so, then we would have $3H_{2k} + 2P_{2k-1} > 2P_{2k+1}$. Next, suppose that $a_{2k+1} = 1$ and $a_{2k} = 3$. Then we would have $2P_{2k+1} + 3H_{2k} > 2P_{2k+3}$.

We finish up this proof by looking at condition (6). If $a_{2k} = a_{2k-2} = 3$, then $3H_{2k} + 3H_{2k-2} > 2P_{2k+1}$. Similarly, if $a_{2k+2} = a_{2k} = 3$, then $3H_{2k+2} + 3H_{2k} > P_{2k+3}$. $\square$

A result similar to the one above is stated below. Its proof is almost identical and is therefore omitted. The following theorem asserts that every positive natural number can be expressed as a sum of the mixed set of numbers comprised of

$$\{P_1, H_2, P_3, H_4, P_5, H_6, \dots\},$$

subject to several conditions.

Theorem 5. *Let $t \in \mathbb{N}^+$. Then*

$$t = \sum_{k=1}^{n} a_k R_k,$$

where

(1) $R_k = P_k$, *if k is odd, and* $R_k = H_k$, *if k is even,*

(2) $R_n \leq t$, *but* $R_{n+1} > t$,

(3) $0 \leq a_1 \leq 2$ *and if* $a_1 = 2$, *then* $a_2 = 0$,

(4) $0 \leq a_{2k+1} \leq 3$ *and* $0 \leq a_{2k} \leq 1$,

(5) for $k \geq 1$, *if* $a_{2k+1} = 3$, *then* $a_{2k} = 0$ *and* $a_{2k+2} = 0$,

(6) if $a_{2k+1} = 3$, *then* $a_{2k-1} < 3$ *and* $a_{2k+3} < 3$.

Let $\alpha = 1 + \sqrt{2}$ and $\beta = \frac{2+\sqrt{2}}{2}$ for the remainder of this section. As in the last section, the following proposition is very helpful.

Proposition 2. *Let P_n be the n^{th} Pell number and H_n be the n^{th} half-companion Pell number. Then, for $k \geq 0$,*

(1) $P_{2k+1} \cdot \alpha = P_{2k+2} + \left(\sqrt{2} - 1\right)^{2k+1}$,

(2) $H_{2k} \cdot \alpha = H_{2k+1} + \sqrt{2}\left(\sqrt{2}-1\right)^{2k}$,

(3) $2P_{2k+1} \cdot \beta = H_{2k+2} + \left(\sqrt{2}-1\right)^{2k+1}$, *and*

(4) $H_{2k} \cdot \beta = P_{2k+1} + \frac{1}{\sqrt{2}}\left(\sqrt{2}-1\right)^{2k}$.

Proof. All four of these statements follow from the two Binet-like formulas:

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} \quad \text{and} \quad H_n = \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}.$$

□

Our next goals are to show that

$$0 \leq \sum_{k \text{ odd}} a_k \left(\sqrt{2}-1\right)^{2k+1} + \sqrt{2} \sum_{k>0, \text{ even}} a_k \left(\sqrt{2}-1\right)^{2k} < 1$$

for a given valid mixed Pell representation of the type in Theorem 5, and

$$0 \leq \sum_{k \text{ odd}} a_k \left(\sqrt{2}-1\right)^{2k+1} + \frac{1}{\sqrt{2}} \sum_{k \text{ even}} a_k \left(\sqrt{2}-1\right)^{2k} < 1$$

for a given valid mixed Pell representation of the type in Theorem 4. Taken together, these two results establish the next theorem.

Theorem 6.

(1) If $R_k = \begin{cases} P_k, & \textit{if } k \textit{ is odd} \\ H_k, & \textit{if } k \textit{ is even} \end{cases}$ *and* $R'_k = \begin{cases} H_k, & \textit{if } k \textit{ is odd} \\ P_k, & \textit{if } k \textit{ is even} \end{cases}$, *then, for* $n \geq 1$,

$$\left\lfloor \left(\sum_{k=1}^{n} a_k R_k \right) \cdot \alpha \right\rfloor = \sum_{k=1}^{n} a_k R'_{k+1}, \textit{ and}$$

(2) If $R_k = \begin{cases} 2P_k, & \textit{if } k \textit{ is odd} \\ H_k, & \textit{if } k \textit{ is even} \end{cases}$ *and* $R'_k = \begin{cases} P_k, & \textit{if } k \textit{ is odd} \\ H_k, & \textit{if } k \textit{ is even} \end{cases}$, *then, for* $n \geq 0$,

$$\left\lfloor \left(\sum_{k=0}^{n} a_k R_k \right) \cdot \beta \right\rfloor = \sum_{k=0}^{n} a_k R'_{k+1}.$$

Proof. For (1), we will show that the largest errors come from representations of the form

$$P_{2n+1} + 2P_{2n-1} + \cdots + 2P_5 + 2P_3 + 2P_1, \text{ or}$$

$$2P_{2n+1} + 2P_{2n-1} + \cdots + 2P_5 + 2P_3 + 2P_1.$$

First note that

$$\begin{aligned}\left(P_{2n+1}+\sum_{k=0}^{n-1}2P_{2k+1}\right)\cdot\alpha &= \left(P_{2n+2}+\sum_{k=0}^{n-1}2P_{2k+2}\right)+\left(\sqrt{2}-1\right)^{2n+1}\\ &\quad +2\sum_{k=0}^{n-1}\left(\sqrt{2}-1\right)^{2k+1}\\ &= \left(P_{2n+2}+\sum_{k=0}^{n-1}2P_{2k+2}\right)+1\\ &\quad -\left(2-\sqrt{2}\right)\left(\sqrt{2}-1\right)^{2n}\end{aligned}$$

and

$$0<\left(1-\left(2-\sqrt{2}\right)\left(\sqrt{2}-1\right)^{2n}\right)<1$$

for all $n\geq 0$.

Note too that

$$\begin{aligned}\left(\sum_{k=0}^{n}2P_{2k+1}\right)\cdot\alpha &= \left(\sum_{k=0}^{n}2P_{2k+2}\right)+2\sum_{k=0}^{n}\left(\sqrt{2}-1\right)^{2k+1}\\ &= \left(\sum_{k=0}^{n}2P_{2k+2}\right)+1-\left(3-2\sqrt{2}\right)\left(\sqrt{2}-1\right)^{2n}\end{aligned}$$

and

$$0<\left(1-\left(3-2\sqrt{2}\right)\left(\sqrt{2}-1\right)^{2n}\right)<1$$

for all $n\geq 0$.

We will show that the error of any representation

$$t=\sum_{k=1}^{n}a_kR_k$$

is at most the error associated with a sum of the form

$$P_{2\ell+1}+\sum_{k=1}^{\ell-1}2P_{2k+1}$$

or of the form

$$\sum_{k=1}^{\ell}2P_{2k+1}$$

which is the largest representation of this type less than or equal to t. The proof of this fact will proceed by induction. In table 2 we provide the representations of the first 41 positive integers to establish our basis steps.

From this table, we can see that the errors of the representations for $t=3,4,5,6$ are less than the error for $t=2$. Similarly, the errors for $t=8,9,10,11$ are less than for $t=7$. We also see that the errors for $t=13,14,\ldots 40$ are all less than the error for $t=12$.

Now let $t=\sum_{k=1}^{n}a_kR_k$ be a given representation, where $t\geq 41$. There are four cases:

(a) $R_n = H_{2m}$ and $a_n = 1$,

(b) $R_n = P_{2m+1}$ and $a_n = 1$,

(c) $R_n = P_{2m+1}$ and $a_n = 2$, and

(d) $R_n = P_{2m+1}$ and $a_n = 3$.

`Case a.` Suppose that $R_n = H_{2m}$ and $a_n = 1$. In this case we cannot have $R_{n-1} = P_{2m-1}$ with $a_{n-1} = 3$. So we have $R_{n-1} = P_{2m-1}$ with $a_{n-1} = 0, 1$ or 2. First suppose that $a_{n-1} = 2$. We cannot have $\sum_{k=1}^{n-1} a_k R_k > \sum_{k=1}^{m-1} 2P_{2k+1}$ since

$$H_{2m} + \sum_{k=1}^{m-1} 2P_{2k+1} = P_{2m+1}.$$

Thus, we must have $\sum_{k=1}^{n-1} a_k R_k < \sum_{k=1}^{m-1} 2P_{2k+1}$ and $\sum_{k=1}^{n-1} a_k R_k > P_{2m-1} + \sum_{k=1}^{m-2} 2P_{2k+1}$. Hence, by induction,

$$\textbf{error}(t - H_{2m}) \quad \le \quad \textbf{error}(P_{2m-1} + \textstyle\sum_{k=1}^{m-2} 2P_{2k+1}).$$

Since the error of the single term H_{2m} is less than the error of the single term P_{2m-1}, we see that

$$\begin{aligned} \textbf{error}(\textstyle\sum_{k=1}^{n} a_k R_k) \quad &\le \quad \textbf{error}(P_{2m-1}) + \textbf{error}(P_{2m-1} + \textstyle\sum_{k=1}^{m-2} 2P_{2k+1}) \\ &= \quad \textbf{error}(\textstyle\sum_{k=1}^{m-1} 2P_{2k+1}). \end{aligned}$$

We now consider the subcase where $R_n = H_{2m}$ and $a_{n-1} = 1$. If

$$\sum_{k=1}^{n-1} a_k R_k > P_{2m-1} + \sum_{k=1}^{m-2} 2P_{2k+1},$$

then

$$\textbf{error}(\textstyle\sum_{k=1}^{n} a_k R_k) \quad \le \quad \textbf{error}(\textstyle\sum_{k=1}^{m-1} 2P_{2k+1}).$$

If

$$\sum_{k=1}^{n-1} a_k R_k \le P_{2m-1} + \sum_{k=1}^{m-2} 2P_{2k+1},$$

then

$$\sum_{k=1}^{n-1} a_k R_k > \sum_{k=1}^{m-2} 2P_{2k+1}.$$

It follows that

$$\textbf{error}(\textstyle\sum_{k=1}^{n} a_k R_k) \quad \le \quad \textbf{error}(\textstyle\sum_{k=1}^{m-1} 2P_{2k+1}).$$

If $R_n = H_{2m}$ and $a_{n-1} = 0$, then choose the largest $\ell > 0$ such that one of the sums $\sum_{k=1}^{\ell} 2P_{2k+1}$ or $P_{2\ell+1} + \sum_{k=1}^{\ell-1} 2P_{2k+1}$ is less than or equal to $\sum_{k=1}^{n-2} a_k R_k$. By induction, the error of $\sum_{k=1}^{n-2} a_k R_k$ is less than the error of one of $\sum_{k=1}^{\ell} 2P_{2k+1}$ or $P_{2\ell+1} + \sum_{k=1}^{\ell-1} 2P_{2k+1}$. As the error of the single term H_{2m} is less than the error of $\sum_{k=\ell}^{m-1} 2P_{2k+1} + P_{2\ell+1}$ or of $\sum_{k=\ell+1}^{m-1} 2P_{2k+1}$, it now follows that

$$\textbf{error}(\textstyle\sum_{k=1}^{n} a_k R_k) \quad \le \quad \textbf{error}(\textstyle\sum_{k=1}^{m-1} 2P_{2k+1}).$$

`Case b.` Suppose that $R_n = P_{2m+1}$ and $a_n = 1$. If $a_{n-1} = 1$, then, $\sum_{k=1}^{n} a_k R_k = P_{2m+1} + H_{2m} + \cdots \ge P_{2m+1} + \sum_{k=1}^{m-1} 2P_{2k+1}$. By induction,

$$\textbf{error}(\textstyle\sum_{k=1}^{n-1} a_k R_k) \quad \le \quad \textbf{error}(\textstyle\sum_{k=1}^{m-1} 2P_{2k+1}).$$

Thus,

$$\textbf{error}(\textstyle\sum_{k=1}^{n} a_k R_k) \quad \le \quad \textbf{error}(P_{2m+1} + \textstyle\sum_{k=1}^{m-1} 2P_{2k+1}).$$

Next, if $a_{n-1} = 0$, then choose the largest $\ell > 0$ such that one of the sums $\sum_{k=1}^{\ell} 2P_{2k+1}$ or $P_{2\ell+1} + \sum_{k=1}^{\ell-1} 2P_{2k+1}$ is less than or equal to $\sum_{k=1}^{n-2} a_k R_k$. By induction, the error of $\sum_{k=1}^{n-2} a_k R_k$ is less than the error of $\sum_{k=1}^{\ell} 2P_{2k+1}$ or $P_{2\ell+1} + \sum_{k=1}^{\ell-1} 2P_{2k+1}$. It follows that

$$\textbf{error}(\textstyle\sum_{k=1}^{n} a_k R_k) \quad \le \quad \textbf{error}(\textstyle\sum_{k=1}^{m-1} 2P_{2k+1}).$$

`Case c.` Suppose that $R_n = P_{2m+1}$ and $a_n = 2$. If $\sum_{k=1}^{n} a_k R_k > \sum_{k=1}^{m} 2P_{2k+1}$, then the sum $P_{2m+1} + \sum_{k=1}^{n-1} a_k R_k$ is greater than $P_{2m+1} + \sum_{k=1}^{m-1} 2P_{2k+1}$. Hence, by induction,

$$\textbf{error}(P_{2m+1} + \textstyle\sum_{k=1}^{n-1} a_k R_k) \quad \le \quad \textbf{error}(P_{2m+1} + \textstyle\sum_{k=1}^{m-1} 2P_{2k+1}).$$

Thus,

$$\textbf{error}(\textstyle\sum_{k=1}^{n} a_k R_k) \quad \le \quad \textbf{error}(\textstyle\sum_{k=1}^{m} 2P_{2k+1}).$$

If $\sum_{k=1}^{n} a_k R_k < \sum_{k=1}^{m} 2P_{2k+1}$, then $P_{2m+1} + \sum_{k=1}^{n-1} a_k R_k$ is less than $P_{2m+1} + \sum_{k=1}^{m-1} 2P_{2k+1}$. But then $P_{2m+1} + \sum_{k=1}^{n-1} a_k R_k$ must be greater than $\sum_{k=1}^{m-1} 2P_{2k+1}$. Thus,

$$\textbf{error}(\textstyle\sum_{k=1}^{n} a_k R_k) \quad \le \quad \textbf{error}(P_{2m+1} + \sum_{k=1}^{m-1} 2P_{2k+1}).$$

`Case d.` Suppose that $R_n = P_{2m+1}$ and $a_n = 3$. First note that if $2P_{2m+1} + \sum_{k=1}^{n-1} a_k R_k > \sum_{k=1}^{m} 2P_{2k+1}$, then $\sum_{k=1}^{n} a_k R_k > 3P_{2m+1} + \sum_{k=1}^{m-1} 2P_{2k+1} = H_{2m+2}$, a contradiction. Hence, $2P_{2m+1} + \sum_{k=1}^{n-1} a_k R_k < \sum_{k=1}^{m} 2P_{2k+1}$, but $2P_{2m+1} + \sum_{k=1}^{n-1} a_k R_k > P_{2m+1} + \sum_{k=1}^{m-1} 2P_{2k+1}$. It follows by induction that

$$\textbf{error}(\textstyle\sum_{k=1}^{n} a_k R_k) \quad \le \quad \textbf{error}(\sum_{k=1}^{m} 2P_{2k+1}).$$

The proof of (2) is similar to the proof of (1) and is omitted.

Therefore, the operations given in (1) and (2) both behave as additive homomorphisms on the specified sums. □

We now turn our attention to the *silver game*, $\mathscr{S}$, given by the infinite octal code

$$\mathbf{0.d_1 d_2 d_3}\ldots,$$

where

$$\mathbf{d}_i = \begin{cases} 1, & \text{if } i \in \mathcal{B}_\alpha \\ 2, & \text{if } i \in \mathcal{B}_\beta. \end{cases}$$

(Recall that $\mathcal{B}_\alpha = \{\lfloor n \cdot (1+\sqrt{2})\rfloor : n \in \mathbb{N}\}$ and $\mathcal{B}_\beta = \{\lfloor n \cdot \left(\frac{2+\sqrt{2}}{2}\right)\rfloor : n \in \mathbb{N}^+\}$.)

The beginning of this octal code looks like:

$$\mathbf{0.21212212122121212212122121212212122121221}\ldots.$$

A plot of the first 10,000 nim-values is shown in figure 2.

The main aim of this section is to prove the following theorem.

Theorem 7. *The $\mathcal{P}$ positions of the silver game occur at the following heap sizes:*

$$0, P_1, H_2, P_3, H_4, P_5, H_6 \ldots, P_{2k+1}, H_{2k+2}, \ldots.$$

Proof. It is easy to see from the game rules that any nonempty heap $\mathscr{S}_k$ with size $k \in \mathcal{B}_\alpha$ has nonzero nim-value. Next, note that, by definition, no heap $\mathscr{S}_k$ with $k \in \mathcal{B}_\beta$ has a move to the empty heap.

Now let $t \in \mathcal{B}_\beta$ with $t \notin \{P_1, H_2, P_3, H_4, P_5, H_6 \ldots, P_{2k+1}, H_{2k+2}, \ldots\}$. Then

$$t = \sum_{k=0}^{n} a_k R'_{k+1} = \left\lfloor \left(\sum_{k=0}^{n} a_k R_k\right) \cdot \beta \right\rfloor,$$

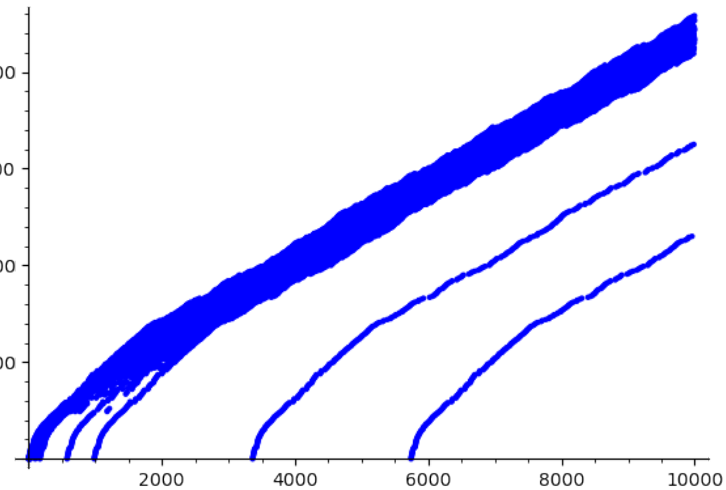

Figure 2: Nim-values for the silver game up to heap size 10,000

by Theorem 6 (2). Notice that if

$$j = \begin{cases} 2P_{2m+1} + \sum\limits_{k=0}^{n-1} a_k R'_{k+1}, & \text{if } R'_{n+1} = P_{2m+1} \text{ and } a_n = 3 \\ P_{2m+1} + \sum\limits_{k=0}^{n-1} a_k R'_{k+1}, & \text{if } R'_{n+1} = P_{2m+1} \text{ and } a_n = 2 \\ \sum\limits_{k=0}^{n-1} a_k R'_{k+1}, & \text{if } R'_{n+1} = P_{2m+1} \text{ or } H_{2m} \text{ and } a_n = 1 \end{cases}$$

then $t - j \in \mathcal{B}_\beta$ is an option. Thus, every non-Pell number and non-half-companion number $t \in \mathcal{B}_\beta$ has a move to a Pell number or a half-companion number in $\mathcal{B}_\beta$.

The last thing that we need to show is that there are no moves of the following types:

(a) $P_{2m+1} \in \mathcal{B}_\beta$ to $P_{2\ell+1} \in \mathcal{B}_\beta$,

(b) $P_{2m+1} \in \mathcal{B}_\beta$ to $H_{2\ell} \in \mathcal{B}_\beta$,

(c) $H_{2m} \in \mathcal{B}_\beta$ to $P_{2\ell+1} \in \mathcal{B}_\beta$, or

(d) $H_{2m} \in \mathcal{B}_\beta$ to $H_{2\ell} \in \mathcal{B}_\beta$.

For (a) we observe that

$$\begin{aligned}
P_{2m+1} - P_{2\ell+1} &= (P_{2m+1} - P_{2m-1}) + (P_{2m-1} - P_{2m-3}) + \cdots + (P_{2\ell+3} - P_{2\ell+1}) \\
&= 2P_{2m} + 2P_{2m-2} + \cdots + 2P_{2\ell+2} \\
&= \left\lfloor \left(\sum_{k=\ell}^{m-1} 2P_{2k+1} \right) \cdot \alpha \right\rfloor .
\end{aligned}$$

For (b) we have

$$\begin{aligned}
P_{2m+1} - H_{2\ell} &= (P_{2m+1} - P_{2m-1}) + (P_{2m-1} - P_{2m-3}) + \cdots + (P_{2\ell+1} - H_{2\ell}) \\
&= 2P_{2m} + 2P_{2m-2} + \cdots + 2P_{2\ell+2} + P_{2\ell} \\
&= \left\lfloor \left(\sum_{k=\ell}^{m-1} 2P_{2k+1} + P_{2\ell-1} \right) \cdot \alpha \right\rfloor .
\end{aligned}$$

Type (c) is similar to type (b):

$$\begin{aligned}
H_{2m} - P_{2\ell+1} &= (H_{2m} - P_{2m-1}) + (P_{2m-1} - P_{2m-3}) + \cdots + (P_{2\ell+3} - P_{2\ell+1}) \\
&= P_{2m} + 2P_{2m-2} + \cdots + 2P_{2\ell+2} \\
&= \left\lfloor \left(P_{2m-1} + \sum_{k=\ell}^{m-2} 2P_{2k+1} \right) \cdot \alpha \right\rfloor .
\end{aligned}$$

Finally, for type (d), we have

$$\begin{aligned}
H_{2m} - H_{2\ell} &= (H_{2m} - H_{2m-2}) + (H_{2m-2} - H_{2m-4}) + \cdots + (H_{2\ell+2} - H_{2\ell}) \\
&= P_{2m} + 2P_{2m-2} + \cdots + 2P_{2\ell+2} + P_{2\ell} \\
&= \left\lfloor \left(P_{2m-1} + \sum_{k=\ell}^{m-2} 2P_{2k+1} + P_{2\ell-1} \right) \cdot \alpha \right\rfloor .
\end{aligned}$$

Thus, every one of the above moves is an element of $\mathcal{B}_\alpha$ and hence there is no legal way to perform such a move. Therefore, the $\mathcal{P}$-positions of $\mathscr{S}$ are precisely the heaps of size

$$0, P_1, H_2, P_3, H_4, P_5, H_6 \ldots, P_{2k+1}, H_{2k+2}, \ldots .$$

□

4. The Bronze Game

In this section we introduce three infinite sequences of nonnegative integers which are associated with the bronze mean $\frac{3+\sqrt{13}}{2}$. The *bronze numbers* are given by

$$B_0 = 0, B_1 = 1, \text{ and } B_{n+1} = 3B_n + B_{n-1}, \text{ for } n \geq 1.$$

Thus, the first few are

$$0, 1, 3, 10, 33, 109, 360, 1189, 3927, 12970.$$

The *companion bronze numbers of the first type* are given by

$$C_0 = 1, C_1 = 1, \text{ and } C_{n+1} = 3C_n + C_{n-1}, \text{ for } n \geq 1.$$

The first few are

$$1, 1, 4, 13, 43, 142, 469, 1549, 5116, 16897.$$

Note that $C_n = B_n + B_{n-1}$.

Lastly, the *companion bronze numbers of the second type* are given by

$$D_0 = 1, D_1 = 2, \text{ and } D_{n+1} = 3D_n + D_{n-1}, \text{ for } n \geq 1.$$

The first few are

$$1, 2, 7, 23, 76, 251, 829, 2738, 9043, 29867.$$

Note that $D_n = 2B_n + B_{n-1}$.

It is also straightforward to check that, except for $B_1 = C_0 = C_1 = D_0 = 1$, the three sets of numbers share no common elements.

We will prove a theorem which asserts that every positive natural number can be expressed as a sum of the mixed set of numbers comprised of

$$\{D_0, (3B_1), (B_1 + C_2), D_2, (3B_3), (B_3 + C_4), \dots\},$$

subject to several conditions.

Theorem 8. *Let $t \in \mathbb{N}^+$. Then*

$$t = \sum_{k=0}^{n} a_k R_k,$$

where

(1) $R_k = D_{2k/3}$ *if* $k \equiv 0 \pmod 3$*,* $R_k = (3B_{(2k+1)/3})$ *if* $k \equiv 1 \pmod 3$*, and* $R_k = (B_{(2k-1)/3} + C_{(2k+2)/3})$ *if* $k \equiv 2 \pmod 3$*,*

(2) $R_n \leq t$*, but* $R_{n+1} > t$*,*

(3) $0 \leq a_0 \leq 2$ *and if* $a_1 = 1$ *or* $a_2 = 1$*, then* $a_0 \leq 1$*,*

(4) if $k > 0$*, then* $0 \leq a_k \leq 4$*, if* $k \equiv 0 \pmod 3$*, and* $0 \leq a_k \leq 1$*, otherwise,*

(5) if $a_k = 4$*, then* $a_{k-1} = 0$*,* $a_{k+1} = 0$*,* $a_{k-3} < 4$*, and* $a_{k+3} < 4$*,*

(6) if $a_{3m+1} = 1$*, then* $a_{3m+2} = 0$*, and, if* $a_{3m+2} = 1$*, then* $a_{3m+1} = 0$*,*

(7) if $a_{3m} = 3$*, then (if* $a_{3m+1} = 1$ *or* $a_{3m+2} = 1$*, then* $a_{3m-1} = a_{3m-2} = 0$*, and vice-versa).*

Proof. As with the other representations in this exposition, a representation from this mixed set of numbers can be constructed in a greedy fashion. For (3), first note that $3D_0 = (3B_1)$. Next, suppose that $a_1 = 1$ and $a_0 = 2$. Then $2D_0 + (3B_1) = (B_1 + C_2)$. Similarly, if $a_2 = 1$ and $a_0 = 2$, then $2D_0 + (B_1 + C_2) = D_2$.

Moving on to (4), if $a_k = 5$, for $k = 3m$ and $m > 0$, then

$$5D_{2m} > (3B_{2m+1}).$$

Hence, $a_k \leq 4$. Now, if $a_k = 2$, for $k = 3m + 1$ or $3m + 2$, then we get

$$2(3B_{2m+1}) = 6B_{2m+1} > (B_{2m+1} + C_{2m+2})$$

or

$$2(B_{2m+1} + C_{2m+2}) > D_{2m+2},$$

respectively. Thus, $a_k \leq 1$, for $k \equiv 1$ or $2 \pmod 3$.

For (5), suppose that $a_k = a_{3m} = 4$ and $a_{k+1} = a_{3m+1} = 1$. But then

$$(3B_{2m+1}) + 4D_{2m} > (B_{2m+1} + C_{2m+2}).$$

Hence, $a_{k+1} = 0$. A similar argument shows that $a_{k-1} = 0$.

Note also, that if $a_k = a_{3m} = 4$ and $a_{k-3} = a_{3m-3} = 4$, then

$$4D_{2m} + 4D_{2m-2} > (3B_{2m+1}).$$

Similarly, if $a_{k+3} = a_{3m+3} = 4$ and $a_k = a_{3m} = 4$, then

$$4D_{2m+2} + 4D_{2m} > (3B_{2m+3}).$$

Now we turn our attention to (6). If $a_{3m+1} = 1$ and $a_{3m+2} = 1$, then

$$(3B_{2m+1}) + (B_{2m+1} + C_{2m+2}) > D_{2m+2}.$$

We finish the proof of this result by establishing (7). If $a_{3m} = 3$, $a_{3m+1} = 1$, and $a_{3m-2} = 1$, then

$$(3B_{2m+1}) + 3D_{2m} + (B_{2m-1} + C_{2m}) > (B_{2m+1} + C_{2m+2}).$$

The other three cases are similar. □

A result similar to the one above is stated below. Its proof is almost identical and is therefore omitted. The following theorem asserts that every positive natural number can be expressed as a sum of the mixed set of numbers comprised of

$$\{B_1, C_2, D_2, B_3, C_4, D_4, B_5, C_6, D_6, \dots\},$$

subject to several conditions.

Theorem 9. *Let $t \in \mathbb{N}^+$. Then*

$$t = \sum_{k=1}^{n} a_k R_k,$$

where

(1) $R_k = B_{(2k+1)/3}$ *if* $k \equiv 1 \pmod 3$, $R_k = C_{(2k+2)/3}$ *if* $k \equiv 2 \pmod 3$, *and* $R_k = D_{2k/3}$ *if* $k \equiv 0 \pmod 3$,

(2) $R_n \le t$, *but* $R_{n+1} > t$,

(3) $0 \le a_1 \le 3$ *and if* $a_1 = 3$, *then* $a_2 = 0$ *and* $a_3 = 0$,

(4) if $k > 1$, *then* $0 \le a_k \le 4$, *if* $k \equiv 1 \pmod 3$, *and* $0 \le a_k \le 1$, *otherwise,*

(5) if $a_k = 4$, *then* $a_{k-1} = 0$, $a_{k+1} = 0$, $a_{k-3} < 4$, *and* $a_{k+3} < 4$,

(6) if $a_{3m} = 1$, *then* $a_{3m+2} = 0$, *and, if* $a_{3m+2} = 1$, *then* $a_{3m} = 0$,

(7) if $a_{3m+1} = 3$, *then (if* $a_{3m} = 1$ *or* $a_{3m+2} = 1$, *then* $a_{3m-1} = a_{3m-3} = 0$, *and vice-versa).*

Let $\alpha = \frac{3+\sqrt{13}}{2}$ and $\beta = \frac{5+\sqrt{13}}{6}$ for the remainder of this section. As in the last two sections, the following proposition is very helpful.

Proposition 3. *Let B_n be the n^{th} bronze number, C_n be the n^{th} companion bronze number of the first type, and D_n be the n^{th} companion bronze number of the second type. Then, for $k \ge 0$,*

(1) $B_{2k+1} \cdot \alpha = B_{2k+2} + \left(\frac{\sqrt{13}-3}{2}\right)^{2k+1}$,

(2) $C_{2k} \cdot \alpha = C_{2k+1} + \frac{13+\sqrt{13}}{2\sqrt{13}} \left(\frac{\sqrt{13}-3}{2}\right)^{2k}$,

(3) $D_{2k} \cdot \alpha = D_{2k+1} + \frac{13-\sqrt{13}}{2\sqrt{13}} \left(\frac{\sqrt{13}-3}{2}\right)^{2k}$,

(4) $(3B_{2k+1}) \cdot \beta = C_{2k+2} + \frac{13-3\sqrt{13}}{2\sqrt{13}} \left(\frac{\sqrt{13}-3}{2}\right)^{2k}$,

(5) $(B_{2k+1}+C_{2k+2})\cdot\beta = D_{2k+2}+\frac{65-17\sqrt{13}}{6\sqrt{13}}\left(\frac{\sqrt{13}-3}{2}\right)^{2k}$, *and*

(6) $D_{2k}\cdot\beta = B_{2k+1}+\frac{13-\sqrt{13}}{6\sqrt{13}}\left(\frac{\sqrt{13}-3}{2}\right)^{2k}$.

Proof. All six of these statements follow from the three Binet-like formulas:

$$\begin{aligned}
B_n &= \tfrac{1}{\sqrt{13}}\left(\left(\tfrac{3+\sqrt{13}}{2}\right)^n-\left(\tfrac{3-\sqrt{13}}{2}\right)^n\right),\\
C_n &= \tfrac{1}{2\sqrt{13}}\left((-1+\sqrt{13})\left(\tfrac{3+\sqrt{13}}{2}\right)^n+(1+\sqrt{13})\left(\tfrac{3-\sqrt{13}}{2}\right)^n\right), \text{ and}\\
D_n &= \tfrac{1}{2\sqrt{13}}\left((1+\sqrt{13})\left(\tfrac{3+\sqrt{13}}{2}\right)^n+(-1+\sqrt{13})\left(\tfrac{3-\sqrt{13}}{2}\right)^n\right).
\end{aligned}$$

□

Our next goals are to show that

$$\begin{aligned}
0 \leq \tfrac{13-\sqrt{13}}{2\sqrt{13}}\textstyle\sum_{k\equiv 0} a_k\left(\tfrac{\sqrt{13}-3}{2}\right)^{2k} &+ \textstyle\sum_{k\equiv 1} a_k\left(\tfrac{\sqrt{13}-3}{2}\right)^{2k+1}+\\
&+\tfrac{13+\sqrt{13}}{2\sqrt{13}}\textstyle\sum_{k\equiv 2} a_k\left(\tfrac{\sqrt{13}-3}{2}\right)^{2k}<1
\end{aligned}$$

for a given valid mixed bronze representation of the type in Theorem 9, and

$$\begin{aligned}
0 \leq \tfrac{13-\sqrt{13}}{6\sqrt{13}}\textstyle\sum_{k\equiv 0} a_k\left(\tfrac{\sqrt{13}-3}{2}\right)^{2k} &+ \tfrac{13-3\sqrt{13}}{2\sqrt{13}}\textstyle\sum_{k\equiv 1} a_k\left(\tfrac{\sqrt{13}-3}{2}\right)^{2k}+\\
&+\tfrac{65-17\sqrt{13}}{6\sqrt{13}}\textstyle\sum_{k\equiv 2} a_k\left(\tfrac{\sqrt{13}-3}{2}\right)^{2k}<1
\end{aligned}$$

for a given valid mixed bronze representation of the type in Theorem 8. Taken together, these two results establish the next theorem.

Theorem 10. *The following hold:*

(1) If

$$R_k=\begin{cases} B_{(2k+1)/3}, & \text{if } k\equiv 1 \pmod 3\\ C_{(2k+2)/3}, & \text{if } k\equiv 2 \pmod 3\\ D_{2k/3}, & \text{if } k\equiv 0 \pmod 3\end{cases}$$

and

$$R'_k=\begin{cases} D_{(2k+1)/3}, & \text{if } k\equiv 1 \pmod 3\\ B_{(2k+2)/3}, & \text{if } k\equiv 2 \pmod 3\\ C_{(2k+3)/3}, & \text{if } k\equiv 0 \pmod 3\end{cases},$$

then, for $n\geq 1$,

$$\left\lfloor\left(\sum_{k=1}^{n} a_k R_k\right)\cdot\alpha\right\rfloor=\sum_{k=1}^{n} a_k R'_{k+1}.$$

(2) If

$$R_k = \begin{cases} D_{2k/3}, & \text{if } k \equiv 0 \pmod 3) \\ 3B_{(2k+1)/3}, & \text{if } k \equiv 1 \pmod 3) \\ B_{(2k-1)/3} + C_{(2k+2)/3}, & \text{if } k \equiv 2 \pmod 3) \end{cases}$$

and

$$R'_k = \begin{cases} D_{2k/3}, & \text{if } k \equiv 0 \pmod 3) \\ B_{(2k+1)/3}, & \text{if } k \equiv 1 \pmod 3) \\ C_{(2k+2)/3}, & \text{if } k \equiv 2 \pmod 3) \end{cases},$$

then, for $n \geq 0$,

$$\left\lfloor \left(\sum_{k=0}^{n} a_k R_k \right) \cdot \beta \right\rfloor = \sum_{k=0}^{n} a_k R'_{k+1}.$$

Proof. For (1), we will show that the largest errors come from representations of the form

$$B_{2n+1} + 3B_{2n-1} + \cdots + 3B_5 + 3B_3 + 3B_1,$$

$$2B_{2n+1} + 3B_{2n-1} + \cdots + 3B_5 + 3B_3 + 3B_1, \quad \text{or}$$

$$3B_{2n+1} + 3B_{2n-1} + \cdots + 3B_5 + 3B_3 + 3B_1.$$

First note that

$$\begin{aligned} \left(B_{2n+1} + \textstyle\sum_{k=0}^{n-1} 3B_{2k+1} \right) \cdot \alpha \quad &= \quad \left(B_{2n+2} + \textstyle\sum_{k=0}^{n-1} 3B_{2k+2} \right) + \left(\tfrac{\sqrt{13}-3}{2} \right)^{2n+1} \\ &\quad + 3 \textstyle\sum_{k=0}^{n-1} \left(\tfrac{\sqrt{13}-3}{2} \right)^{2k+1} \\ &= \quad \left(B_{2n+2} + \textstyle\sum_{k=0}^{n-1} 3B_{2k+2} \right) + 1 \\ &\quad - \tfrac{5-\sqrt{13}}{2} \left(\tfrac{\sqrt{13}-3}{2} \right)^{2n} \end{aligned}$$

and

$$0 < \left(1 - \frac{5-\sqrt{13}}{2} \left(\frac{\sqrt{13}-3}{2} \right)^{2n} \right) < 1 \quad \text{for all} \quad n \geq 0.$$

Secondly, we have that

$$\begin{aligned} \left(2B_{2n+1} + \textstyle\sum_{k=0}^{n-1} 3B_{2k+1} \right) \cdot \alpha \quad &= \quad \left(2B_{2n+2} + \textstyle\sum_{k=0}^{n-1} 3B_{2k+2} \right) + 2 \left(\tfrac{\sqrt{13}-3}{2} \right)^{2n+1} \\ &\quad +3 \textstyle\sum_{k=0}^{n-1} \left(\tfrac{\sqrt{13}-3}{2} \right)^{2k+1} \\ &= \quad \left(2B_{2n+2} + \textstyle\sum_{k=0}^{n-1} 3B_{2k+2} \right) + 1 - \\ &\quad (4-\sqrt{13}) \left(\tfrac{\sqrt{13}-3}{2} \right)^{2n} \end{aligned}$$

and

$$0 < \left(1 - (4 - \sqrt{13}) \left(\frac{\sqrt{13} - 3}{2}\right)^{2n}\right) < 1 \quad \text{for all} \quad n \geq 0.$$

Lastly, note that

$$\begin{aligned} \left(\textstyle\sum_{k=0}^{n} 3B_{2k+1}\right) \cdot \alpha &= \left(\textstyle\sum_{k=0}^{n} 3B_{2k+2}\right) + 3 \textstyle\sum_{k=0}^{n} \left(\frac{\sqrt{13}-3}{2}\right)^{2k+1} \\ &= \left(\textstyle\sum_{k=0}^{n} 3B_{2k+2}\right) + 1 - \frac{11-3\sqrt{13}}{2} \left(\frac{\sqrt{13}-3}{2}\right)^{2n} \end{aligned}$$

and

$$0 < \left(1 - \frac{11 - 3\sqrt{13}}{2} \left(\frac{\sqrt{13} - 3}{2}\right)^{2n}\right) < 1 \quad \text{for all} \quad n \geq 0.$$

As we have done with the other two metal means, we will show that the error of any representation $t = \sum_{k=1}^{n} a_k R_k$ is at most the error associated with one of the three sums shown above, the sum which is the largest representation of this type less than or equal to t. The proof of this fact will proceed by induction.

In table 3 we provide the representations of the first 142 positive integers to establish our basis steps.

From this table, we can see that the errors for $t = 4, 5, \ldots, 12$ are less than for $t = 3$. We also see that the errors for $t = 14, 15, \ldots 22$ are all less than the error for $t = 13$. Likewise, the errors for $t = 24, 25, \ldots, 32$ are less than for $t = 23$ and the errors for $t = 34, 35, \ldots, 141$ are less than for $t = 33$.

Now let $t = \sum_{k=1}^{n} a_k R_k$ be a given representation, where $t \geq 142$. There are six cases:

(a) $R_n = C_{2m}$ and $a_n = 1$,

(b) $R_n = D_{2m}$ and $a_n = 1$,

(c) $R_n = B_{2m+1}$ and $a_n = 1$,

(d) $R_n = B_{2m+1}$ and $a_n = 2$,

(e) $R_n = B_{2m+1}$ and $a_n = 3$, and

(f) $R_n = B_{2m+1}$ and $a_n = 4$.

`Case a.` Suppose that $R_n = C_{2m}$ and $a_n = 1$. The possibilities for a_{n-1} in this case are $a_{n-1} = 0, 1, 2$ or 3. We first suppose that $a_{n-1} = 3$. If

$$\sum_{k=1}^{n-1} a_k R_k \geq 3B_{2m-1} + 3B_{2m-3} + \cdots + 3B_3 + 3B_1,$$

then

$$\sum_{k=1}^{n} a_k R_k \geq C_{2m} + 3B_{2m-1} + 3B_{2m-3} + \cdots + 3B_3 + 3B_1 = D_{2m},$$

a contradiction.

Hence, we must have

$$\begin{aligned} 3B_{2m-1} + 3B_{2m-3} + \cdots + 3B_3 + 3B_1 &> \textstyle\sum_{k=1}^{n-1} a_k R_k \\ &\geq 2B_{2m-1} + 3B_{2m-3} + \cdots + 3B_3 + 3B_1. \end{aligned}$$

This implies that

$$\mathbf{error}\left(\sum_{k=1}^{n-1} a_k R_k\right) \leq \mathbf{error}(2B_{2m-1} + 3B_{2m-3} + \cdots + 3B_3 + 3B_1).$$

Therefore,

$$\begin{aligned} \mathbf{error}\left(\textstyle\sum_{k=1}^{n} a_k R_k\right) &\leq \mathbf{error}(C_{2m} + 2B_{2m-1} + 3B_{2m-3} + \cdots + 3B_3 + 3B_1) \\ &\leq \mathbf{error}(3B_{2m-1} + 3B_{2m-3} + \cdots + 3B_3 + 3B_1). \end{aligned}$$

Next, we consider the case where $a_{n-1} = 2$. Note that we either have

$$\sum_{k=1}^{n-1} a_k R_k > 2B_{2m-1} + 3B_{2m-3} + \cdots + 3B_3 + 3B_1$$

or

$$\sum_{k=1}^{n-1} a_k R_k \leq 2B_{2m-1} + 3B_{2m-3} + \cdots + 3B_3 + 3B_1.$$

In the former case, we have, as before

$$\begin{aligned} \mathbf{error}\left(\textstyle\sum_{k=1}^{n} a_k R_k\right) &\leq \mathbf{error}(C_{2m} + 2B_{2m-1} + 3B_{2m-3} + \cdots + 3B_3 + 3B_1) \\ &\leq \mathbf{error}(3B_{2m-1} + 3B_{2m-3} + \cdots + 3B_3 + 3B_1). \end{aligned}$$

In the latter case, we have

$$\begin{aligned} \mathbf{error}\left(\textstyle\sum_{k=1}^{n} a_k R_k\right) &\leq \mathbf{error}(C_{2m} + B_{2m-1} + 3B_{2m-3} + \cdots + 3B_3 + 3B_1) \\ &\leq \mathbf{error}(3B_{2m-1} + 3B_{2m-3} + \cdots + 3B_3 + 3B_1), \end{aligned}$$

since

$$\frac{13-\sqrt{13}}{2\sqrt{13}}\left(\frac{\sqrt{13}-3}{2}\right)^{2m} < 2\left(\frac{\sqrt{13}-3}{2}\right)^{2m-1}.$$

If $a_{n-1} = 1$, then, using previous logic, we see that

$$\textbf{error}\left(\sum_{k=1}^{n} a_k R_k\right) \leq \textbf{error}(2B_{2m-1} + 3B_{2m-3} + \cdots + 3B_3 + 3B_1),$$

which implies that

$$\textbf{error}\left(\sum_{k=1}^{n} a_k R_k\right) \leq \textbf{error}(3B_{2m-1} + 3B_{2m-3} + \cdots + 3B_3 + 3B_1).$$

Our last subcase is $a_{n-1} = 0$. In this case, we find the closest sum of the form

$$3B_{2\ell+1} + 3B_{2\ell-1} + \cdots + 3B_3 + 3B_1,$$

$$2B_{2\ell+1} + 3B_{2\ell-1} + \cdots + 3B_3 + 3B_1,$$

or of the form

$$B_{2\ell+1} + 3B_{2\ell-1} + \cdots + 3B_3 + 3B_1,$$

which is less than or equal to $\sum_{k=1}^{n-2} a_k R_k$. Induction and the process of swapping the error of C_{2m} for the error of one B_{2m-1} term now gives us our desired result.

`Case b.` This case is similar to case (a).

`Case c.` Suppose that $R_n = B_{2m+1}$ and $a_n = 1$. We either have

$$\sum_{k=1}^{n} a_k R_k \geq B_{2m+1} + 3B_{2m-1} + \cdots + 3B_3 + 3B_1$$

or

$$\sum_{k=1}^{n} a_k R_k < B_{2m+1} + 3B_{2m-1} + \cdots + 3B_3 + 3B_1.$$

The former case implies that

$$\textbf{error}\left(\sum_{k=1}^{n-1} a_k R_k\right) \leq \textbf{error}(3B_{2m-1} + \cdots + 3B_3 + 3B_1),$$

which then implies that

$$\textbf{error}\left(\sum_{k=1}^{n} a_k R_k\right) \leq \textbf{error}(B_{2m+1} + 3B_{2m-1} + \cdots + 3B_3 + 3B_1).$$

In the latter case we have

$$\sum_{k=1}^{n-1} a_k R_k < 3B_{2m-1} + \cdots + 3B_3 + 3B_1.$$

We choose the largest $\ell \geq 1$ such that

$$\sum_{k=1}^{n-1} a_k R_k \geq B_{2\ell-1} + 3B_{2\ell-3} + \cdots + 3B_3 + 3B_1,$$

or

$$\sum_{k=1}^{n-1} a_k R_k \geq 2B_{2\ell-1} + 3B_{2\ell-3} + \cdots + 3B_3 + 3B_1,$$

or

$$\sum_{k=1}^{n-1} a_k R_k \geq 3B_{2\ell-1} + 3B_{2\ell-3} + \cdots + 3B_3 + 3B_1.$$

In any of these cases, it follows by induction that

$$\mathbf{error}\left(\sum_{k=1}^{n-1} a_k R_k\right) \leq \mathbf{error}(c_{2\ell-1}B_{2\ell-1} + 3B_{2\ell-3} + \cdots + 3B_3 + 3B_1),$$

where $1 \leq c_{2\ell-1} \leq 3$. Hence, it follows that

$$\mathbf{error}\left(\sum_{k=1}^{n} a_k R_k\right) \leq \mathbf{error}(3B_{2m-1} + \cdots + 3B_3 + 3B_1),$$

as

$$\left(\frac{\sqrt{13}-3}{2}\right)^{2m+1} < 3\left(\frac{\sqrt{13}-3}{2}\right)^{2m-1} + \ \cdots \ + 3\left(\frac{\sqrt{13}-3}{2}\right)^{2\ell+1} + (3 - c_{2\ell-1})\left(\frac{\sqrt{13}-3}{2}\right)^{2\ell-1}.$$

`Case d.` Suppose that $R_n = B_{2m+1}$ and $a_n = 2$. If

$$\sum_{k=1}^{n} a_k R_k \geq 2B_{2m+1} + 3B_{2m-1} + \cdots + 3B_3 + 3B_1,$$

then

$$\mathbf{error}\left(\sum_{k=1}^{n-1} a_k R_k\right) \leq \mathbf{error}(3B_{2m-1} + \cdots + 3B_3 + 3B_1).$$

Thus,

$$\mathbf{error}\left(\sum_{k=1}^{n} a_k R_k\right) \leq \mathbf{error}(2B_{2m+1} + 3B_{2m-1} + \cdots + 3B_3 + 3B_1).$$

If

$$\sum_{k=1}^{n} a_k R_k < 2B_{2m+1} + 3B_{2m-1} + \cdots + 3B_3 + 3B_1,$$

then,

$$\sum_{k=1}^{n-1} a_k R_k < 3B_{2m-1} + \cdots + 3B_3 + 3B_1.$$

As in case (c), we choose the largest $\ell \geq 1$ such that

$$\sum_{k=1}^{n-1} a_k R_k \geq B_{2\ell-1} + 3B_{2\ell-3} + \cdots + 3B_3 + 3B_1,$$

or

$$\sum_{k=1}^{n-1} a_k R_k \geq 2B_{2\ell-1} + 3B_{2\ell-3} + \cdots + 3B_3 + 3B_1,$$

or

$$\sum_{k=1}^{n-1} a_k R_k \geq 3B_{2\ell-1} + 3B_{2\ell-3} + \cdots + 3B_3 + 3B_1.$$

Hence, it follows that

$$\mathbf{error}\left(\sum_{k=1}^{n} a_k R_k\right) \leq \mathbf{error}(B_{2m+1} + 3B_{2m-1} + \cdots + 3B_3 + 3B_1).$$

`Case e.` This case is similar to case (d).
`Case f.` This case is also similar to case (d).

As in the silver case, the proof of (2) is similar to the proof of (1) and is omitted.

Therefore, the operations given in (1) and (2) both behave as additive homomorphisms on the specified sums. □

We now turn our attention to the *bronze game*, $\mathscr{B}$, given by the infinite octal code

$$\mathbf{0.d_1 d_2 d_3} \ldots,$$

where

$$\mathbf{d}_i = \begin{cases} 1, & \text{if } i \in \mathcal{B}_\alpha \\ 2, & \text{if } i \in \mathcal{B}_\beta. \end{cases}$$

(Recall that $\mathcal{B}_\alpha = \{\lfloor n \cdot \left(\frac{3+\sqrt{13}}{2}\right) \rfloor : n \in \mathbb{N}\}$ and $\mathcal{B}_\beta = \{\lfloor n \cdot \left(\frac{5+\sqrt{13}}{6}\right) \rfloor : n \in \mathbb{N}^+\}$.) The beginning of this octal code looks like:

$$\mathbf{0.2212212212221221221222122122122212212212 2}\ldots.$$

A plot of the first 10,000 nim-values is shown in figure 3.

The main aim of this section is to prove the following theorem.

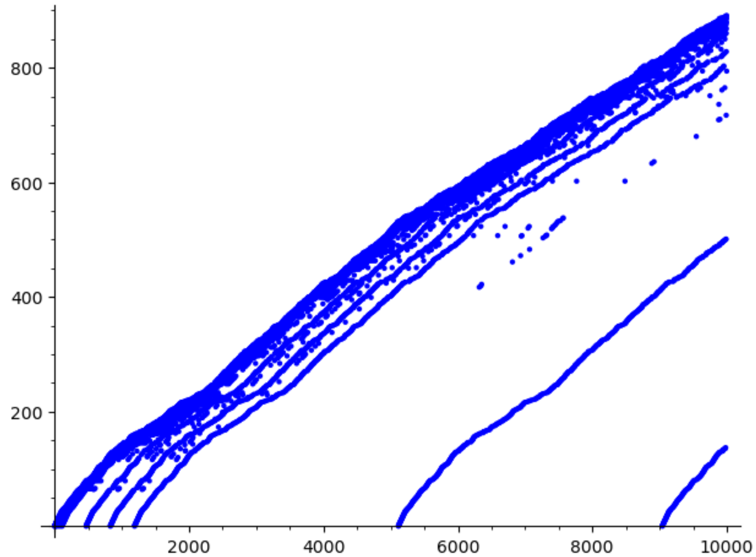

Figure 3: Nim-values for the bronze game up to heap size 10,000

Theorem 11. *The $\mathcal{P}$ positions of the bronze game occur at the following heap sizes:*

$$0, B_1, C_2, D_2, B_3, C_4, D_4, B_5, C_6, D_6 \ldots, B_{2k+1}, C_{2k+2}, D_{2k+2} \ldots.$$

Proof. It is easy to see from the game rules that any nonempty heap $\mathscr{B}_k$ with size $k \in \mathcal{B}_\alpha$ has nonzero nim-value. Next, note that, by definition, no heap $\mathscr{B}_k$ with $k \in \mathcal{B}_\beta$ has a move to the empty heap.

Now let $m \in \mathcal{B}_\beta$ with $m \notin \{B_1, C_2, D_2, B_3, C_4, D_4, \ldots, B_{2k+1}, C_{2k+2}, D_{2k+2} \ldots\}$. Then

$$m = \sum_{k=0}^{n} a_k R'_{k+1} = \left\lfloor \left(\sum_{k=0}^{n} a_k R_k \right) \cdot \beta \right\rfloor,$$

by Theorem 10 (2). Notice that if

$$j = \begin{cases} 3B_{2t+1} + \sum\limits_{k=0}^{n-1} a_k R'_{k+1}, & \text{if } R'_{n+1} = B_{2t+1} \text{ and } a_n = 4 \\ 2B_{2t+1} + \sum\limits_{k=0}^{n-1} a_k R'_{k+1}, & \text{if } R'_{n+1} = B_{2t+1} \text{ and } a_n = 3 \\ B_{2t+1} + \sum\limits_{k=0}^{n-1} a_k R'_{k+1}, & \text{if } R'_{n+1} = B_{2t+1} \text{ and } a_n = 2 \\ \sum\limits_{k=0}^{n-1} a_k R'_{k+1}, & \text{if } R'_{n+1} = B_{2t+1}, C_{2t}, \text{ or } D_{2t} \text{ and } a_n = 1 \end{cases}$$

then $m - j \in \mathcal{B}_\beta$ is an option.

Thus, every non-bronze number and non-companion number $m \in \mathcal{B}_\beta$ has a move to a bronze number or a companion number in $\mathcal{B}_\beta$.

The last thing that we need to show is that there are no moves of the following types:

(a) $B_{2t+1} \in \mathcal{B}_\beta$ to $B_{2\ell+1} \in \mathcal{B}_\beta$,

(b) $B_{2t+1} \in \mathcal{B}_\beta$ to $C_{2\ell} \in \mathcal{B}_\beta$,

(c) $B_{2t+1} \in \mathcal{B}_\beta$ to $D_{2\ell} \in \mathcal{B}_\beta$,

(d) $C_{2t} \in \mathcal{B}_\beta$ to $B_{2\ell+1} \in \mathcal{B}_\beta$,

(e) $C_{2t} \in \mathcal{B}_\beta$ to $C_{2\ell} \in \mathcal{B}_\beta$,

(f) $C_{2t} \in \mathcal{B}_\beta$ to $D_{2\ell} \in \mathcal{B}_\beta$,

(g) $D_{2t} \in \mathcal{B}_\beta$ to $B_{2\ell+1} \in \mathcal{B}_\beta$,

(h) $D_{2t} \in \mathcal{B}_\beta$ to $C_{2\ell} \in \mathcal{B}_\beta$, or

(i) $D_{2t} \in \mathcal{B}_\beta$ to $D_{2\ell} \in \mathcal{B}_\beta$.

For (a) we observe that

$$\begin{aligned} B_{2t+1} - B_{2\ell+1} &= (B_{2t+1} - B_{2t-1}) + (B_{2t-1} - B_{2t-3}) + \cdots + (B_{2\ell+3} - B_{2\ell+1}) \\ &= 3B_{2t} + 3B_{2t-2} + \cdots + 3B_{2\ell+2} \\ &= \left\lfloor \left(\sum_{k=\ell}^{t-1} 3B_{2k+1} \right) \cdot \alpha \right\rfloor . \end{aligned}$$

For (b) we have

$$\begin{aligned} B_{2t+1} - C_{2\ell} &= (B_{2t+1} - B_{2t-1}) + (B_{2t-1} - B_{2t-3}) + \cdots + (B_{2\ell+1} - C_{2\ell}) \\ &= 3B_{2t} + 3B_{2t-2} + \cdots + 3B_{2\ell+2} + 2B_{2\ell} \\ &= \left\lfloor \left(\sum_{k=\ell}^{t-1} 3B_{2k+1} + 2B_{2\ell-1} \right) \cdot \alpha \right\rfloor . \end{aligned}$$

For (c) we have

$$\begin{aligned} B_{2t+1} - D_{2\ell} &= (B_{2t+1} - B_{2t-1}) + (B_{2t-1} - B_{2t-3}) + \cdots + (B_{2\ell+1} - D_{2\ell}) \\ &= 3B_{2t} + 3B_{2t-2} + \cdots + 3B_{2\ell+2} + B_{2\ell} \\ &= \left\lfloor \left(\sum_{k=\ell}^{t-1} 3B_{2k+1} + B_{2\ell-1} \right) \cdot \alpha \right\rfloor . \end{aligned}$$

For (d) we have

$$\begin{aligned}
C_{2t} - B_{2\ell+1} &= (C_{2t} - B_{2t-1}) + (B_{2t-1} - B_{2t-3}) + \cdots + (B_{2\ell+3} - B_{2\ell+1}) \\
&= B_{2t} + 3B_{2t-2} + \cdots + 3B_{2\ell+2} \\
&= \left\lfloor \left(B_{2t-1} + \sum_{k=\ell}^{t-2} 3B_{2k+1} \right) \cdot \alpha \right\rfloor .
\end{aligned}$$

For (e) we have

$$\begin{aligned}
C_{2t} - C_{2\ell} &= (C_{2t} - B_{2t-1}) + (B_{2t-1} - B_{2t-3}) \\
&\quad + \cdots + (B_{2\ell+3} - B_{2\ell+1}) + (B_{2\ell+1} - C_{2\ell}) \\
&= B_{2t} + 3B_{2t-2} + \cdots + 3B_{2\ell+2} + 2B_{2\ell} \\
&= \left\lfloor \left(B_{2t-1} + \sum_{k=\ell}^{t-2} 3B_{2k+1} + 2B_{2\ell-1} \right) \cdot \alpha \right\rfloor .
\end{aligned}$$

For (f) we have

$$\begin{aligned}
C_{2t} - D_{2\ell} &= (C_{2t} - B_{2t-1}) + (B_{2t-1} - B_{2t-3}) \\
&\quad + \cdots + (B_{2\ell+3} - B_{2\ell+1}) + (B_{2\ell+1} - D_{2\ell}) \\
&= B_{2t} + 3B_{2t-2} + \cdots + 3B_{2\ell+2} + B_{2\ell} \\
&= \left\lfloor \left(B_{2t-1} + \sum_{k=\ell}^{t-2} 3B_{2k+1} + B_{2\ell-1} \right) \cdot \alpha \right\rfloor .
\end{aligned}$$

For (g) we have

$$\begin{aligned}
D_{2t} - B_{2\ell+1} &= (D_{2t} - B_{2t-1}) + (B_{2t-1} - B_{2t-3}) + \cdots + (B_{2\ell+3} - B_{2\ell+1}) \\
&= 2B_{2t} + 3B_{2t-2} + \cdots + 3B_{2\ell+2} \\
&= \left\lfloor \left(B_{2t-1} + \sum_{k=\ell}^{t-2} 3B_{2k+1} \right) \cdot \alpha \right\rfloor .
\end{aligned}$$

For (h) we have

$$\begin{aligned}
D_{2t} - C_{2\ell} &= (D_{2t} - B_{2t-1}) + (B_{2t-1} - B_{2t-3}) \\
&\quad + \cdots + (B_{2\ell+3} - B_{2\ell+1}) + (B_{2\ell+1} - C_{2\ell}) \\
&= 2B_{2t} + 3B_{2t-2} + \cdots + 3B_{2\ell+2} + 2B_{2\ell} \\
&= \left\lfloor \left(B_{2t-1} + \sum_{k=\ell}^{t-2} 3B_{2k+1} + 2B_{2\ell-1} \right) \cdot \alpha \right\rfloor .
\end{aligned}$$

For (i) we have

$$\begin{aligned} D_{2t} - D_{2\ell} &= (D_{2t} - B_{2t-1}) + (B_{2t-1} - B_{2t-3}) \\ &\quad + \cdots + (B_{2\ell+3} - B_{2\ell+1}) + (B_{2\ell+1} - D_{2\ell}) \\ &= 2B_{2t} + 3B_{2t-2} + \cdots + 3B_{2\ell+2} + B_{2\ell} \\ &= \left\lfloor \left(2B_{2t-1} + \sum_{k=\ell}^{t-2} 3B_{2k+1} + B_{2\ell-1} \right) \cdot \alpha \right\rfloor . \end{aligned}$$

Thus, every one of the above moves is an element of $\mathcal{B}_\alpha$ and hence there is no legal way to perform such a move. Therefore, the $\mathcal{P}$-positions of $\mathscr{B}$ are precisely the heaps of size

$$0, B_1, C_2, D_2, B_3, C_4, D_4, \ldots, B_{2k+1}, C_{2k+2}, D_{2k+2} \ldots .$$

□

5. Epilogue

For the reader who has made it this far, you may be wondering if anything can be said about the structure of the non-zero nim-values. It appears that, for a fixed nim-value, after an initial pre-period, the pattern of heap sizes that achieve this nim-value is a shifted version of the pattern of the $\mathcal{P}$-positions in each respective game. In figure 4 we provide plots of pre-period length versus nim-value.

From the looks of each plot, it seems unlikely that one would be able to determine which heap sizes achieve a particular nim-value.

It also does appear from empirical data that similar results hold for the metallic means beyond the three we have considered. It does become unclear, in general, how to determine the sets of numbers needed for the representations similar to the ones illustrated in Theorem 10 (2).

Finally, it could be asked why the infinite octal codes that were studied in this paper were chosen. Our goal was to stray from NIM (**0.333**...) as little as possible, but in an interesting way. Other variants were tried. They were either too messy or too orderly.

References

[1] M. Albert, R. Nowakowski, and D. Wolfe, *Lessons in Play: An Introduction to Combinatorial Game Theory*, AK Peters, Natick, 2007.

[2] J-P. Allouche and J. Shallit, *Automatic Sequences: Theory, Applications, Generalizations*, Cambridge Univ. Press, Cambridge, 2003.

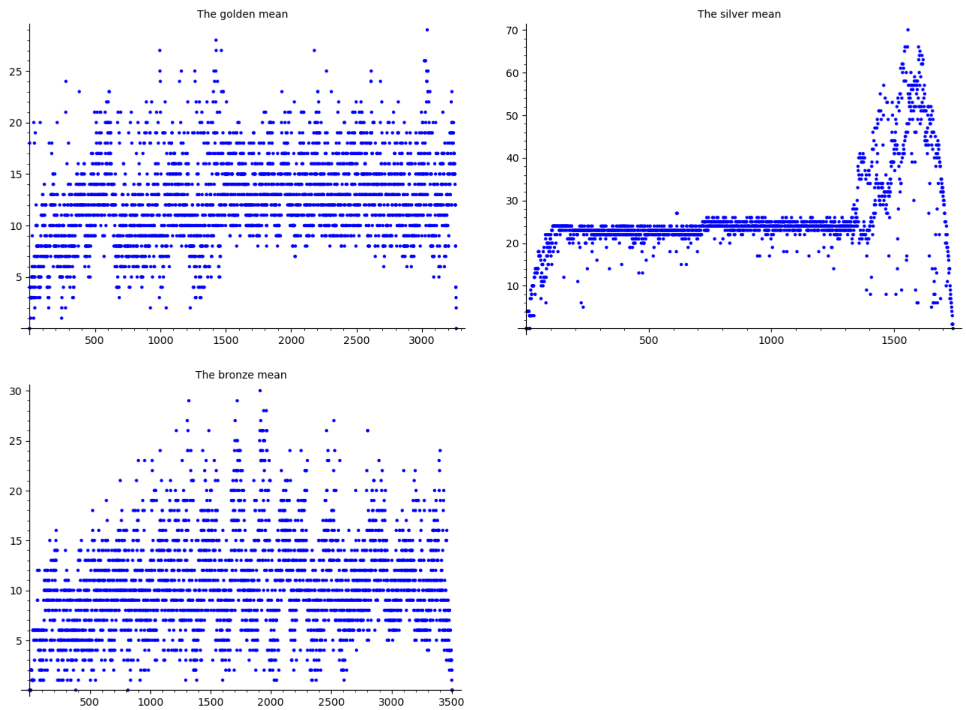

Figure 4: Length of pre-period as a function of nim-value for non-zero nim-values

[3] R. B. Austin, *Impartial and Partizan Games*, M.Sc. Thesis, The University of Calgary, 1976.

[4] S. Beatty, Problem 3173, *Amer. Math. Monthly* **33** (3) (1926), 159.

[5] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways*, Academic Press, London, 1982.

[6] J. H. Conway, *On Numbers and Games*, Academic Press, London, 1976.

[7] V. W. de Spinadel, The family of metallic means. *Symmetry Cult. Sci.* **10** (3) (1999), 317–338.

[8] A. Flammenkamp, Sprague-Grundy values of octal games (2018), wwwhomes.uni-bielefeld.de/achim/octal.html

[9] A. Gangolli and T. Plambeck, A note on periodicity in some octal games, *Internat. J. Game Theory* **18** (1989), 311–320.

[10] R. K. Guy and R. J. Nowakowski, Unsolved problems in combinatorial game theory, in *Games of No Chance*, Math. Sci. Res. Inst. Publ. **29**, 329-337, Cambridge Univ. Press, Cambridge, 1996.

[11] R. K. Guy and C. A. B. Smith, The g-values of various games, *Proc. Cambridge Philos. Soc.* **52** (1956), 514–526.

[12] A. N. Siegel, *Combinatorial Game Theory*, American Math. Soc., Providence, 2013.

[13] W. A. Wythoff, A modification of the game of nim, *Nieuw Arch. Wiskd.* **7** (2) (1907), 199–202.

$\mathbf{t}$	$\sum \mathbf{a_k R_k}$	$(\sum \mathbf{a_k R_k}) \cdot \alpha$	$\sum \mathbf{a_k R'_{k+1}}$	**Error**
1	$\mathbf{P_1}$	2.41421356237309	2	**0.414213562373095**
2	$\mathbf{2P_1}$	4.82842712474619	4	**0.828427124746190**
3	H_2	7.24264068711929	7	0.242640687119286
4	$H_2 + P_1$	9.65685424949238	9	0.656854249492381
5	P_3	12.0710678118655	12	0.0710678118654755
6	$P_3 + P_1$	14.4852813742386	14	0.485281374238571
7	$\mathbf{P_3 + 2P_1}$	16.8994949366117	16	**0.899494936611665**
8	$P_3 + H_2$	19.3137084989848	19	0.313708498984761
9	$P_3 + H_2 + P_1$	21.7279220613579	21	0.727922061357857
10	$2P_3$	24.1421356237310	24	0.142135623730951
11	$2P_3 + P_1$	26.5563491861040	26	0.556349186104047
12	$\mathbf{2P_3 + 2P_1}$	28.9705627484771	28	**0.970562748477143**
13	$2P_3 + H_2$	31.3847763108502	31	0.384776310850238
14	$2P_3 + H_2 + P_1$	33.7989898732233	33	0.798989873223331
15	$3P_3$	36.2132034355964	36	0.213203435596427
16	$3P_3 + P_1$	38.6274169979695	38	0.627416997969522
17	H_4	41.0416305603426	41	0.0416305603426181
18	$H_4 + P_1$	43.4558441227157	43	0.455844122715714
19	$H_4 + 2P_1$	45.8700576850888	45	0.870057685088806
20	$H_4 + H_2$	48.2842712474619	48	0.284271247461902
21	$H_4 + H_2 + P_1$	50.6984848098350	50	0.698484809834998
22	$H_4 + P_3$	53.1126983722081	53	0.112698372208094
23	$H_4 + P_3 + P_1$	55.5269119345812	55	0.526911934581186
24	$H_4 + P_3 + 2P_1$	57.9411254969543	57	0.941125496954285
25	$H_4 + P_3 + H_2$	60.3553390593274	60	0.355339059327378
26	$H_4 + P_3 + H_2 + P_1$	62.7695526217005	62	0.769552621700477
27	$H_4 + 2P_3$	65.1837661840736	65	0.183766184073569
28	$H_4 + 2P_3 + P_1$	67.5979797464467	67	0.597979746446661
29	P_5	70.0121933088198	70	0.0121933088197608
30	$P_5 + P_1$	72.4264068711929	72	0.426406871192853
31	$P_5 + 2P_1$	74.8406204335660	74	0.840620433565952
32	$P_5 + H_2$	77.2548339959390	77	0.254833995939045
33	$P_5 + H_2 + P_1$	79.6690475583121	79	0.669047558312137
34	$P_5 + P_3$	82.0832611206852	82	0.0832611206852363
35	$P_5 + P_3 + P_1$	84.4974746830583	84	0.497474683058329
36	$P_5 + P_3 + 2P_1$	86.9116882454314	86	0.911688245431428
37	$P_5 + P_3 + H_2$	89.3259018078045	89	0.325901807804520
38	$P_5 + P_3 + H_2 + P_1$	91.7401153701776	91	0.740115370177612
39	$P_5 + 2P_3$	94.1543289325507	94	0.154328932550712
40	$P_5 + 2P_3 + P_1$	96.5685424949238	96	0.568542494923804
41	$\mathbf{P_5 + 2P_3 + 2P_1}$	98.9827560572969	98	**0.982756057296903**

Table 2: Mixed-Pell representations of the first 41 positive integers

t	$\sum \mathbf{a_k R_k}$	$(\sum \mathbf{a_k R_k}) \cdot \alpha$	$\sum \mathbf{a_k R'_{k+1}}$	**Error**
1	$\mathbf{B_1}$	3.30277563773199	3	**0.302775637731995**
2	$\mathbf{2B_1}$	6.60555127546399	6	**0.605551275463989**
3	$\mathbf{3B_1}$	9.90832691319598	9	**0.908326913195983**
4	C_2	13.2111025509280	13	0.211102550927978
5	$C_2 + B_1$	16.5138781886600	16	0.513878188659973
6	$C_2 + 2B_1$	19.8166538263920	19	0.816653826391967
7	D_2	23.1194294641240	23	0.119429464123963
8	$D_2 + B_1$	26.4222051018560	26	0.422205101855956
9	$D_2 + 2B_1$	29.7249807395880	29	0.724980739587952
10	B_3	33.0277563773199	33	0.0277563773199461
11	$B_3 + B_1$	36.3305320150519	36	0.330532015051940
12	$B_3 + 2B_1$	39.6333076527839	39	0.633307652783934
13	$\mathbf{B_3 + 3B_1}$	42.9360832905159	42	**0.936083290515928**
14	$B_3 + C_2$	46.2388589282479	46	0.238858928247925
15	$B_3 + C_2 + B_1$	49.5416345659799	49	0.541634565979919
16	$B_3 + C_2 + 2B_1$	52.8444102037119	52	0.844410203711913
17	$B_3 + D_2$	56.1471858414439	56	0.147185841443907
18	$B_3 + D_2 + B_1$	59.4499614791759	59	0.449961479175904
19	$B_3 + D_2 + 2B_1$	62.7527371169079	62	0.752737116907895
20	$2B_3$	66.0555127546399	66	0.0555127546398921
21	$2B_3 + B_1$	69.3582883923719	69	0.358288392371882
22	$2B_3 + 2B_1$	72.6610640301039	72	0.661064030103880
23	$\mathbf{2B_3 + 3B_1}$	75.9638396678359	75	**0.963839667835877**
24	$2B_3 + C_2$	79.2666153055679	79	0.266615305567868
25	$2B_3 + C_2 + B_1$	82.5693909432999	82	0.569390943299865
26	$2B_3 + C_2 + 2B_1$	85.8721665810318	85	0.872166581031856
27	$2B_3 + D_2$	89.1749422187638	89	0.174942218763853
28	$2B_3 + D_2 + B_1$	92.4777178564958	92	0.477717856495850
29	$2B_3 + D_2 + 2B_1$	95.7804934942278	95	0.780493494227841
30	$3B_3$	99.0832691319598	99	0.0832691319598382
31	$3B_3 + B_1$	102.386044769692	102	0.386044769691829
32	$3B_3 + 2B_1$	105.688820407424	105	0.688820407423826
33	$\mathbf{3B_3 + 3B_1}$	108.991596045156	108	**0.991596045155823**
34	$3B_3 + C_2$	112.294371682888	112	0.294371682887814
35	$3B_3 + C_2 + B_1$	115.597147320620	115	0.597147320619811
36	$3B_3 + C_2 + 2B_1$	118.899922958352	118	0.899922958351809
37	$3B_3 + D_2$	122.202698596084	122	0.202698596083792
38	$3B_3 + D_2 + B_1$	125.505474233816	125	0.505474233815789
39	$3B_3 + D_2 + 2B_1$	128.808249871548	128	0.808249871547787
40	$4B_3$	132.111025509280	132	0.111025509279784
41	$4B_3 + B_1$	135.413801147012	135	0.413801147011782
42	$4B_3 + 2B_1$	138.716576784744	138	0.716576784743765
43	C_4	142.019352422476	142	0.0193524224757624
44	$C_4 + B_1$	145.322128060208	145	0.322128060207760
45	$C_4 + 2B_1$	148.624903697940	148	0.624903697939757
46	$C_4 + 3B_1$	151.927679335672	151	0.927679335671755
47	$C_4 + C_2$	155.230454973404	155	0.230454973403738
48	$C_4 + C_2 + B_1$	158.533230611136	158	0.533230611135735
49	$C_4 + C_2 + 2B_1$	161.836006248868	161	0.836006248867733
50	$C_4 + D_2$	165.138781886600	165	0.138781886599730

t	$\sum \mathbf{a_k R_k}$	$(\sum \mathbf{a_k R_k}) \cdot \alpha$	$\sum \mathbf{a_k R'_{k+1}}$	Error
51	$C_4 + D_2 + B_1$	168.441557524332	168	0.441557524331728
52	$C_4 + D_2 + 2B_1$	171.744333162064	171	0.744333162063711
53	$C_4 + B_3$	175.047108799796	175	0.0471087997957085
54	$C_4 + B_3 + B_1$	178.349884437528	178	0.349884437527706
55	$C_4 + B_3 + 2B_1$	181.652660075260	181	0.652660075259703
56	$C_4 + B_3 + 3B_1$	184.955435712992	184	0.955435712991701
57	$C_4 + B_3 + C_2$	188.258211350724	188	0.258211350723684
58	$C_4 + B_3 + C_2 + B_1$	191.560986988456	191	0.560986988455682
59	$C_4 + B_3 + C_2 + 2B_1$	194.863762626188	194	0.863762626187679
60	$C_4 + B_3 + D_2$	198.166538263920	198	0.166538263919676
61	$C_4 + B_3 + D_2 + B_1$	201.469313901652	201	0.469313901651674
62	$C_4 + B_3 + D_2 + 2B_1$	204.772089539384	204	0.772089539383657
63	$C_4 + 2B_3$	208.074865177116	208	0.0748651771156545
64	$C_4 + 2B_3 + B_1$	211.377640814848	211	0.377640814847652
65	$C_4 + 2B_3 + 2B_1$	214.680416452580	214	0.680416452579649
66	$C_4 + 2B_3 + 3B_1$	217.983192090312	217	0.983192090311647
67	$C_4 + 2B_3 + C_2$	221.285967728044	221	0.285967728043630
68	$C_4 + 2B_3 + C_2 + B_1$	224.588743365776	224	0.588743365775628
69	$C_4 + 2B_3 + C_2 + 2B_1$	227.891519003508	227	0.891519003507625
70	$C_4 + 2B_3 + D_2$	231.194294641240	231	0.194294641239622
71	$C_4 + 2B_3 + D_2 + B_1$	234.497070278972	234	0.497070278971620
72	$C_4 + 2B_3 + D_2 + 2B_1$	237.799845916704	237	0.799845916703617
73	$C_4 + 3B_3$	241.102621554436	241	0.102621554435615
74	$C_4 + 3B_3 + B_1$	244.405397192168	244	0.405397192167584
75	$C_4 + 3B_3 + 2B_1$	247.708172829900	247	0.708172829899581
76	D_4	251.010948467632	251	0.0109484676315788
77	$D_4 + B_1$	254.313724105364	254	0.313724105363576
78	$D_4 + 2B_1$	257.616499743096	257	0.616499743095574
79	$D_4 + 3B_1$	260.919275380828	260	0.919275380827571
80	$D_4 + C_2$	264.222051018560	264	0.222051018559569
81	$D_4 + C_2 + B_1$	267.524826656292	267	0.524826656291566
82	$D_4 + C_2 + 2B_1$	270.827602294024	270	0.827602294023563
83	$D_4 + D_2$	274.130377931756	274	0.130377931755561
84	$D_4 + D_2 + B_1$	277.433153569488	277	0.433153569487530
85	$D_4 + D_2 + 2B_1$	280.735929207220	280	0.735929207219527
86	$D_4 + B_3$	284.038704844952	284	0.0387048449515248
87	$D_4 + B_3 + B_1$	287.341480482684	287	0.341480482683522
88	$D_4 + B_3 + 2B_1$	290.644256120416	290	0.644256120415520
89	$D_4 + B_3 + 3B_1$	293.947031758148	293	0.947031758147517
90	$D_4 + B_3 + C_2$	297.249807395880	297	0.249807395879515
91	$D_4 + B_3 + C_2 + B_1$	300.552583033612	300	0.552583033611512
92	$D_4 + B_3 + C_2 + 2B_1$	303.855358671344	303	0.855358671343510
93	$D_4 + B_3 + D_2$	307.158134309076	307	0.158134309075507
94	$D_4 + B_3 + D_2 + B_1$	310.460909946807	310	0.460909946807476
95	$D_4 + B_3 + D_2 + 2B_1$	313.763685584539	313	0.763685584539473
96	$D_4 + 2B_3$	317.066461222271	317	0.0664612222714709
97	$D_4 + 2B_3 + B_1$	320.369236860003	320	0.369236860003468
98	$D_4 + 2B_3 + 2B_1$	323.672012497735	323	0.672012497735466
99	$D_4 + 2B_3 + 3B_1$	326.974788135467	326	0.974788135467463
100	$D_4 + 2B_3 + C_2$	330.277563773199	330	0.277563773199461

$\mathbf{t}$	$\sum \mathbf{a_k R_k}$	$(\sum \mathbf{a_k R_k}) \cdot \alpha$	$\sum \mathbf{a_k R'_{k+1}}$	**Error**
101	$D_4 + 2B_3 + C_2 + B_1$	333.580339410931	333	0.580339410931458
102	$D_4 + 2B_3 + C_2 + 2B_1$	336.883115048663	336	0.883115048663456
103	$D_4 + 2B_3 + D_2$	340.185890686395	340	0.185890686395453
104	$D_4 + 2B_3 + D_2 + B_1$	343.488666324127	343	0.488666324127422
105	$D_4 + 2B_3 + D_2 + 2B_1$	346.791441961859	346	0.791441961859420
106	$D_4 + 3B_3$	350.094217599591	350	0.0942175995914170
107	$D_4 + 3B_3 + B_1$	353.396993237323	353	0.396993237323414
108	$D_4 + 3B_3 + 2B_1$	356.699768875055	356	0.699768875055412
109	B_5	360.002544512787	360	0.00254451278740930
110	$B_5 + B_1$	363.305320150519	363	0.305320150519407
111	$B_5 + 2B_1$	366.608095788251	366	0.608095788251404
112	$B_5 + 3B_1$	369.910871425983	369	0.910871425983402
113	$B_5 + C_2$	373.213647063715	373	0.213647063715399
114	$B_5 + C_2 + B_1$	376.516422701447	376	0.516422701447368
115	$B_5 + C_2 + 2B_1$	379.819198339179	379	0.819198339179366
116	$B_5 + D_2$	383.121973976911	383	0.121973976911363
117	$B_5 + D_2 + B_1$	386.424749614643	386	0.424749614643360
118	$B_5 + D_2 + 2B_1$	389.727525252375	389	0.727525252375358
119	$B_5 + B_3$	393.030300890107	393	0.0303008901073554
120	$B_5 + B_3 + B_1$	396.333076527839	396	0.333076527839353
121	$B_5 + B_3 + 2B_1$	399.635852165571	399	0.635852165571350
122	$B_5 + B_3 + 3B_1$	402.938627803303	402	0.938627803303348
123	$B_5 + B_3 + C_2$	406.241403441035	406	0.241403441035345
124	$B_5 + B_3 + C_2 + B_1$	409.544179078767	409	0.544179078767314
125	$B_5 + B_3 + C_2 + 2B_1$	412.846954716499	412	0.846954716499312
126	$B_5 + B_3 + D_2$	416.149730354231	416	0.149730354231309
127	$B_5 + B_3 + D_2 + B_1$	419.452505991963	419	0.452505991963307
128	$B_5 + B_3 + D_2 + 2B_1$	422.755281629695	422	0.755281629695304
129	$B_5 + 2B_3$	426.058057267427	426	0.0580572674273014
130	$B_5 + 2B_3 + B_1$	429.360832905159	429	0.360832905159299
131	$B_5 + 2B_3 + 2B_1$	432.663608542891	432	0.663608542891296
132	$B_5 + 2B_3 + 3B_1$	435.966384180623	435	0.966384180623294
133	$B_5 + 2B_3 + C_2$	439.269159818355	439	0.269159818355263
134	$B_5 + 2B_3 + C_2 + B_1$	442.571935456087	442	0.571935456087260
135	$B_5 + 2B_3 + C_2 + 2B_1$	445.874711093819	445	0.874711093819258
136	$B_5 + 2B_3 + D_2$	449.177486731551	449	0.177486731551255
137	$B_5 + 2B_3 + D_2 + B_1$	452.480262369283	452	0.480262369283253
138	$B_5 + 2B_3 + D_2 + 2B_1$	455.783038007015	455	0.783038007015250
139	$B_5 + 3B_3$	459.085813644747	459	0.0858136447472475
140	$B_5 + 3B_3 + B_1$	462.388589282479	462	0.388589282479245
141	$B_5 + 3B_3 + 2B_1$	465.691364920211	465	0.691364920211242
142	$\mathbf{B_5 + 3B_3 + 3B_1}$	468.994140557943	468	**0.994140557943240**

Table 3: Mixed-bronze representations of the first 142 positive integers