CYCLE GAMES ON GRAPHS

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Abstract

We define two 2-player impartial games on a simple graph called the Make-A-Cycle (MAC) game and the Avoid-A-Cycle (AAC) game. Given a simple graph Γ and starting vertex v, two players alternate forming a path in Γ without backtracking. The first player to return to a previously visited vertex wins the MAC game or, respectively, loses the AAC game. We give a complete description of winning strategies for the AAC game. For the game MAC, we show winning strategies for several families of graphs including complete and complete bipartite graphs, wheel graphs, stacked prism graphs, and some generalized Petersen graphs. Moreover, we provide a complete description of winning strategies on a join of two simple graphs.

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1. Introduction

In [7], the two player combinatorial games `REL` (Relator Achievement) and `RAV` (Relator Avoidance) were defined by authors Gates and Kelvey. These games take a finite group G and generating set S, and have two players alternate building a word in the group G out of letters from S. When playing `REL`, the first player to form a relator wins. When playing `RAV`, the first player to form a relator loses. Both games can be visualized via the Cayley graph $\Gamma(G, S)$, where the game words correspond to forming a path in the graph. Hence, winning the game `REL` amounts to making a cycle in the Cayley graph, whereas winning the game `RAV` is equivalent to avoiding a cycle.

In [7], it was noted that one could drop the requirements of starting with a finite group G and instead examine the above games in a purely graph-theoretic way. In this paper, we aim to do just that. Given a simple graph and a starting vertex, we ask if there are winning strategies on particular families of graphs. We begin in Section 2 by extending the definition of `REL` and `RAV` to general graphs. We call these new games `MAC` for Make-A-Cycle and `AAC` for Avoid-A-Cycle to distinguish them from the relator games `REL` and `RAV`. We examine a few special examples in Section 2 and then proceed to discuss winning strategies for various graph families: complete and complete k-partite graphs (Section 3), wheel graphs (Section 4), stacked prism graphs (Section 5), and generalized Petersen graphs (Section 6).

While searching for literature on combinatorial games on graphs related to `MAC` and `AAC`, we discovered, through [4] and [5], the game of Generalized Geography, which has important bearing on the game of `AAC`. This combinatorial game is played on a directed graph and consists of two players moving a token on the graph between adjacent vertices. On each player's turn, after moving the token from one vertex to another, the starting vertex of the move is deleted from the graph. This game is based off a classic children's game where players alternate naming cities from around the world, with the next city name beginning with the letter that ended the previous city name and where repetition is not allowed.

There is extensive research in the literature on "geography" games and many studies that examine variants of the game. One such variant is Undirected Geography which is played on undirected graphs [5]. This variant is equal to the game `AAC` and hence results from [5] allow us to easily describe winning strategies for the `AAC` game for many graphs (see Section 2). In particular, we describe in Section 6 winning strategies for the `AAC` game on graph joins and certain subgraphs of graph joins. Our results for the generalized Petersen graphs in Section 6 are then an application of these results.

Also in Section 6, we give a complete description of winning strategies for `MAC` on the graph join of two simple graphs before closing with a discussion in Section 7 of some code that was useful in developing intuition for several of our results and, finally, some open questions in Section 8.

2. Defining the Games

A graph Γ is comprised of a *vertex set* $V\Gamma$ and an *edge set* $E\Gamma$, where $E\Gamma$ is a subset of $V\Gamma \times V\Gamma$. For vertices $v, w \in V\Gamma$, if $(v, w) \in E\Gamma$, then we say (v, w) is an *edge* in the graph Γ. For a vertex v, we denote by $E(v)$ the set of edges containing v. A *path* p in Γ is represented by its natural sequence of vertices, which we write as $p = v_0v_1 \ldots v_n$, such that $(v_i, v_{i+1}) \in E\Gamma$ for $i = 0, 1, \ldots, n-1$. A *cycle* in Γ is a path $p = v_0v_1 \ldots v_n$, such that $v_0 = v_n$ and v_i is unique for $i = 1, 2, \ldots, n-1$.

Let Γ be a simple graph, that is, a graph with no loops and no multiple edges, and let v be a vertex of Γ. We define the games Make-A-Cycle and Avoid-A-Cycle, denoted by $\texttt{MAC}(\Gamma, v)$ and $\texttt{AAC}(\Gamma, v)$, as follows.

Definition 2.1. Let $v_0 = v$. Player 1 begins by choosing an edge $e_1 \in E(v_0)$ with $e_1 = (v_0, v_1)$, which we will commonly phrase as Player 1 "moves to the vertex v_1." Let $p_1 = v_0v_1$ denote the game path after turn 1. Then Player 2 chooses an edge $e_2 \in E(v_1)$ such that $e_2 \neq e_1$ and with $e_2 = (v_1, v_2)$. This forms the game path $p_2 = v_0v_1v_2$. In general, on turn n, the current player begins with the game path

$$p_{n-1} = v_0v_1 \ldots v_{n-1}, \text{ where } e_i = (v_{i-1}, v_i).$$

The current player then chooses $e_n = (v_{n-1}, v_n) \in E(v_{n-1})$ where $e_n \neq e_{n-1}$, forming the path $p_n = v_0v_1 \ldots v_{n-1}v_n$. If a player forms p_n such that $v_n = v_k$ for some k with $0 \leq k \leq n-1$, then that player wins the Make-A-Cycle game $\texttt{MAC}(\Gamma, v)$. The Avoid-A-Cycle game, denoted by $\texttt{AAC}(\Gamma, v)$, is the misère version of Make-A-Cycle; that is, a player loses if they form p_n such that $v_n = v_k$ for some k. For both games, if at any point there are no legal moves, then the current player loses.

Looking at the definition of $\texttt{REL}$ and $\texttt{RAV}$ in [7], it is immediately clear that the games $\texttt{MAC}$ and $\texttt{AAC}$ are generalizations of these. That is, if a graph Γ is the Cayley graph for some finite group G and generating set S, then $\texttt{MAC}(\Gamma, v)$ and $\texttt{AAC}(\Gamma, v)$ are exactly the games $\texttt{REL}(G, S)$ and $\texttt{RAV}(G, S)$, respectively. That is, these games will have identical game trees, and hence will be isomorphic games [1, p.66].

Theorem 2.2. *If $\Gamma = \Gamma(G, S)$ for a finite group G and generating set S, then we have for any vertex v,*

$$\texttt{MAC}(\Gamma, v) \cong \texttt{REL}(G, S),$$
$$\texttt{AAC}(\Gamma, v) \cong \texttt{RAV}(G, S).$$

Note that Cayley graphs are *vertex-transitive*, meaning the automorphism group of the graph acts transitively on the set of vertices. Because of this, it does not matter at which vertex the games of REL and RAV start. However, for general graphs, there could be different outcomes for MAC or AAC depending on the chosen starting vertex. In the case that a graph Γ is vertex-transitive, we will write $\texttt{MAC}(\Gamma)$ and $\texttt{AAC}(\Gamma)$ instead of $\texttt{MAC}(\Gamma, v)$ and $\texttt{AAC}(\Gamma, v)$, respectively, since the starting vertex is irrelevant. If, instead, a graph is not vertex transitive, then we can use the symmetries of the graph to narrow down starting positions so as to not have to examine a different game for every vertex in a graph.

We denote the automorphism group of a graph Γ by $\mathrm{Aut}(\Gamma)$. If v is a vertex of Γ, then the orbit $\mathcal{O}(v)$ is defined as $\mathcal{O}(v) = \{\alpha(v) \mid \alpha \in \mathrm{Aut}(\Gamma)\}$.

Theorem 2.3. *Let Γ be a simple graph and let $v \in V\Gamma$. Let $\mathcal{O}(v)$ denote the orbit of v under the action of $\mathrm{Aut}(\Gamma)$. Then if $w \in \mathcal{O}(v)$, then $\texttt{MAC}(\Gamma, v) \cong \texttt{MAC}(\Gamma, w)$ and $\texttt{AAC}(\Gamma, v) \cong \texttt{AAC}(\Gamma, w)$.*

If Γ is a vertex-transitive graph, the orbit of any vertex is the entire graph Γ. Thus, we have the following corollary.

Corollary 2.4. *If Γ is a vertex-transitive graph, then $\texttt{MAC}(\Gamma, v) \cong \texttt{MAC}(\Gamma, w)$ and $\texttt{AAC}(\Gamma, v) \cong \texttt{AAC}(\Gamma, w)$ for all $v, w \in V\Gamma$.*

Example 2.5. As a simple example of the non-vertex transitive case, consider the class of *star graphs* S_k. These are the complete bipartite graphs $K_{1,k}$ ($k > 2$) and consist of a central vertex connected to k vertices, forming a star-like shape (see Figure 1).

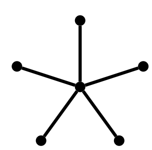

Figure 1: Example of star graph S_5.

There are two kinds of orbits for the automorphism group of a star graph. The central vertex has trivial orbit whereas the outer vertices all lie in the same orbit via a rotational symmetry. Hence, one only needs to consider the games MAC and AAC with two different starting positions: the central vertex or an outer vertex.

If u denotes the central vertex in S_k, we have that Player 1 wins both $\texttt{MAC}(S_k, u)$ and $\texttt{AAC}(S_k, u)$, since Player 2 will have no legal move to make on their turn. If v is any outer vertex of S_k, then Player 2 wins $\texttt{MAC}(S_k, v)$ and $\texttt{AAC}(S_k, v)$ since Player 1 will have no legal move on their second turn.

Example 2.6. Consider the family of *prism graphs* D_n (see Figure 2). These graphs are the Cayley graphs for the dihedral groups. These graphs are vertex-transitive; hence it suffices to consider just one starting vertex. In [7], it is shown that Player

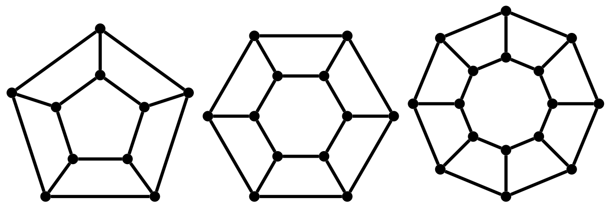

Figure 2: Prism graphs D_5, D_6, and D_8.

1 has a winning strategy for $\texttt{MAC}(D_n)$ if and only if n is odd or $n \equiv 2 \pmod 6$. For $\texttt{AAC}(D_n)$, Player 1 has a winning strategy if and only if n is even.

In Section 5, we show winning strategies for a family of graphs called *stacked prism graphs* which generalizes prism graphs.

2.1. Avoid-A-Cycle and Matchings

As mentioned in Section 1, the game of `AAC` is equal to the Undirected Geography Game, referred to as Undirected Vertex Geography in [5]. The first theorem in [5] provides a classification for when a player has a winning strategy. However, to understand it, we first require a few preliminary definitions from graph theory. We also include some graph theory definitions that we use later in the paper when discussing specific graphs and the `AAC` game.

Given a simple graph Γ, a *matching* M is a set of edges in Γ such that no two edges in M share a vertex. If v is contained in an edge of a matching M, then we say that M *saturates* v. A matching M is said to be *maximum* if it is a matching with maximum cardinality among all possible matchings in Γ. A *perfect matching* is a maximum matching M such that every vertex of Γ is saturated while a *near-perfect matching* is a matching of Γ such that exactly one vertex is not saturated. Note that a perfect matching is necessarily a maximum matching. Moreover, a finite graph with a perfect matching necessarily contains an even number of vertices.

The following definitions will be useful in Section 3. Given a matching M in Γ, an *alternating path* is a path such that every second edge is contained in M [8]. An *augmenting path* is an alternating path that begins and ends at vertices not contained in the matching. The following theorem of Berge [2] connects the property of a matching being maximum with its augmenting paths.

Theorem 2.7 (Berge's Theorem)**.** *Given a graph Γ and matching M, the matching M is maximum if and only if there is no augmenting path with respect to M.*

We can now state a result from [5] that characterizes winning strategies for the `AAC` game.

Theorem 2.8 (Theorem 1.1, [5])**.** *Let Γ be a simple graph and v a vertex in Γ. Then Player 1 has a winning strategy for $\mathtt{AAC}(\Gamma, v)$ if and only if every maximum matching in Γ saturates v.*

Proof. We first suppose that every maximum matching in Γ saturates v, and let M be one such matching. Then Player 1's strategy is to only choose edges contained in M. Since M saturates v, Player 1 can clearly do this on their first turn. Assume that on some turn, Player 1 is unable to choose an edge in M. That is, on the previous turn, Player 2 moved to the current vertex w, which is not saturated by M. Let p denote the current game path from starting vertex v to w and note that p is of even length. We can form a matching M' by

$$M' = (p - M) \cup (M - p).$$

That is, M' is formed by taking the edges traversed so far by Player 2 and joining them with the edges still left in M that Player 1 has not yet used. Since the path p was of even length, the cardinality of M' is the same as the cardinality of M. Hence, M' is a maximum matching, but M' does not saturate v, which contradicts our hypothesis. Hence, Player 1 is always able to choose an edge in M for their turn and, because the graph Γ is finite, Player 2 will necessarily be forced to make a cycle.

For the forward direction, assume that there exists a maximum matching M that does not saturate v. Then Player 2 will proceed to only choose the edges in M. This is a winning strategy by the same argument as above - should Player 2 be unable to choose an edge in M, then we can construct a new matching M' by means of symmetric difference. But this time, the path to the current vertex will be of odd length (as it is currently Player 2's turn), and hence $|M'| > |M|$, contradicting the maximum cardinality of M. $\square$

After one starts playing AAC on various graphs, one will notice that the game is generally one of vertex exhaustion. Hence, winning strategies naturally turn into one player or the other creating a maximum matching in the graph. Indeed, the authors found winning strategies for several graphs and graph families before learning of the result in [5], but all of the discovered winning strategies were seen to be formations of maximum matchings that contained the starting vertex. We encourage the reader to play a few games of AAC for the graphs in each section before thinking about matchings. After a while, one will see how natural maximum matchings are to the AAC game.

Recall that a Cayley graph $\Gamma(G, S)$ is both connected and vertex-transitive. The following result concerning connected vertex-transitive graphs is well-known (see [8] Theorem 3.5.1).

Theorem 2.9. *Let Γ be a connected, vertex-transitive graph. If Γ has an even number of vertices, then Γ has a perfect matching. If Γ has an odd number of* *vertices, then for every vertex v, there exists a near-perfect matching with v not saturated.*

As stated in Theorem 2.2, the `RAV` game is isomorphic to the game `AAC`. Therefore, the following corollary for `RAV` follows immediately from Theorem 2.8 and Theorem 2.9.

Corollary 2.10. *For the game RAV(G, S), Player 1 has a winning strategy if the number of vertices of $\Gamma(G, S)$ is even and Player 2 has a winning strategy if the number of vertices is odd.*

Theorem 2.8 makes it clear that `AAC` is a graph-theoretic game. However, we have yet to determine a connection between a graph-theoretic result and a general winning strategy for the Make-A-Cycle game.

3. Complete Graphs and Complete k-partite Graphs

Let K_n denote the complete graph on n vertices. We note that K_n is vertex-transitive and thus omit the choice of vertex and write `MAC`(K_n) and `AAC`(K_n) for the Make-A-Cycle and Avoid-A-Cycle games, respectively.

Theorem 3.1.

1. *Player 1 has a winning strategy for MAC(K_n) if $n \geq 2$ while Player 2 has a winning strategy if $n = 1$.*

2. *Player 1 has a winning strategy for AAC(K_n) if n is even while Player 2 has a winning strategy for AAC(K_n) if n is odd.*

Proof. First, consider the game `MAC`(K_n). We consider the cases where $n = 1$, $n = 2$, and $n \geq 3$ separately. Suppose that $n = 1$. Since there is only one vertex, Player 1 is unable to make a move at the start of the game, resulting in a Player 2 win. If $n = 2$, then Player 1 moves to the other vertex from the starting vertex, and Player 2 is unable to move since backtracking is disallowed. Thus, Player 1 wins. Now suppose $n \geq 3$ and the game begins at the vertex v. Player 1 will move to some vertex $w \neq v$. Since backtracking is not allowed, Player 2 cannot move to v and also is unable to complete a cycle, so they must move to some vertex $u \notin \{v, w\}$. Player 1 can then win by returning to the vertex v on their second turn.

Now consider the game `AAC`(K_n). Note that K_n is connected and vertex-transitive, thus satisfying the hypotheses of Theorem 2.9. Thus, if n is even, then there exists a maximum matching in K_n which saturates every vertex. Therefore, by Theorem 2.8, Player 1 has a winning strategy for `AAC`(K_n) when n is even. On the other hand, if n is odd and the game has starting vertex v, then by Theorem 2.9 there exists a maximum matching in K_n which saturates every vertex except for v. Thus, by Theorem 2.8 Player 2 has a winning for $\texttt{AAC}(K_n)$ when n is odd. $\square$

We note that Theorem 3.1 follows quickly from Theorem 2.6 in [7] and Theorem 2.2 since the same argument for REL on complete Cayley graphs applies here. However, we note that the cases where $n = 1$ or $n = 2$ are not covered in that theorem but are covered above in detail.

Let $\Gamma = K(n_1, n_2, \ldots, n_k)$ denote a complete k-partite graph ($k \geq 2$) with disjoint vertex sets of sizes n_1, n_2, ... , n_k where $1 \leq n_1 \leq n_2 \leq \cdots \leq n_k$. We first prove results for $\texttt{MAC}(\Gamma, v)$, with v the starting vertex.

Theorem 3.2. *Let v be a vertex in Γ.*

1. *If v lies in a part of size 2 or greater, then Player 2 has a winning strategy for $\texttt{MAC}(\Gamma, v)$.*
2. *If v lies in a part of size 1, then Player 1 has a winning strategy for $\texttt{MAC}(\Gamma, v)$.*

Proof. Consider statement (1) where v lies in a part of size 2 or greater. Suppose Player 1 moves from v to the vertex u on their first turn, where u necessarily lies in a part disjoint from v because Γ is k-partite. Then Player 2 will choose to move to a vertex v', where v' is a vertex in the same part as v. Since there are no edges between vertices in the same part, Player 1 cannot move to v, and, by the rules of the game, also cannot backtrack. From here, they lose if there is no legal move; otherwise, they must choose some new vertex w that is not in the same part as v. Then Player 2 wins on the subsequent turn by returning to v.

Now consider statement (2) where v lies in a part of size 1. Suppose Player 1 moves from v to a vertex u for their first turn. Note that Player 2 cannot choose a vertex in the same part as v on their turn. If $k = 2$, this means that they have no legal move, so Player 1 wins. If $k > 2$, then Player 2 is able to move to some vertex $w \notin \{u, v\}$. However, Player 1 wins on the following turn by returning to v. $\square$

Now we analyze the game AAC for complete k-partite graphs. Due to Theorem 2.8, we seek to determine the maximum matchings for such graphs. For a complete k-partite graph Γ, this is a combinatorial problem that depends on the sizes of the parts of Γ.

Theorem 3.3. *Let $\Gamma = K(n_1, n_2, \ldots, n_k)$ be a complete k-partite graph with $1 \leq n_1 \leq n_2 \leq \ldots \leq n_k$ and let $n = \sum_{i=1}^{k} n_i$ be the total number of vertices in Γ. There are two cases:*

1. *Suppose $n_k \leq \frac{n}{2}$. Then Player 1 has a winning strategy for $\texttt{AAC}(\Gamma, v)$, for any vertex v if and only if n is even.*

2. *Suppose $n_k > \frac{n}{2}$. Then Player 1 has a winning strategy for* $\texttt{AAC}(\Gamma, v)$ *if and only if $v \in V_i$ for some $i \neq k$.*

Proof. Let V_1, V_2, …, V_k denote the vertex sets of each part of Γ. Then we label the vertices of V_1 by the integers $\{0, 1, \ldots, n_1 - 1\}$, in increasing order. We label the vertices of V_2 by $\{n_1, n_1 + 1, \ldots, n_1 + n_2 - 1\}$. In general, the vertices of V_{i+1}, where $i \geq 1$, will be labeled by the integers $\{s_i, s_i + 1, \ldots, s_i + n_{i+1} - 1\}$, where $s_i = \sum_{j=1}^{i} n_j$.

`Case 1:` $n_k \leq \frac{n}{2}$. Let $m = \lfloor \frac{n}{2} \rfloor$. We define a maximum matching M as follows:

$$M = \{(0, m), (1, m + 1), \ldots, (j, m + j), \ldots, (m - 1, 2m - 1)\}.$$

We note that these chosen edges must exist since $n_i \leq m$ for all $1 \leq i \leq k$ and that the edges are disjoint. We also note that $2m - 1 = n - 1$ if n is even and $2m - 1 = n - 2$ if n is odd.

If n is even, then M is a perfect matching and hence maximum. Therefore any maximum matching is perfect, so every vertex is saturated by every maximum matching. Thus, by Theorem 2.8 Player 1 has a winning strategy for $\texttt{AAC}(\Gamma, v)$ for any vertex v if n is even.

Now suppose n is odd. Then the matching M is a near-perfect matching missing only the vertex labeled $n - 1$. Hence, by Theorem 2.8, we can say that Player 2 has a winning strategy for $\texttt{AAC}(\Gamma, n - 1)$. But in order to show that Player 2 has a winning strategy for $\texttt{AAC}(\Gamma, v)$ for any vertex v, we must demonstrate a near-perfect matching that does not saturate v. To do this, we adapt M to create a new matching M_ℓ, which is a near-perfect matching that misses the vertex labeled ℓ. We form M_ℓ by changing the edge $(u, w) \in M$ to $(u + (\ell + 1), w + (\ell + 1)) \in M_\ell$, where addition is done modulo n. This results in

$$\begin{aligned} M_\ell = \{&(\ell + 1, m + (\ell + 1)), (\ell + 1, m + 1 + (\ell + 1)), \ldots \\ &\ldots, (j + (\ell + 1), m + j + (\ell + 1)), \ldots, (m - 1 + (\ell + 1), 2m - 1 + (\ell + 1))\}. \end{aligned}$$

This proves that Player 2 has a winning strategy for $\texttt{AAC}(\Gamma, v)$ for any vertex v if n is odd.

An example of a matching M with $n = 13$ and a new matching M_ℓ is given in Figure 3. For this example, we see that $n - 1 = 12$ is unmatched in M in the left figure. The right figure of Figure 3 shows M_ℓ with $\ell = 4$, which leaves 4 unmatched.

`Case 2:` $n_k > \frac{n}{2}$. We define a maximum matching N as follows:

$$N = \{(0, s_{k-1}), (1, s_{k-1} + 1), \ldots, (j, s_{k-1} + j), \ldots, (s_{k-1} - 1, 2s_{k-1} - 1)\}.$$

Essentially, since more than half of the vertices lie in V_k, we can find disjoint edges between each vertex not in V_k and a corresponding vertex in V_k. We show the

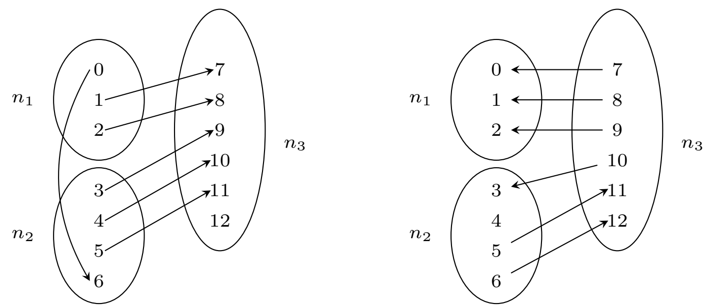

Figure 3: An example of the matching M (left) and shifted matching M_ℓ, where $\ell = 4$, (right) from the proof of Theorem 3.3.

matching N is maximum by Berge's Theorem 2.7. We first note that any augmenting path must begin and end in V_k. Since all edges in N are of the form (v, w) where $v \in V_i$, $i \neq k$, and $w \in V_k$ and any augmenting path is alternating, we observe that our augmenting path must be of even length. However, augmenting paths must be of odd length, so there can be no augmenting path corresponding to N. Thus, the matching N is maximum. Note that by relabeling the vertices within V_k, we can arrange for any vertex from V_k to be left unsaturated by this maximum matching. Thus Player 2 has a winning strategy for $\texttt{AAC}(\Gamma, v)$ if $v \in V_k$.

On the other hand, suppose $v \notin V_k$ and that L is a maximum matching which does not saturate v. Then, since there are no edges between vertices in V_k, there are at most $s_{k-1} - 1$ vertices in V_k which are saturated by L. Since $s_{k-1} - 1 < n_k$, there must exist some $w \in V_k$ such that w is not saturated by L. Thus, we may add the edge (v, w) to L, which contradicts the assumption that L is maximum. Thus there is no maximum matching which does not saturate v, so Player 1 has a winning strategy for $\texttt{AAC}(\Gamma, v)$ if $v \notin V_k$. □

4. Wheel Graphs

We define the *wheel graph* W_n of order $n \geq 4$ as containing a cycle of length $n - 1$ and where each vertex in the cycle is connected to a universal vertex u at the center of the graph. This formation can be considered as the graph join of a singleton graph and the $n - 1$ cycle graph. A general discussion of MAC and AAC for general graph joins can be found in Section 6.2 below.

We denote the vertices of W_n by v_i for $i = 0, 1, \ldots, n-2$ for those on the cycle and we denote the central vertex by u. Then the edges of W_n are given by (u, v_i) and (v_i, v_{i+1}) for $i = 0, 1, \ldots, n-2$, where the addition in the subscript is taken modulo $n - 1$. See Figure 4 for a general example of a Wheel graph.

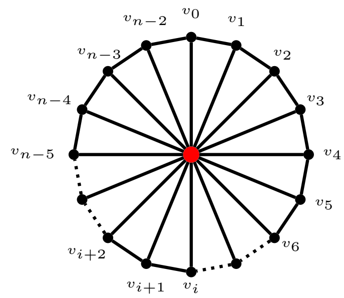

Figure 4: A general wheel graph W_n.

It is clear that there are two orbits for the action of $\mathrm{Aut}(W_n)$ on W_n. That is, $\mathcal{O}(v_0) = \{v_0, v_1, \ldots, v_{n-2}\}$ and $\mathcal{O}(u) = \{u\}$. Hence, without loss of generality, we need only consider the starting locations u and v_0 for our games.

Theorem 4.1. *Player 1 has a winning strategy for* MAC(W_n, u) *for any* $n \geq 4$.

Proof. We know that every v_i for $0 \leq i \leq n-2$ is connected by an edge to the universal vertex u. Thus, when Player 1 starts off by moving from u to any v_i, regardless of the subsequent move by Player 2, Player 1 is always able to make a cycle by traveling back to u. Therefore, Player 1 always wins on the third move of the game. □

Theorem 4.2. *Player 1 has a winning strategy for* MAC(W_n, v_0) *if n is even. If n is odd, then Player 2 has a winning strategy for* MAC(W_n, v_0).

Proof. We begin by making the observation that if Player 1 begins by moving to either v_1 or v_{n-2}, then both players will only choose to move along edges on the cycle part of the wheel graph. Indeed, if, after this first turn, a player moves from a vertex v_j to the central vertex u, then the next player will be able to move back to v_0 and create a cycle, thus winning the game. Now we will discuss the winning strategy, separated into cases by parity.

Suppose n is even. Then Player 1 will move to v_1 and, as discussed above, Player 2 is forced to move along the outer cycle. Since n is even, the outer cycle of the wheel is of odd length, so Player 1 will be the first to return to v_0 and win.

If n is odd and Player 1 opens with a move from v_0 to v_1 or v_{n-2}, then Player 2 will be the first to return to v_0 and win by the same reasoning outlined above. If Player 1 opens the game by moving from v_0 to u, then Player 2 can win on their second turn of the game. Indeed, since $n \geq 4$, Player 2 can move to v_i where $i \neq 0, 1, n-2$. Player 1 cannot backtrack to u and thus moves to either v_{i-1} or v_{i+1}. Player 2 then wins on the following turn by moving back to the central vertex u. □

Now we consider the game $\texttt{AAC}(W_n, v)$, with v any vertex in W_n. As mentioned at the end of Section 2 with Theorem 2.8, we can determine winning strategies by examining maximum matchings of W_n that contain v.

Theorem 4.3. *For the game $\texttt{AAC}(W_n, v)$, Player 1 has a winning strategy if and only if n is even.*

Proof. When n is odd, we have two kinds of near-perfect matchings of W_n, up to isomorphism. These are given by the following sets of edges:

1. $\{(v_0, v_1), (v_2, v_3), \ldots, (v_{n-3}, v_{n-2})\}$, and
2. $\{(u, v_1), (v_2, v_3), \ldots, (v_{n-3}, v_{n-2})\}$.

If the game begins at the vertex u, then a matching of the first kind is a near-perfect matching that misses u. Hence, Player 2 has a winning strategy of only choosing edges in this matching. If the game begins at v_0, then Player 2's strategy is to use the second matching (2) above. See Figure 5.

When n is even, there is, up to isomorphism, one perfect matching of W_n, given by the set of edges

$$\{(u, v_0), (v_1, v_2), (v_3, v_4), \ldots, (v_{n-3}, v_{n-2})\}.$$

See Figure 6. $\square$

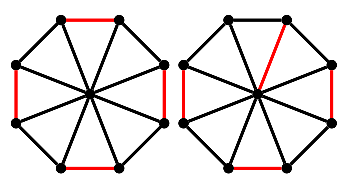

Figure 5: The two kinds of maximum matchings, up to isomorphism, in W_9. The edges of the matchings are red.

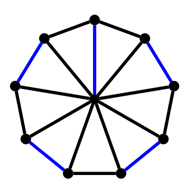

Figure 6: The single maximum matching, up to isomorphism, in W_{10}. The edges of the matching are blue.

5. Stacked Prism Graphs

We denote by $\mathrm{SP}(n, m)$ a nesting of an m number of n-gons. Such graphs are known as *stacked prism graphs*, and we note that $\mathrm{SP}(n, m) = C_n \square P_m$, the product of an n-cycle graph with a path graph of length m.

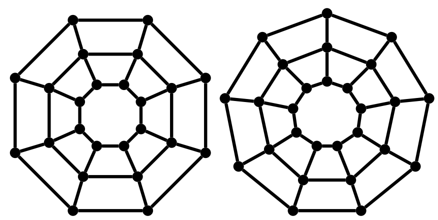

Figure 7: Graphs of $\mathrm{SP}(8, 3)$ and $\mathrm{SP}(9, 3)$.

Note that the graphs $\mathrm{SP}(n, 2)$ are the same as the family of prism graphs, that is, Cayley graphs for dihedral groups. The family of graphs $\mathrm{SP}(n, m)$ can be viewed as a natural extension of the dihedral Cayley graphs.

In this section, we examine the Make-A-Cycle game for the family of stacked prism graphs of depth three, $\mathrm{SP}(n, 3)$, like the ones pictured in Figure 7. For the game AAC, we will show winning strategies for any general graph $\mathrm{SP}(n, m)$.

In order to discuss the cycle games on the $\mathrm{SP}(n, m)$ graphs, let us first introduce some notation for the vertices. We label each vertex by an ordered pair of integers (i, j) where $0 \leq i \leq n-1$ and $0 \leq j \leq m-1$. The integer i denotes where on an n-gon the vertex lies and the j denotes the level of n-gon, with $j = 0$ corresponding to the outermost n-gon and $j = m-1$ the innermost n-gon. See Figure 8 for an example.

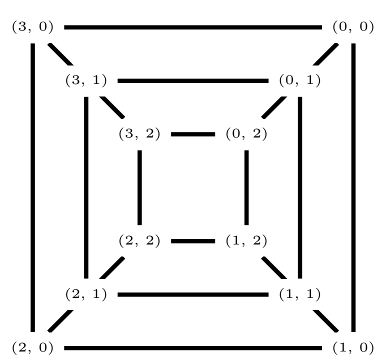

Figure 8: Example of labeling scheme for the stacked prism graph $\mathrm{SP}(4, 3)$.

5.1. Make-A-Cycle for Stacked Prism Graphs $\mathbf{SP}(n,3)$

In this section we classify the winning strategies for the Make-A-Cycle game on the graph family $\mathrm{SP}(n,3)$. There are only two orbits of starting vertices to consider. The vertices of the central n-gon, $(0,1),(1,1),\ldots,(n-1,1)$ comprise one orbit. The innermost and outermost n-gon vertices form the second orbit. Thus, when considering games on $\mathrm{SP}(n,3)$, we only need to consider two starting positions. Without loss of generality, we will consider starting vertices $(0,0)$ and $(0,1)$.

Remark 5.1. Notice that the graphs $\mathrm{SP}(n,3)$ are comprised of squares: the vertices labeled $(i,0),(i+1,0),(i+1,1),(i,1)$ and $(i,1),(i+1,1),(i+1,2),(i,2)$ for $i=0,1,\ldots,n-1$. If two sides of a square have been traversed, say $(i,0)$ to $(i+1,0)$ and $(i+1,0)$ to $(i+1,1)$, then the next player will not move to $(i,1)$, that is, will not traverse the third side of a square, since the previous player will then make a cycle by traversing the final edge of the square.

Theorem 5.2. *Suppose n is odd. Then Player 1 has a winning strategy for $\mathtt{MAC}(\mathrm{SP}(n,3),v)$ for any vertex v.*

Proof. By the symmetries of $\mathrm{SP}(n,3)$, we only need to consider two cases of starting positions. Without loss of generality, we consider the starting vertex v to be one of $(0,0)$ or $(0,1)$.
`Case 1:` $v=(0,0)$. Player 1 begins by moving to $(1,0)$. Note that Player 1 will necessarily move to vertices (i,j) where $i+j$ is odd. Player 1's strategy is to move from vertices (i,j) to $(i+1,j)$ unless there is a winning move otherwise. By this strategy and by Remark 5.1, all moves will result in i non-decreasing, and i will increase by 1 at a minimum of every two moves. Hence, Player 1 will eventually arrive at vertex $(n,0)=(0,0)$ and win; or Player 2 will arrive at vertex $(n,1)$ and Player 1 moves to $(n,0)=(0,0)$ to win; or Player 1 arrives first at $(n,2)=(0,2)$. In this last case, Player 2 must choose to move to $(n+1,2)=(1,2)$. Now let $(i_0,0)$ denote the vertex from which Player 2 first moved off the 0-gon; that is, Player 2 moved to $(i_0,1)$. Necessarily, $0<i_0\leq n-1$ and i_0 is odd. We note that if Player 2 moves from $(k,2)$ to $(k,1)$ for $0\leq k<i_0$, then Player 1 subsequently has a winning move to $(k,0)$, which has previously been reached by the assumption on i_0. Thus, Player 2 will move from $(k,2)$ to $(k+1,2)$, where k is odd. Given this observation and Player 1's strategy, both players will move along the innermost n-gon until Player 1 reaches the vertex $(n+i_0-1,2)=(i_0-1,2)$. Player 2 then has two options: they can move to $(n+i_0-1,1)=(i_0-1,1)$ whereby Player 1 will win by moving to $(i_0,1)$ or Player 2 can move to $(n+i_0,2)=(i_0,2)$ whereby Player 1 will again win at $(i_0,1)$.
`Case 2:` $v=(0,1)$. In this case, Player 1 will necessarily move to vertices (i,j) where $i+j$ is even. Player 1 uses the same strategy in this case: their first move will be to $(1,1)$ and they will only move from vertices of the form (i,j) to $(i+1,j)$.

As in the previous case, all moves will result in i non-decreasing and i will increase by 1 at a minimum of every two moves. Hence, Player 1 will arrive at $(n, 1)$ and win; or Player 2 arrives first to $(n, 0) = (0, 0)$ or to $(n, 2) = (0, 2)$. In these latter cases, Player 1 will move to $(n, 1) = (0, 1)$ to win. □

Remark 5.3. Theorem 5.2 can be generalized to any $\mathrm{SP}(n, m)$ with n odd. One can draw an $\mathrm{SP}(n, m)$ graph as a grid of nm points labeled (i, j), with $0 \le i \le n-1$ and $0 \le j \le m-1$. The vertices $(n-1, j)$ are then connected to the vertices $(n, j) = (0, j)$ using our modulo n notation, forming a cylinder. The general case then differs from the $m = 3$ case in the following way: the game path can pass by the starting vertex, either above or below, multiple times instead of at most once in the case of Theorem 5.2. However, the parity of moves is the same in the general case for Player 1 and Player 2. We can denote the game vertices here more generally by $(kn + i, j)$, where k denotes how many times the game path has wound around the graph (that is, passed by the starting vertex). Then Player 1, by their strategy, still only moves to vertices of the form (a, b) with $a + b$ odd and Player 2 still only moves to vertices (a, b) with $a + b$ even. Therefore, the same ideas as in the proof of Theorem 5.2 will apply to the general case.

For the even cases, Player 2 has a winning strategy if the starting vertex is $(0, 0)$ or $(0, 2)$. Their winning strategy is one of mirroring.

Theorem 5.4. *Player 2 has a winning strategy for* $\mathtt{MAC}(\mathrm{SP}(n, 3), v)$ *if n is even and* $v \in \{(0, 0), (0, 2)\}$.

Proof. Player 2 will implement a mirroring strategy. That is, if Player 1 moves from (i, j) to $(i \pm 1, j)$, then Player 2 moves from $(i \pm 1, j)$ to $(i \pm 2, j)$. If Player 1 moves from (i, j) to $(i, 1)$, then Player 2 moves from $(i, 1)$ to $(i, j + 2 \pmod 2)$ for $j \in \{0, 2\}$. It follows that for $j \in \{0, 2\}$, Player 1 moves only to (i, j) for odd i while Player 2 moves only to (i, j) for even i, and for $j = 1$, Player 1 moves only to (i, j) for even i.

We will show that Player 2's strategy is a winning strategy by contradiction. First, assume Player 1 has a winning strategy. By the observation above, Player 1 can only win by moving to vertices of the form $(i, 1)$, with i even, or (i, j) with i odd and $j \in \{0, 2\}$.

Suppose Player 1 wins at $(i, 1)$, where i is even. This implies that Player 1 moved to $(i, 1)$ from (i, j) where $j \in \{0, 2\}$. Since this is a winning move, $(i, 1)$ was previously visited by Player 1. The first time it was visited, it must have been from $(i, 0)$ or $(i, 2)$, and then Player 2 would have moved to the other. Thus, upon reaching $(i, 1)$ a second time, Player 1 must have moved from $(i, 0)$ or $(i, 2)$, both of which were previously visited. Therefore Player 2 won the previous turn, a contradiction.

Now suppose Player 1 wins at (i, j), where i is odd and $j \in \{0, 2\}$. The first time (i, j) was visited, it must have been from $(i-1, j)$ or $(i+1, j)$, and then Player 2 would have moved to the other. Thus, upon reaching (i, j) a second time, Player 1 must have moved from $(i-1, j)$ or $(i+1, j)$, both of which were previously visited. Therefore Player 2 won the previous turn, a contradiction. □

Remark 5.5. Note that Theorem 5.4 is generalizable to graphs $\mathrm{SP}(n, m)$ where n is even, m is odd, and $v \in \{(0, 2k) \mid k = 1, 2, \ldots, m-1\}$. The proof is almost identical to the one above with necessary changes to allow for $m > 3$. Namely, the restrictions to $j = 1$ or $j \in \{0, 2\}$ now become j odd or j even, respectively.

We now consider the case where n is even and the starting vertex is $v = (0, 1)$, which is slightly more involved than the other cases. We begin with two lemmas.

Lemma 5.6. *Consider the game* MAC$(\mathrm{SP}(n, 3), v)$ *with n even and $v = (0, 1)$. If Player 1's first move is to $(1, 1)$, then Player 2 has a winning strategy.*

Proof. If Player 1 begins by moving from $(0, 1)$ to $(1, 1)$, then Player 2's strategy is the same as Player 1's winning strategy from the proof of Theorem 5.2. □

Similar to the dihedral groups, we have a lemma describing forced moves on the $\mathrm{SP}(n, 3)$ family of graphs.

Remark 5.7. In the following lemma and its proof, we denote players by Player A and Player B instead of Player 1 and Player 2, where $\{\mathrm{A}, \mathrm{B}\} = \{1, 2\}$. We denote the players in this manner due to the fact that, in different cases, Player 1 and Player 2 will each take on the role of Player "B" when applying this lemma. For several other results in this paper, we also denote players as Player A or Player B for similar reasons.

Lemma 5.8. *For the game* MAC$(\mathrm{SP}(n, 3), v)$*, let $B \in \{1, 2\}$ and suppose Player B moves from $(i-1, j)$ to (i, j) where $j \in \{0, 2\}$ and that no vertices (p, q) with $i - 1 < p < n - 1$ and $0 \leq q \leq 2$ have been previously visited. Then Player B can guarantee moving from $(i+3, j+2 \pmod 4)$ to $(i+4, j+2 \pmod 4)$.*

Proof. We have two cases. Suppose Player A moves from (i, j) to $(i+1, j)$. Then Player B moves inside to $(i+1, 1)$. Player A cannot move to $(i, 1)$, lest Player B win back at (i, j). Regardless then of Player A's move, either to $(i+2, 1)$ or to $(i+1, j+2 \pmod 4)$, Player B can guarantee arriving at $(i+2, j+2 \pmod 4)$. So as to not complete the third side of a square, Player A is forced to move from $(i+2, j+2 \pmod 4)$ to $(i+3, j+2 \pmod 4)$. Hence, Player B moves from $(i+3, j+2 \pmod 4)$ to $(i+4, j+2 \pmod 4)$.

The other case where Player A moves from (i, j) to $(i, 1)$ is similar. See Figure 9. □

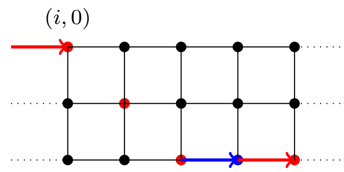

Figure 9: Example scheme for guaranteed vertices reached by Player B (in red).

Theorem 5.9. *Suppose n is even. For the game* MAC$(\mathrm{SP}(n,3), v)$ *with $v = (0,1)$, we have the following:*

1. *If $n \equiv 0, 4,$ or $6 \pmod 8$, then Player 2 has a winning strategy.*
2. *If $n \equiv 2 \pmod 8$, then Player 1 has a winning strategy.*

Proof. By Lemma 5.6, Player 1's first move is, without loss of generality, to move from $(0,1)$ to $(0,2)$. Without loss of generality, Player 2 will move from $(0,2)$ to $(1,2)$. Player 1's next move is forced to be from $(1,2)$ to $(2,2)$. We consider cases for each even n modulo 8.

`Case 1:` $n \equiv 0 \pmod 8$. Player 2 moves from $(2,2)$ to $(3,2)$. By Lemma 5.8 with $i-1=2$, Player 2 can guarantee moving from $(6,0)$ to $(7,0)$. By applying Lemma 5.8 repeatedly, Player 2 can guarantee moving from $(n-2,0)$ to $(n-1,0)$, since $n-1 \equiv 7 \pmod 8$. Regardless of Player 1's next move, to either $(n-1,1)$ or $(0,0)$, Player 2 can win by returning to $(0,1)$.

`Case 2:` $n \equiv 2 \pmod 8$. We can apply Lemma 5.8 twice to see that Player 1 guarantees moving from $(9,2)$ to $(10,2)$. By repeatedly applying Lemma 5.8, we see that Player 1 guarantees moving from $(n-1,2)$ to $(0,2)$, winning the game.

`Case 3:` $n \equiv 4 \pmod 8$. Starting with Player 2's move from $(0,2)$ to $(1,2)$, repeated applications of Lemma 5.8 guarantees Player 2 moving from $(n-4,2)$ to $(n-3,2)$ since $n-3 \equiv 1 \pmod 8$. Player 2 can then guarantee moving to $(n-2,1)$ on their following turn. Since the game began at $(0,1) = (n,1)$, Player 1 is forced to move to $(n-2,0)$. Player 2 then guarantees moving to $(n-1,0)$ and will win by returning to $(0,1)$ on their next move.

`Case 4:` $n \equiv 6 \pmod 8$. By one application of Lemma 5.8 with $i=1$, Player 2 guarantees moving from $(4,0)$ to $(5,0)$. Now repeatedly applying Lemma 5.8, Player 2 guarantees moving from $(n-2,0)$ to $(n-1,0)$ since $n-1 \equiv 5 \pmod 8$. Just as in Case (1), Player 2 can win by returning to $(0,1)$. $\square$

5.2. Avoid-A-Cycle for Stacked Prism Graphs

For the Avoid-A-Cycle game, we provide winning strategies for any general stacked prism graph $\mathrm{SP}(n,m)$, regardless of the starting vertex. By Theorem 2.8, we know that winning strategies amount to examining the maximum matchings in $\mathrm{SP}(n,m)$. We summarize the results in the following theorem.

Theorem 5.10. *For the game $\mathtt{AAC}(\mathrm{SP}(n,m))$, if n or m is even, then Player 1 has a winning strategy. If n and m are both odd, then Player 2 has a winning strategy.*

Proof. If both m and n are odd, for any vertex v, there exists a maximum matching not containing v. Let $v=(k,\ell)$ where $0 \le k \le n-1$ and $0 \le \ell \le m-1$. We describe a near-perfect matching which misses v below depending on the parity of ℓ. First, we introduce some notation that will be used. For each $0 \le j \le m-1$, let $e^H_{i,j}$ denote the "horizontal" edge joining vertices (i,j) and $(i+1,j)$, and, for each $0 \le i \le n-1$, let $e^V_{i,j}$ denote the "vertical" edge joining vertices (i,j) and $(i,j+1)$.
`Case 1:` ℓ is even. We then define the matching M by

$$M = \{e^H_{i,j} \mid i = k+s \pmod n, s \text{ odd}, 1 \le s \le n-2 \text{ and } 0 \le j \le m-1\}$$

$$\bigcup \{e^V_{i,j} \mid i = k \text{ and } j \text{ even for } 0 \le j \le \ell-2, \text{ and } j \text{ odd for } \ell+1 \le j \le m-2\}.$$

Then M is a near-perfect matching which misses the vertex $v=(k,\ell)$. See Figure 10.
`Case 2:` ℓ is odd. Then for the vertical edges, we get in our matching

$$\{e^V_{k-1,j} \mid 0 \le j \le \ell-1, j \text{ even}\} \cup \{e^V_{k-1,j} \mid \ell+2 \le j \le m-2, j \text{ odd}\}$$

and

$$\{e^V_{k,j} \mid 0 \le j \le \ell-3, j \text{ even}\} \cup \{e^V_{k,j} \mid \ell+2 \le j \le m-2, j \text{ odd}\}$$

and

$$\{e^V_{k+1,j} \mid 0 \le j \le \ell-3, j \text{ even}\} \cup \{e^V_{k+1,j} \mid \ell \le j \le m-2, j \text{ odd}\}.$$

For the horizontals, we get

$$\{e^H_{k,\ell-1}, e^H_{k-1,\ell+1}\} \cup \{e^H_{i,j} \mid 0 \le j \le m-1, i = k+s \pmod n, s \text{ even}, 2 \le s \le n-3\}.$$

This yields a near-perfect matching which misses the vertex $v=(k,\ell)$. Thus Player 2 has a winning strategy.

If either m or n are even, then a perfect matching exists. We describe these below. See also Figure 11 for two examples.

`Case 1:` n is even. Then define a perfect matching M by,

$$M = \{e^H_{i,j} \mid i \text{ even}, 0 \le i \le n-2 \text{ and } 0 \le j \le m-1\}.$$

`Case 2:` n is odd and m is even. Then define a perfect matching M by

$$M = \{e^H_{i,j}, e^V_{0,k} \mid i \text{ odd}, 1 \le i \le n-2 \text{ and } 0 \le j \le m-1, k \text{ even and } 0 \le k \le n-2\}.$$

Since any maximum matching will contain all vertices of the graph, Player 1 has a winning strategy when either n or m is even. $\square$

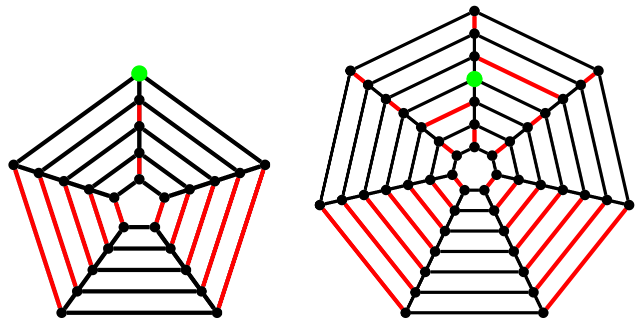

Figure 10: Example of near-perfect matchings on SP(5, 5) (left) and SP(7, 7) (right) with the unsaturated vertices marked green.

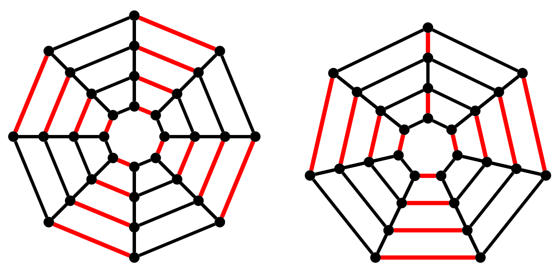

Figure 11: Example of perfect matchings on SP(8, 4) (left) and SP(7, 4) (right).

6. Generalized Petersen Graphs and Graph Joins

Let n and k be integers such that $2 \leq 2k \leq n$ and $n \geq 3$. We denote by $\mathrm{GP}(n, k)$ the class of *generalized Petersen graphs*. These are graphs containing $2n$ vertices denoted by u_i and v_i for $i = 0, 1, \ldots, n-1$. The edges are given by (u_i, u_{i+1}), (u_i, v_i), and (v_i, v_{i+k}) for $i = 0, 1, \ldots, n-1$, and where the addition in the subscript is taken modulo n. See Figure 12.

The generalized Petersen graphs can be seen visually as the joining of an outer n-gon with an inner (n, k) star polygon. Note also that $\mathrm{GP}(n, k) = \mathrm{GP}(n, n-k)$. See Figure 13 for three examples.

Generalized Petersen graphs were defined by Watkins in [10] although a special subclass was first studied by Coxeter ([3]). See also [6]. Although the above authors excluded the case of $2k = n$ for the family $\mathrm{GP}(n, k)$, we include it here as the graphs $\mathrm{GP}(2n, n)$ have a nice structure that allow for an analysis of our games (see Section 6.3).

The family of graphs $\mathrm{GP}(n, 1)$ are called *prism graphs* and are isomorphic to the Cayley graphs for the dihedral groups D_n. Hence, by Theorem 2.2 and the results in [7], the outcomes for $\mathtt{MAC}(\mathrm{GP}(n, 1))$ and $\mathtt{AAC}(\mathrm{GP}(n, 1))$ are already known.

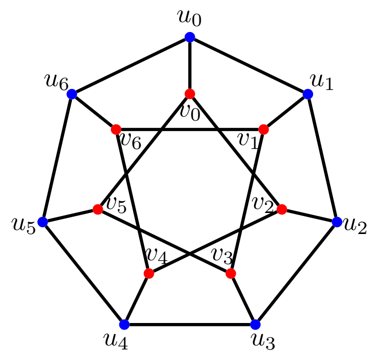

Figure 12: An example of the generalized Petersen graph GP$(7, 2)$.

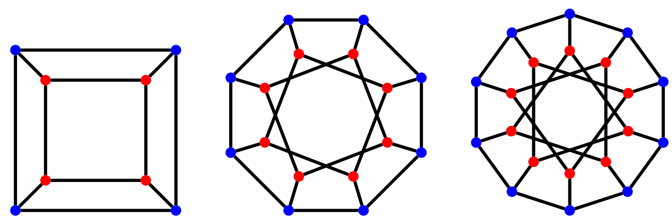

Figure 13: Generalized Petersen graphs: GP$(4, 1)$, GP$(8, 2)$, GP$(10, 3)$.

The family GP$(n, 2)$ starts at $n = 5$ with the famous Petersen graph. The Petersen graph is vertex-transitive; hence the choice of starting vertex for MAC or AAC on the Petersen graph is irrelevant. In general, the graphs GP(n, k) are vertex-transitive if and only if $k^2 \equiv \pm 1 \pmod{n}$ (see [6]). Therefore, since $4 \equiv \pm 1 \pmod{n}$ implies either $n \mid 3$ or $n \mid 5$, GP$(n, 2)$ is vertex-transitive if and only if $n = 5$. We will show winning strategies for MAC(GP$(n, 2)$) and AAC(GP(n, k)) for all k in the next two sections.

6.1. Make-A-Cycle for Generalized Petersen Graphs GP$(n, 2)$

The vertices of the GP$(n, 2)$ graphs, for $n > 5$, lie in one of two orbits under the action of the automorphism group of the graph. These are indicated by the labeling of the vertices themselves: the outer vertices $u_0, u_1, \ldots, u_{n-1}$ lie in one orbit, while the inner vertices $v_0, v_1, \ldots, v_{n-1}$ lie in the other orbit. Hence, without loss of generality, there are two starting positions for the MAC game on GP$(n, 2)$ graphs with $n > 5$ (for $n = 5$ all starting positions are equivalent because the graph is vertex-transitive). We will always consider starting vertices of either u_0 or v_0 for the Make-A-Cycle game.

Due to the structure of the inner star polygon in the GP$(n, 2)$ graphs, an interesting pattern of forced moves can occur for the Make-A-Cycle games. This is detailed in Lemma 6.1 below. Both players are able to employ this lemma to create winning strategies on $\mathrm{GP}(n, 2)$ graphs, which depends on the value of n modulo 5. Note that in this lemma we denote players by A and B, where {A,B} $= \{1, 2\}$, because, in different cases of n modulo 5, Player 1 and Player 2 will each take on the role of Player "A" (see Remark 5.7).

Lemma 6.1 (Mod Five Lemma)**.** *For the game* $\mathtt{MAC}(\mathrm{GP}(n,2))$*, suppose for some* i*,* $0 \le i < n-3$*, that no vertices* u_k *or* v_k *have been previously visited for* $i < k < n$*. Let* $\{A, B\} = \{1, 2\}$*.*

1. *If Player A moves from* u_i *to* v_i *and Player B moves from* v_i *to* v_{i+2}*, then Player A can guarantee the following moves:*

$$\begin{aligned} &u_j \to v_j \textit{ for } j \equiv i \pmod 5,\\ &v_j \to u_j \textit{ for } j \equiv i+2 \pmod 5,\\ &u_j \to u_{j+1} \textit{ for } j \equiv i+3 \pmod 5, \end{aligned}$$

   for $i \le j < n$*.*

2. *If Player A moves from* v_i *to* u_i *and Player B moves from* u_i *to* u_{i+1}*, then Player A can guarantee the following moves:*

$$\begin{aligned} &v_j \to u_j \textit{ for } j \equiv i \pmod 5,\\ &u_j \to u_{j+1} \textit{ for } j \equiv i+1 \pmod 5,\\ &u_j \to v_j \textit{ for } j \equiv i+3 \pmod 5, \end{aligned}$$

   for $i \le j < n$*.*

Proof. We prove statement (1) as the proof of statement (2) uses a similar argument. Suppose Player A moves from u_i to v_i and Player B moves from v_i to v_{i+2}. Then Player A moves from v_{i+2} to u_{i+2}. If Player B moves from u_{i+2} to u_{i+1}, then Player A wins at u_i. Therefore, Player B must move from u_{i+2} to u_{i+3}. Player A then chooses to move to u_{i+4}. Note that Player B cannot move to v_{i+4} since Player A would then win at v_{i+2}. Therefore, Player B moves to u_{i+5}.

Player A then chooses to move from u_{i+5} to v_{i+5}. From here Player B is forced to move to v_{i+7} since a move to v_{i+3} allows Player A to win at u_{i+3}. Note that the moves from u_{i+5} to v_{i+5} and then v_{i+5} to v_{i+7} precisely satisfy the conditions of the lemma starting at $i+5$ instead of i. Thus, the above argument may be repeated to guarantee the specified moves for all j with $i \le j \le n$. $\square$

Below in Figure 14 is an example of how a player could use the strategy of Lemma 6.1.

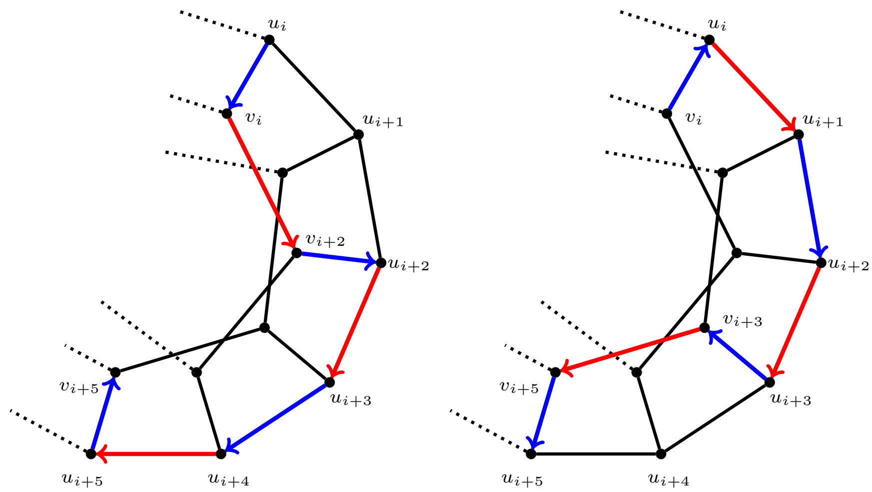

Figure 14: The moves described in Lemma 6.1 statement (1) (left) and Lemma 6.1 statement (2) (right), where blue denotes player A and red denotes player B.

Theorem 6.2. *Player 1 has a winning strategy for* MAC(GP$(n, 2)$) *for* $n \not\equiv 0 \pmod 5$ *starting at either vertex* u_0 *or* v_0*.*

Proof. We first consider the starting vertex u_0. Player 1 begins by moving from u_0 to v_0. Without loss of generality, assume Player 2 moves from v_0 to v_2. We show that repeated applications of Lemma 6.1 statement (1) by Player 1 will lead to their winning the game. We consider the four cases of $n \pmod 5$, where $n \not\equiv 0 \pmod 5$, separately.

`Case 1:` $n \equiv 1 \pmod 5$. Given the first two moves outlined above, Player 1 can repeatedly apply Lemma 6.1 statement (1) to guarantee moving from u_j to u_{j+1} with $j \equiv 3 \pmod 5$. Since $n \equiv 1 \pmod 5$, then $n - 3 \equiv 3 \pmod 5$. Thus Player 1 guarantees moving from u_{n-3} to u_{n-2}. Regardless of Player 2's next move, Player 1 will win at either u_0 or v_0.

`Case 2:` $n \equiv 2 \pmod 5$. Applying Lemma 6.1 statement (1), Player 1 guarantees moving from u_j to v_j with $j \equiv 0 \pmod 5$. Hence, Player 1 guarantees moving from u_{n-7} to v_{n-7} since $n \equiv 2 \pmod 5$ implies $n - 7 \equiv 0 \pmod 5$.

From v_{n-7}, Player 2 is forced to move to v_{n-5}. Player 1 will now move to v_{n-3}. If Player 2 moves to u_{n-3}, then Player 1 will move to u_{n-2} and win on their next turn at either v_0 or u_0.

If Player 2 moves instead to v_{n-1}, then Player 1 will move to v_1, which has not been previously visited. If $n > 7$, then regardless of Player 2's next move, Player 1 will win at either u_0 or u_3, the latter of which has been visited due to the first application of Lemma 6.1 statement (1).

If $n = 7$, then u_3 will not have been previously visited and thus Player 2 will move to v_3. Thereafter Player 1 will move to u_3. Regardless of Player 2's next move, Player 1 will then win on their next turn at either v_2 or v_4.

`Case 3:` $n \equiv 3 \pmod 5$. Applying Lemma 6.1 statement (1), Player 1 guarantees moving from u_j to v_j with $j \equiv 0 \pmod 5$. Hence, Player 1 will move from u_{n-3} to v_{n-3} since $n - 3 \equiv 0 \pmod 5$. Player 2 is forced to move to v_{n-1}. Then as in the previous case, Player 1 will move to v_1 and win the game on their next turn.

`Case 4:` $n \equiv 4 \pmod 5$. Applying Lemma 6.1 statement (1), Player 1 guarantees moving from u_j to v_j with $j \equiv 0 \pmod 5$. Hence, Player 1 is guaranteed to move from u_{n-4} to v_{n-4}. Regardless of Player 2's next move, Player 1 wins on their next turn at v_0 or u_{n-6}.

We now consider the starting vertex v_0. Player 1 will begin by moving from v_0 to u_0. Without loss of generality, we assume Player 2 moves from u_0 to u_1. We show that Player 1 has a winning strategy via repeated applications of Lemma 6.1 statement (2) which again depends on $n \pmod 5$.

`Case 1:` $n \equiv 1 \pmod 5$. Since Player 1 moved from v_0 to u_0 and Player 2 moved from u_0 to u_1, Player 1 can guarantee moving from u_j to u_{j+1} where $j \equiv 1 \pmod 5$, by repeated applications of Lemma 6.1 statement (2). Hence, Player 1 is guaranteed to move from u_{n-5} to u_{n-4} because $n \equiv 1 \pmod 5$, thus $n-5 \equiv 1 \pmod 5$. Player 2 is forced to move from u_{n-4} to u_{n-3}. Player 1 will move from u_{n-3} to u_{n-2}, and regardless of Player 2's next move, Player 1 will win at either u_0 or v_0.

`Case 2:` $n \equiv 2 \pmod 5$. Through repeated applications of Lemma 6.1 statement (2), Player 1 can guarantee moving from v_j to u_j with $j \equiv 0 \pmod 5$. Hence, Player 1 can guarantee moving from v_{n-2} to u_{n-2} since $n \equiv 2 \pmod 5$. Regardless of Player 2's next move, Player 1 will win the game at either u_{n-4} or u_0.

`Case 3:` $n \equiv 3 \pmod 5$. Applying Lemma 6.1 statement (2), Player 1 can guarantee moving from u_j to v_j with $j \equiv 3 \pmod 5$. Hence, Player 1 can guarantee moving from u_{n-5} to v_{n-5} since $n \equiv 3 \pmod 5$. Player 2 is then forced to move from v_{n-5} to v_{n-3} and Player 1 will choose to move from v_{n-3} to v_{n-1}. Since v_1 has not yet been visited, Player 2 cannot win on this turn. Regardless of their move, Player 1 wins on the subsequent turn at u_0 or u_1.

`Case 4:` $n \equiv 4 \pmod 5$. Again, Player 1 uses Lemma 6.1 statement (2) to guarantee moving from v_j to u_j with $j \equiv 0 \pmod 5$. Hence, Player 1 is guaranteed to make a move from v_{n-4} to u_{n-4} since $n \equiv 4 \pmod 5$. Player 2 is then forced to move from u_{n-4} to u_{n-3}. Player 1 will choose to move from u_{n-3} to u_{n-2} and will win on their next move at either u_0 or v_0.

Since we have now shown that Player 1 has a winning strategy regardless of the starting vertex, this completes the proof. □

Now we examine `MAC`$(\mathrm{GP}(n, 2))$ for the case of $n \equiv 0 \pmod 5$. Player 2 has a winning strategy for this case. To prove this, we must consider all of Player 1's opening moves. In order to counter all of these moves, Player 2 needs one additional tool, which we state as an additional lemma.

Lemma 6.3 (Zigzag Lemma)**.** *Consider the game* $\mathtt{MAC}(\mathrm{GP}(n,2))$ *and let* A*,* $B \in \{1,2\}$*.*

1. *Suppose* $4 \leq i \leq n-5$ *and that no vertices* u_k *or* v_k *have been visited for* $i-3 \leq k \leq i+4$ *as well as* v_{i+5}*. Suppose Player A moves from* v_{i-4} *to* v_{i-2}*, Player B moves from* v_{i-2} *to* v_i*, and then Player A moves from* v_i *to* u_i*. If Player B then moves from* u_i *to* u_{i-1}*, then Player A can guarantee moving from* v_{i+5} *to* u_{i+5} *with the two preceding moves being from* v_{i+5-4} *to* v_{i+5-2} *and then from* v_{i+5-2} *to* v_{i+5}*.*

2. *Suppose* $4 \leq i \leq n-5$ *and that no vertices* u_k *or* v_k *have been visited for* $i-3 \leq k \leq i+4$*. Suppose Player B moves from* v_{i-4} *to* v_{i-2} *and then Player A moves from* v_{i-2} *to* v_i*. If Player B moves from* v_i *to* u_i*, then Player A can guarantee moving from* v_{i+5-2} *to* v_{i+5} *with the preceding move being from* v_{i+5-4} *to* v_{i+5-2}*.*

Proof. Suppose for $4 \leq i \leq n-5$ Player A moves from v_{i-4} to v_{i-2}, followed by Player B moving from v_{i-2} to v_i and Player A moving from v_i to u_i. Now suppose Player B moves from u_i to u_{i-1}. Then Player A will choose to move from u_{i-1} to v_{i-1}. Note that if Player B moves to v_{i-3}, then Player A can move to u_{i-3}, thereby forcing a win on their following move from either u_{i-4} to v_{i-4} or from u_{i-2} to v_{i-2}. Thus Player B is forced to move from v_{i-1} to v_{i+1}. Player A then chooses to move to v_{i+3}. If Player B moves from v_{i+3} to u_{i+3}, then Player A can move to u_{i+2} and win on their next turn by returning to v_i or either of u_i or v_{i+1}. Hence, Player B is forced to move from v_{i+3} to v_{i+5}. Finally Player A can now move from v_{i+5} to u_{i+5} with the previous two moves being from v_{i+5-4} to v_{i+5-2} and from v_{i+5-2} to v_{i+5} as claimed. This proves statement (1).

For the second case, suppose that Player B moves from v_{i-4} to v_{i-2} and that Player A subsequently moves from v_{i-2} to v_i. If Player B moves from v_i to u_i, then Player A will choose to move from u_i to u_{i-1}, and the sequence of moves that follows is the same as in case statement (1). □

The sequence of moves that results from Lemma 6.3 statement (1) and Lemma 6.3 statement (2) are shown in Figure 15. Note that not all edges have been drawn for clarity purposes. The name Zigzag Lemma arises from the changes in direction of the game path along the vertices $v_i, u_i, u_{i-1}, v_{i-1}, v_{i+1}$.

Remark 6.4. Note that one can repeatedly apply Lemma 6.3 provided that Player B continues to satisfy the "if condition", that is, if Player B moves from u_i to u_{i-1} (statement 1) or moves from v_i to u_i (statement 2).

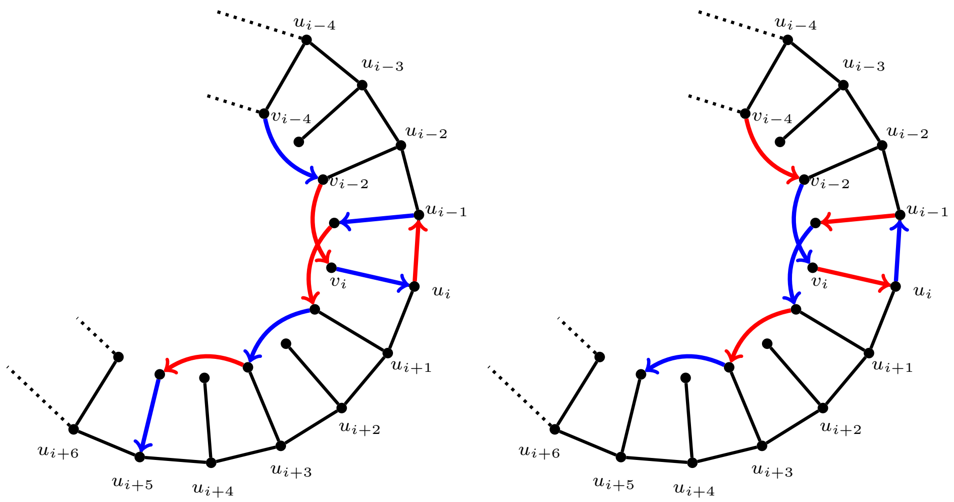

Figure 15: The sequence of moves in Lemma 6.3 statement (1) (left) and Lemma 6.3 statement (2) (right), where Player A moves are colored blue and Player B moves are colored red.

Remark 6.5. Note that both strategies from Lemma 6.1 and Lemma 6.3 work "in reverse" as well. If the subscripts are decreasing (increasing) in the original statement, then they are increasing (decreasing) in the "reverse" statement.

For example, for Lemma 6.1 statement (1), let $0 < i < n-3$ and assume for some integer k_0 that no vertices u_k or v_k have been previously visited for $0 < k_0 \leq k < i$. If Player A moves from u_i to v_i and then Player B moves from v_i to v_{i-2}, then Player A can guarantee the moves

$$\begin{aligned} u_j &\to v_j \text{ for } j \equiv i \pmod 5, \\ v_j &\to u_j \text{ for } j \equiv i-2 \pmod 5, \\ u_j &\to u_{j-1} \text{ for } j \equiv i-3 \pmod 5, \end{aligned}$$

for $0 < k_0 \leq j \leq i$.

We additionally note that the case where $i = 1$ is a special case. If the game begins with the moves $v_0 \to v_2 \to u_2 \to u_1 \to v_1$, then Player 2 can apply Lemma 6.1 statement (1) in reverse as Player A despite not satisfying the condition for k_0.

The other results for Lemma 6.1 and Lemma 6.3 applied in reverse can be written out similarly.

Theorem 6.6. *Player 2 has a winning strategy for* MAC(GP$(n, 2)$) *for* $n \equiv 0 \pmod 5$ *regardless of starting vertex.*

Proof. Recall that, without loss of generality, the game begins at either u_0 or v_0.

First we assume the starting vertex is v_0, and we examine the following two cases for the first move: Player 1 moves from v_0 to v_2 and Player 1 moves from v_0 to u_0.
`Case 1:` Player 1 moves from v_0 to v_2. Note Player 1 could move from v_0 to v_{n-2}, but without loss, we assume they move to v_2. Player 2 will then move from v_2 to u_2. If Player 1 then moves from u_2 to u_3, then Player 2 can move from u_3 to u_4 and use Lemma 6.1 statement (2) to guarantee moving from u_i to u_{i+1} for $i \equiv 3 \pmod 5$. In particular, Player 2 guarantees moving from u_{n-2} to u_{n-1} since $n \equiv 0 \pmod 5$. Regardless of Player 1's next move, Player 2 will win at v_0 or v_{n-3}, the latter of which is guaranteed to be previously visited, also by Lemma 6.1 statement (2).

Suppose instead that Player 1 moves from u_2 to u_1. From here, Player 2 is forced to move to v_1. Player 1 then has two choices, which we will analyze separately.
`Subcase 1a:` Player 1 moves to v_3. Then Player 2 moves to v_5, which is winning if $n = 5$ (in this case, $v_5 = v_0$). If $n > 5$, Player 1 is forced to move to v_7. Indeed, if Player 1 moves from v_5 to u_5, then Player 2 will move from u_5 to u_4 and will win on their next turn at one of v_3, u_2, or v_2. Thus, Player 1 must move from v_5 to v_7 and then Player 2 will choose to move to u_7. If Player 1 moves to u_8, then as above, Player 2 will move from u_8 to u_9 and use repeated applications of Lemma 6.1 statement (2) to win at v_0 or v_{n-3}.

Suppose instead that Player 1 moves from u_7 to u_6. Then by applying Lemma 6.3 statement (1) with $i = 7$ and Player A being Player 2, Player 2 can guarantee moving from v_{12} to u_{12} with the two preceding moves being from v_8 to v_{10} (by Player 2), which is winning if $n = 10$, and then from v_{10} to v_{12} (by Player 1), if $n > 10$. If Player 1 continues to move from u_j to u_{j-1} for $j \equiv 2 \pmod 5$, then repeated applications of Lemma 6.3 statement (1) ensure Player 2 moves from v_{j+5} to u_{j+5}. In this case, Player 2 will eventually move from v_{n-3} to u_{n-3}. Player 1, by our assumption, moves to u_{n-4}. Note that this satisfies the hypotheses of Lemma 6.3 statement (1) with Player 2 as Player A, except that $i = n-3$ is too large. However, we note that the "two preceding moves" in the statement of Lemma 6.3 statement (1) are still guaranteed with $i = n-3$, so Player 2 is guaranteed to move from v_{n-2} to v_0 since $n-2 \equiv 3 \pmod 5$. Thus, Player 1 must move from u_j to u_{j+1} for some $j \equiv 2 \pmod 5$. In this case, we can apply Lemma 6.1 statement (2) as before with Player A being Player 2. Hence, Player 2 guarantees moving from u_{n-2} to u_{n-1} and wins on their subsequent turn at v_0 or v_{n-3}.
`Subcase 1b:` Player 1 moves from v_1 to v_{n-1}. Then since the previous move is from u_1 to v_1, Lemma 6.1 statement (1) applies in reverse (see Remark 6.5, specifically the special case when $i = 1$). Hence, Player 2 can guarantee moving from v_i to u_i for $i \equiv 4 \pmod 5$. In doing so, Player 2 wins on the turn following their move from v_4 to u_4. Indeed, Player 1 moves to either u_3 or u_5 on their following turn, whereby Player 2 completes a cycle at u_2 or u_6, respectively.

`Case 2:` Player 1 begins by moving from v_0 to u_0. We first show that Player 1 loses if they use Lemma 6.1 at their first opportunity to do so, which implies that Player 1 must at some point deviate from the guaranteed moves in Lemma 6.1. We then show that, given any deviation from this strategy by Player 1, Player 2 will have a winning strategy.

If Player 1 moves from v_0 to u_0 and Player 2 moves from u_0 to u_1, then Lemma 6.1 statement (2) applies with Player 1 as Player A. Thus, Player 1 ensures the following moves:

$$\begin{aligned} &v_i \to u_i \text{ for } i \equiv 0 \pmod 5,\\ &u_i \to u_{i+1} \text{ for } i \equiv 1 \pmod 5,\\ &u_i \to v_i \text{ for } i \equiv 3 \pmod 5, \end{aligned}$$

for all $0 \leq i < n$. Should Player 1 execute this strategy, they will eventually move from u_{n-2} to v_{n-2}. This is a losing move since Player 2 can then move to v_0 on the next move. Thus, we may assume that Player 1 applies Lemma 6.1 statement (2) until a certain point in the game and then deviates from this strategy. Thus, there are 3 cases, one for each of the move types listed above. See Figure 16 for a visual aid on the three cases, which are labeled red, blue, and green. A solid line denotes the guaranteed move from Lemma 6.1 and the dotted line denotes Player 1's deviation, with i an integer satisfying $i > 0$ and $i \equiv 0 \pmod 5$.

`Subcase 2a:` For some $i > 0$ with $i \equiv 0 \pmod 5$, Player 1 moves from v_i to v_{i+2} instead of v_i to u_i. Player 2 then moves from v_{i+2} to v_{i+4}.

If Player 1 moves from v_{i+4} to u_{i+4}, then Player 2 moves to u_{i+3} and we apply Lemma 6.3 statement (2) with Player 2 as Player A so that Player 2 guarantees moving from v_{i+7} to v_{i+9}. If Player 1 continues to move from v_j to u_j where $j \equiv 4 \pmod 5$, then Lemma 6.3 statement (2) will guarantee Player 2 moves from v_{j+3} to v_{j+5}. Suppose Player 1 moves from v_j to u_j for all $j \equiv 4 \pmod 5$ with $j \geq i+4$. Then Player 2 wins by moving from v_{n-3} to v_{n-1} since Player 1 will then move to either v_1, losing at u_1 the next turn, or to u_{n-1}, losing at u_0 the next turn.

Thus, Player 1 must move from v_j to v_{j+2} for some $i + 4 \leq j \leq n - 1$, where $j \equiv 4 \pmod 5$. Player 2 will then move from v_{j+2} to u_{j+2}, where we note $j + 2 \equiv 1 \pmod 5$. If Player 1 continues to move from u_k to u_{k-1}, with $k \equiv 1 \pmod 5$ and $k \geq j + 2$, then Lemma 6.3 statement (1) will guarantee Player 2 moves from v_{k+5} to u_{k+5}.

As above, if Player 1 moves from u_k to u_{k-1} for all $k \equiv 1 \pmod 5$ with $k \geq j+2$, then Player 2 guarantees moving from v_{n-4} to u_{n-4}. By assumption, Player 1 will move from u_{n-4} to u_{n-5}. This again satisfies the hypotheses of Lemma 6.3 statement (1) with Player 2 as Player A, except that $i = n - 4$ is too large, but the "two preceding moves" in the statement of Lemma 6.3 statement (1) are still guaranteed with $i = n - 4$, so Player 2 is guaranteed to move from v_{n-3} to v_{n-1} and wins on their next turn at u_0 or u_1.

Finally, Player 1 must move from u_k to u_{k+1}, for some $k \geq j + 2$, where $k \equiv 1$

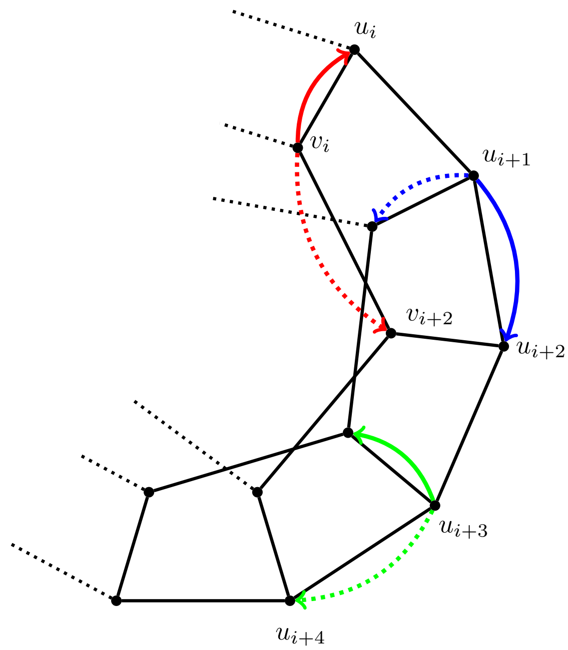

Figure 16: The three cases of Player 1 deviating from the forced moves of Lemma 6.1.

(mod 5). Then by Lemma 6.1 statement (2) Player 2 can guarantee moving from u_ℓ to $u_{\ell+1}$ for all $\ell \geq k+1$ and $\ell \equiv 2 \pmod 5$. Hence Player 2 eventually guarantees moving from u_{n-3} to u_{n-2} and will win on their next turn at either u_0 or v_0.

`Subcase 2b:` Player 1 moves from u_i to v_i for some $1 \leq i \leq n-4$ with $i \equiv 1 \pmod 5$. If $i \geq 6$, then Player 2 moves to v_{i-2}. Regardless of Player 1's next move, Player 2 will win at a previously visited vertex (say v_{i-6} or u_{i-3}).

If $i = 1$, then Player 2 moves to v_{n-1}. Player 1 is then forced to v_{n-3} and Player 2 then moves to v_{n-5}. If $n = 5$, then Player 2 wins on this move.

Now if $n > 5$ and Player 1 moves from v_{n-5} to u_{n-5}, then applying Lemma 6.3 statement (2) in reverse ensures Player 2 moves from v_{n-8} to v_{n-10}. If Player 1 continues to move from v_j to u_j where $j \equiv 0 \pmod 5$, then Lemma 6.3 statement (2) will guarantee that Player 2 moves from v_{j-3} to v_{j-5}.

If Player 1 moves from v_j to u_j for all $j \equiv 0 \pmod 5$, with $j \leq n-5$, then Player 2 will move from v_2 to v_0 to win. Thus, for some $5 \leq j \leq n-5$ with $j \equiv 0 \pmod 5$, Player 1 will move from v_j to v_{j-2}. Player 2 will then move to u_{j-2}.

From this point, if Player 1 always moves from u_k to u_{k+1} where $k \equiv 3 \pmod 5$, then applying Lemma 6.3 statement (1) in reverse will guarantee that Player 2 moves from v_{k-1} to v_{k-3}. Player 2 then wins by reaching v_0 from v_2. Thus, at some point Player 1 must move from u_k to u_{k-1} for some $3 \leq k \leq j-2$, where

$k \equiv 3 \pmod 5$. Then by applying Lemma 6.1 statement (2) in reverse, Player 2 guarantees being able to move from u_ℓ to $u_{\ell-1}$ where $\ell \leq k-1$ and $\ell \equiv 2 \pmod 5$. Player 2 thus wins by moving from u_2 to u_1.

`Subcase 2c:` Player 1 moves from u_i to u_{i+1} for some i with $3 \leq i \leq n-2$ and $i \equiv 3 \pmod 5$. Then Player 2 moves from u_{i+1} to v_{i+1}. Player 1 is then forced to move from v_{i+1} to v_{i+3} since u_{i-1} has been previously visited in this case. Hence, by Lemma 6.1 statement (1), Player 2 guarantees moving from u_j to u_{j+1} for $j \equiv 2 \pmod 5$, with $j \geq i+4$. In particular, Player 2 moves from u_{n-3} to u_{n-2}. If Player 1 moves to u_{n-1}, then Player 2 wins at u_0. If Player 1 moves to v_{n-2}, then Player 2 wins at v_0.

The case with starting vertex u_0 is very similar to the case above. A detailed proof of this case can be found in Appendix A. □

6.2. Graph Joins

Before we state the winning strategies for $\texttt{AAC}(\mathrm{GP}(n,k))$, we define a more general graph structure, which we call the straight join. We will prove a result for straight joins and then obtain a winning strategy for $\texttt{AAC}(\mathrm{GP}(n,k))$ as a corollary.

However, we first recall the more general notion of a graph join. Given two graphs Γ_1 and Γ_2, the *graph join* $\Gamma_1 + \Gamma_2$ is a disjoint union of both graphs, together with new edges joining every vertex in Γ_1 with every vertex in Γ_2. For example, the wheel graph W_n is a graph join $C_{n-1} + K_1$.

Definition 6.7. Let Γ_1 and Γ_2 be finite simple graphs with vertex sets $V\Gamma_1$ and $V\Gamma_2$, respectively. Suppose $|V\Gamma_1| = |V\Gamma_2| = n$ and vertex sets $V\Gamma_1 = \{v_0^1, v_1^1, \ldots, v_{n-1}^1\}$ and $V\Gamma_2 = \{v_0^2, v_1^2, \ldots, v_{n-1}^2\}$. Then we define the *straight join* of the graphs Γ_1 and Γ_2 to be the graph Γ with vertex set $V\Gamma = V\Gamma_1 \cup V\Gamma_2$ and edge set

$$E\Gamma = E\Gamma_1 \cup E\Gamma_2 \cup \{(v_i^1, v_i^2) \mid i = 0, 1, \ldots, n-1\}.$$

We use the notation $\Gamma_1 \leftrightarrow \Gamma_2$ to denote the straight join.

Observe that the straight join is a subgraph of the join of two graphs which have the same number of vertices. We consider a couple of examples. The family of dihedral Cayley graphs or prism graphs is readily seen to be the family of straight joins $C_n \leftrightarrow C_n$, $n \geq 3$, where C_n denotes the cycle graph on n vertices. Also, the family of graphs $\mathrm{GP}(n,2)$ is formed by a straight join of C_n and an $(n,2)$-star polygon.

We can now state and prove a general result for the Avoid-A-Cycle game played on any straight-join graph.

Theorem 6.8. *Suppose $\Gamma = \Gamma_1 \leftrightarrow \Gamma_2$. Then Player 1 has a winning strategy for $\texttt{AAC}(\Gamma)$.*

Proof. By Definition 6.7, every vertex v^1 in Γ_1 is matched with a vertex v^2 in Γ_2 via the edges (v^1, v^2). These edges thus create a perfect matching of $\Gamma_1 \leftrightarrow \Gamma_2$. Hence, Player 1 has a winning strategy by Theorem 2.8. □

Corollary 6.9. *Player 1 has a winning strategy for* AAC$(\mathrm{GP}(n,k), v)$ *for any starting vertex* v.

Note that the specific strategy for Player 1 on AAC$(\mathrm{GP}(n,k))$ is to only choose edges of the form (u_i, v_i). This is the exact same strategy utilized in [7, Theorem 3.9], which makes sense as the dihedral Cayley graphs are straight join graphs.

Since the new edges created in $\Gamma_1 \leftrightarrow \Gamma_2$ form a perfect matching, Theorem 6.8 tells us more generally how Player 1 can win AAC on any graph join.

Corollary 6.10. *Suppose* $\Gamma = \Gamma_1 + \Gamma_2$ *where* $2 \leq |\Gamma_1| = |\Gamma_2| < \infty$. *Then Player 1 has a winning strategy for* AAC(Γ).

Proof. Let $n = |\Gamma_1| = |\Gamma_2|$. We can enumerate the vertices of each graph by $V\Gamma_1 = \{v_0^1, v_1^1, \ldots, v_{n-1}^1\}$ and $V\Gamma_2 = \{v_0^2, v_1^2, \ldots, v_{n-1}^2\}$. Then the edges $(v_i^1, v_i^2), i = 0, 1, \ldots, n-1$, form a perfect matching in $\Gamma_1 + \Gamma_2$ just as in Theorem 6.8. □

Now consider the following question: if we know a given player has a winning strategy for Make-A-Cycle or Avoid-A-Cycle on one of the graphs in a graph join $\Gamma_1 + \Gamma_2$, can we leverage this knowledge to conclude something about the games on $\Gamma_1 + \Gamma_2$? For the Make-A-Cycle game, we can determine exactly which player has a winning strategy.

Theorem 6.11. *Suppose* Γ_1 *and* Γ_2 *are simple graphs both of cardinality at least two. Fix a starting vertex* s_i *in* Γ_i, *where* $i \in \{1, 2\}$. *Then Player 1 has a winning strategy for* MAC$(\Gamma_1 + \Gamma_2, s_i)$ *if and only if Player 1 has a winning strategy for* MAC(Γ_i, s_i) *or if all vertices in* Γ_i *are adjacent to* s_i.

Proof. Without loss of generality, we fix $i = 1$. We first prove the backward direction, where Player 1 has a winning strategy for MAC(Γ_1, s_1). Then their strategy for MAC$(\Gamma_1 + \Gamma_2, s_1)$ will be to execute their strategy on Γ_1 until/unless Player 2 moves to a vertex in Γ_2. Should that happen, Player 1 will move back to s_1 to win, as, by the definition of join, there necessarily exists an edge between s_1 and every vertex in Γ_2. Note that this move is not a backtrack since the first move of the game is within Γ_1. Hence, Player 1 has a winning strategy on the graph join $\Gamma_1 + \Gamma_2$.

Now we consider the case where all vertices in Γ_1 are adjacent to s_1. Then Player 1 will move to some vertex, say $v_1 \in \Gamma_1$. Then Player 2 has two options: they can either move in Γ_1 or move to Γ_2. If $|\Gamma_1| \geq 3$, then Player 2 may move to some vertex $u_1 \notin \{s_1, v_1\}$ with $u_1 \in \Gamma_1$. However, since u_1 is adjacent to s_1, Player 1 wins on the subsequent turn. If Player 2 moves to some vertex $v_2 \in \Gamma_2$, then Player

1 wins on the subsequent turn by returning to s_1, which is adjacent to all vertices in Γ_2 by the definition of the graph join.

We now prove the forward direction. Suppose Player 2 has a winning strategy on $\texttt{MAC}(\Gamma_1, s_1)$ and there exists a vertex v_1 in Γ_1 not adjacent to s_1. As long as Player 1 moves within Γ_1, Player 2 can implement their winning strategy. If Player 1 moves to Γ_2 after their first move, then Player 2 can win back at s_1 in the same manner as Player 1 did in the case above. Hence, suppose Player 1 moves from s_1 to a vertex v_2 in Γ_2 on the first move of the game. Player 2 will then move to v_1 on the following move. Since Player 1 cannot move to s_1 and cannot backtrack to v_2, they cannot win on their next turn. If Player 1 moves to a vertex in Γ_1, Player 2 will win on their next turn by moving back to v_2, and if Player 1 moves to a vertex in Γ_2, then Player 2 will win on their next turn by moving back to s_1. □

We can see from Theorem 6.11 that the Make-A-Cycle game on graph joins is very similar to Make-A-Cycle games on complete graphs due to the saturation of vertices in each graph created by the join. For the Avoid-A-Cycle game, we have similar results if we assume we are working with Cayley graphs.

Theorem 6.12. *Suppose Γ_1 and Γ_2 are both Cayley graphs having cardinality at least two. Fix a starting vertex s_1 in Γ_1. Then Player 1 has a winning strategy for $\texttt{AAC}(\Gamma_1 + \Gamma_2, s_1)$ if and only if $|\Gamma_1| \equiv |\Gamma_2| \pmod 2$.*

Proof. Recall by Theorem 2.9 that a Cayley graph of even cardinality has a perfect matching, while a Cayley graph with odd cardinality always has a near-perfect matching for any vertex v such that v is unsaturated.

Suppose $|\Gamma_1| \equiv |\Gamma_2| \pmod 2$. If $|\Gamma_1|$ and $|\Gamma_2|$ are both even, then each graph has a perfect matching, say M_1 and M_2 for Γ_1 and Γ_2, respectively. Then $M_1 \cup M_2$ is a perfect matching for $\Gamma_1+\Gamma_2$ and thus Player 1 wins $\texttt{AAC}(\Gamma_1+\Gamma_2, s_1)$ by Theorem 2.8. Now suppose Γ_1 and Γ_2 are both of odd cardinality. Let N_1 be a near-perfect matching of Γ_1 that does not saturate s_1. Pick some vertex s_2 in Γ_2 and let N_2 be a near-perfect matching not containing s_2. Then the set of edges $N_1 \cup N_2 \cup \{(s_1, s_2)\}$ is a perfect matching in $\Gamma_1 + \Gamma_2$, and therefore Player 1 has a winning strategy again by Theorem 2.8.

Now consider the case where $|\Gamma_1| \not\equiv |\Gamma_2| \pmod 2$. Without loss of generality, suppose $|\Gamma_1|$ is even and $|\Gamma_2|$ is odd. Take a perfect matching M in Γ_1 and a near-perfect matching N missing the vertex s_2 in the Cayley graph Γ_2. Note that $M \cup N$ is a near-perfect matching in $\Gamma_1 + \Gamma_2$. Since M is perfect, the starting vertex s_1 is saturated by an edge $e = (s_1, v_1) \in M$. Let $e' = (s_2, v_1)$, one of the new edges formed by the join $\Gamma_1 + \Gamma_2$. Define a new matching M' by:

$$M' = (M - \{e\}) \cup (N \cup \{e'\}).$$

We now have that M' is a near-perfect matching in $\Gamma_1 + \Gamma_2$ that does not saturate s_1. Thus, by Theorem 2.8, Player 2 has a winning strategy. □

6.3. Make-A-Cycle for Graph Family $\mathbf{GP(2n, n)}$

In this section we examine winning strategies for the graphs $\mathrm{GP}(2n, n)$. Recall that the $\mathrm{GP}(2n, n)$ graphs were not originally defined to be generalized Petersen graphs. Indeed, this subclass of graphs is markedly different from the other $\mathrm{GP}(n, k)$ graphs. See Figure 17. This results in a notably different strategy for the Make-A-Cycle game in comparison to the strategy on other generalized Petersen graphs.

Like the other generalized Petersen graphs, the graphs $\mathrm{GP}(2n, n)$ have the same two general starting positions for our cycle games. That is, without loss of generality, we will consider the games to start at either u_0 or v_0.

Player 1 has a winning strategy for the Avoid-A-Cycle game on $\mathrm{GP}(2n, n)$ by Theorem 6.8. However, as stated above, the Make-A-Cycle game is different than when played on the $\mathrm{GP}(n, 2)$ generalized Petersen graphs. While winning strategies for $\mathtt{MAC}(\mathrm{GP}(n, 2))$ depended on the value of n modulo 5, we will see that the winning strategies for $\mathtt{MAC}(\mathrm{GP}(2n, n))$ depend on the value of n modulo 6.

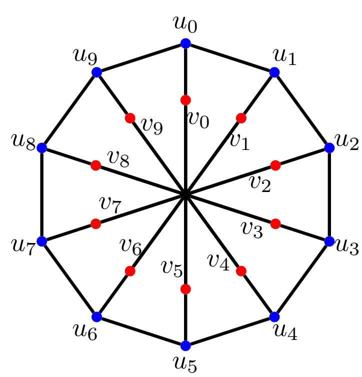

Figure 17: The graph $\mathrm{GP}(10, 5)$. Note that there is no vertex in the center.

Remark 6.13. Let A, B $\in \{1, 2\}$ and consider the game $\mathtt{MAC}(\mathrm{GP}(2n, n))$. Note that if Player A starts their turn at a vertex u_i and chooses the edge (u_i, v_i), then the next two moves are forced:

$$u_i \xrightarrow{A} v_i \xrightarrow{B} v_{n+i} \xrightarrow{A} u_{n+i}.$$

Hence, by choosing the edge (u_i, v_i), Player A has *effectively moved* from the vertex u_i to u_{n+i}. We use the notation $u_i \twoheadrightarrow u_{n+i}$ to denote an effective move for proofs in this section.

If a player effectively moves from a vertex u_i to u_{n+i}, then the subsequent move is often forced. We summarize when this occurs in the following lemma, which we refer to as the Bow Tie Lemma because the sequence of moves resembles a bow tie on the graph. See Figure 18.

Lemma 6.14 (Bow Tie Lemma). *Consider the game* $\mathtt{MAC}(\mathrm{GP}(2n, n))$ *and let* A, $B \in \{1, 2\}$. *For* $1 \leq i \leq 2n - 1$, *suppose Player B moves from* u_{i-1} *to* u_i *followed by Player A effectively moving from* u_i *to* u_{n+i}. *Assuming neither player can win on their next turn, then the next two moves are forced: Player B moves from* u_{n+i} *to* u_{n+i+1} *and Player A moves from* u_{n+i+1} *to* u_{n+i+2}.

Proof. Assuming the moves in the statement of the lemma, suppose Player B moves to u_{n+i-1}. Then Player A can effectively move to u_{i-1}, which has previously been visited by assumption. Thus Player B must move to u_{n+i+1}. See the left graph in Figure 18, where Player A moves are colored in blue and Player B in red.

If Player A effectively moves from u_{n+i+1} to u_{i+1}, then Player B moves to u_i to complete a cycle. Hence, Player A is forced to move from u_{n+i+1} to u_{n+i+2}. See the right graph in Figure 18. □

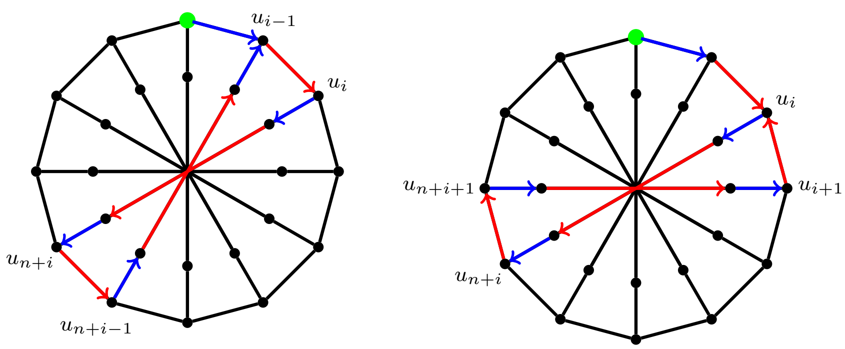

Figure 18: Examples of an application of Lemma 6.14, which we refer to as a *bow tie move*. Player A moves are blue and Player B red.

The following lemma relies heavily upon Lemma 6.14 and states that, given a set of preceding moves, a player can guarantee moving to a specific vertex in $\mathrm{GP}(2n, n)$. In particular, this vertex is six vertices farther along the outside of the graph in the same direction as these preceding moves. Because of this, we have named it the Forward Six Lemma, and it is this lemma that is the reason we will see winning strategies depend on n modulo 6.

Lemma 6.15 (Forward Six Lemma). *Consider the game* $\mathtt{MAC}(\mathrm{GP}(2n, n))$ *and let* k *be an integer with* $0 \leq k \leq 2n - 8$ *and let* A, $B \in \{1, 2\}$. *Suppose Player B moves from* u_{k-2} *to* u_{k-1} *and Player A moves* u_{k-1} *to* u_k. *Assuming vertices* u_i *with index* $i \in \{k+1, k+2, \ldots, k+6\} \cup \{n+k-1, n+k, n+k+1, \ldots, n+k+5\}$ *have not been previously visited, then Player A can guarantee reaching* u_{k+6} *by way of* u_{k+5} *with the previous move being Player B moving from* u_{k+4} *to* u_{k+5}.

Proof. To show that Player A can guarantee reaching u_{k+6}, we must show two cases depending on Player B's choices.

`Case 1:` Player B effectively moves from u_k to u_{n+k}. By Lemma 6.14, the next two moves are forced and hence Player B must move to u_{n+k+2}. Then Player A will choose to move to u_{n+k+3} and there are two subcases.

`Subcase 1a:` Player B moves to u_{n+k+4}. Then Player A will effectively move to u_{k+4}. By Lemma 6.14, Player A will move from u_{k+5} to u_{k+6} on their next turn. See the left image Figure 19.

`Subcase 1b:` Player B effectively moves to u_{k+3}. Then the next two moves are forced and Player B must move to u_{k+5}. Hence, Player A chooses to move from u_{k+5} to u_{k+6} as desired.

`Case 2:` Player B moves from u_k to u_{k+1}. Then Player A will choose to move to v_{k+1} and hence will effectively move to u_{n+k+1}. By Lemma 6.14, the next two moves are forced to be Player B moving from u_{n+k+1} to u_{n+k+2} followed by Player A moving to u_{n+k+3}. Now there are two cases exactly as above and, in both cases, Player A is guaranteed to move from u_{k+5} to u_{k+6}. See the right image of Figure 19 for an example of one of the cases. ☐

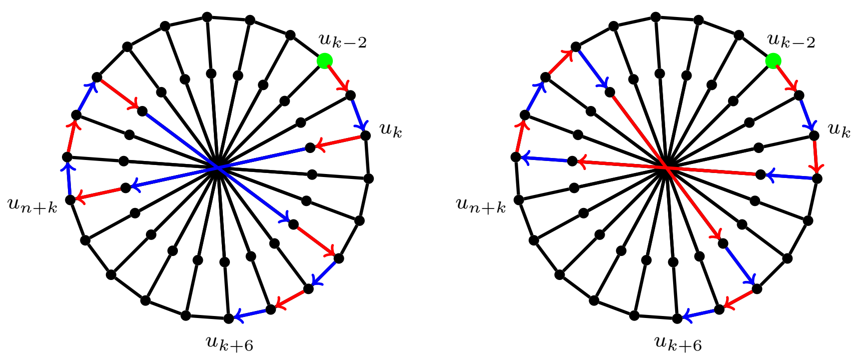

Figure 19: Examples of two of the cases in Lemma 6.15. Player A moves are blue and Player B red.

Theorem 6.16. *Player 1 has a winning strategy for* MAC(GP$(2n, n), u_0$) *if and only if* $n \not\equiv 1 \pmod 6$ *and* $n \not\equiv 3 \pmod 6$*. Player 1 has a winning strategy for* MAC(GP$(2n, n), v_0$) *if and only if* $n \not\equiv 3 \pmod 6$*.*

Proof. For the cases of $2 \leq n \leq 9$, one can verify by hand or with computer code (see Section 7) that the theorem statement holds. Thus, we suppose that $n \geq 10$ for the remainder of the proof. The key element here is Lemma 6.15, which allows a player to guarantee moving from a vertex u_i to u_{i+6}; hence the modulo 6 condition in the statement of the theorem.

First, we consider the starting vertex u_0. Suppose n is even. Player 1 first moves to u_1. We now consider two cases depending on Player 2's first move.

`Case 1:` Player 2 chooses to move from u_1 to u_2. Then Player 1 will choose to move from u_2 to u_3. Then by Lemma 6.15, Player 1 can guarantee moving from u_8 to u_9. By repeated applications of Lemma 6.15, Player 1 is able to guarantee moving to vertices of the form u_x where $x \equiv 3 \pmod 6$ and $x < n$.
`Subcase 1a:` $n \equiv 0 \pmod 6$. Then Player 1 guarantees first reaching u_{n-3}. If Player 2 moves to u_{n-2} then, by Table 1, Player 1 wins at u_0. Otherwise, by Table 2 with Player 1 as Player A, we see that Player 1 still wins, either at u_0 or back at u_{n-3}.
`Subcase 1b:` $n \equiv 2 \pmod 6$. Then Player 1 can guarantee moving from u_{n-6} to u_{n-5} (since $n-5 \equiv 3 \pmod 6$). If Player 2 moves to u_{n-4}, then Player 1 will move to u_{n-3} and by the previous case we get that Player 1 wins. Should Player 2 effectively move to u_{2n-5}, then by Lemma 6.14, the next two moves are forced so that Player 2 moves from u_{2n-4} to u_{2n-3}. Player 1 will choose to move to u_{2n-2}. If Player 2 moves to u_{2n-1}, then Player 1 will win at u_0. Otherwise, Player 2 effectively moves to u_{n-2} and we see by Table 1 that Player 1 wins at u_0.
`Subcase 1c:` $n \equiv 4 \pmod 6$. Then Player 1 can guarantee moving from u_{n-2} to u_{n-1}. Hence, by Table 1, Player 1 wins.
`Case 2:` Player 2 effectively moves from u_1 to u_{n+1}. Then by Lemma 6.14, Player 1 is forced to move to u_{n+2} and Player 2 is forced to move to u_{n+3}. Player 1 will then choose to move to u_{n+4}. By repeated applications of Lemma 6.15, Player 1 is able to guarantee moving to vertices of the form u_{n+x} where $x \equiv 4 \pmod 6$ (and $4 \leq x \leq n$).
`Subcase 2a:` $n \equiv 0 \pmod 6$. Then Player 1 guarantees moving from u_{2n-3} to u_{2n-2}. From this point, Player 1 wins since Player 2 either moves to u_{2n-1}, whereby Player 1 wins at u_0, or Player 2 effectively moves to u_{n-2}, whereby Player 1 wins at u_0 by the moves in Table 1.
`Subcase 2b:` $n \equiv 2 \pmod 6$. Then Player 1 guarantees moving from u_{2n-5} to u_{2n-4}. If Player 2 moves to u_{2n-3}, then Player 1 will move to u_{2n-2} and win as in the previous case. If Player 2 effectively moves to u_{n-4}, then, by Lemma 6.14, Player 1 will move to u_{n-3} and Player 2 to u_{n-2}. Hence, Player 1 wins by Table 1.
`Subcase 2c:` $n \equiv 4 \pmod 6$. Then since $n+4 \equiv 2 \pmod 6$ and $2n \equiv 2 \pmod 6$, Player 1 guarantees moving from u_{2n-7} to u_{2n-6}. Note that we cannot reapply Lemma 6.15, since $2n-6 > 2n-8$. However, by the proof of Lemma 6.15, Player 1, as Player A, can still guarantee reaching u_{2n-2}. From here, Player 2 can either move to u_{2n-1}, whereby Player 1 will win at u_0, or effectively move to u_{n-2}. In the latter case, Player 2 is forced to move to u_n by Lemma 6.14, and Player 1 then wins by effectively moving to u_0.

Now suppose that n is odd. If Player 1 moves to u_1, then Player 2 can move to u_2 and repeatedly apply Lemma 6.15 to win via a similar argument to the even

Player 2 moves to u_{n-2} by	Player 1 moves	Player 1 wins by
$u_{n-3} \xrightarrow{2} u_{n-2}$ or, $u_{2n-2} \overset{2}{\twoheadrightarrow} u_{n-2}$	$u_{n-2} \xrightarrow{1} u_{n-1}$	$u_{n-1} \xrightarrow{2} u_n \overset{1}{\twoheadrightarrow} u_0$ or, $u_{n-1} \overset{2}{\twoheadrightarrow} u_{2n-1} \xrightarrow{1} u_0$

Table 1: Moves for $\texttt{MAC}(\mathrm{GP}(2n, n), u_0)$, where n is even.

From u_{n-3} Player B moves	Player A moves	Player A wins by
$u_{n-3} \overset{B}{\twoheadrightarrow} u_{2n-3}$	$u_{2n-3} \xrightarrow{A} u_{2n-2}$	$u_{2n-2} \xrightarrow{B} u_{2n-1} \xrightarrow{A} u_0$ or, $u_{2n-2} \overset{B}{\twoheadrightarrow} u_{n-2} \xrightarrow{A} u_{n-3}$

Table 2: Moves used in the $n \equiv 0, 1, 3 \pmod 6$ cases for $\texttt{MAC}(\mathrm{GP}(2n, n), u_0)$.

case for Player 1. Thus we may assume Player 1 effectively moves to u_n instead. Without loss of generality, Player 2 moves to u_{n+1}. If Player 1 effectively moves to u_1, then Player 2 will win at u_0. Hence Player 1 will move to u_{n+2}. There are now three cases depending on n modulo 6.

`Case 1:` $n \equiv 1 \pmod 6$. Then Player 2 will effectively move from u_{n+2} to u_2. By Lemma 6.14, Player 1 moves to u_3 and Player 2 to u_4. By repeated applications of Lemma 6.15, Player 2 can guarantee moving from u_{n-4} to u_{n-3} since $n - 3 \equiv 4 \pmod 6$.

If Player 1 moves to u_{n-2}, then Player 2 will effectively move to u_{2n-2}. By Lemma 6.14, Player 1 is forced to u_{2n-1} and Player 2 wins at u_0. Otherwise, Player 1 effectively moves to u_{2n-3} from u_{n-3}. Using Table 2 with Player 2 as Player A, we see that Player 2 wins at u_0 or u_{n-3}.

`Case 2:` $n \equiv 3 \pmod 6$. Then Player 2 will move from u_{n+2} to u_{n+3}. Since $n + 3 \equiv 2n \equiv 0 \pmod 6$, repeated applications of Lemma 6.15 guarantee that Player 2 moves from u_{2n-7} to u_{2n-6}. Note that we cannot reapply Lemma 6.15, since $2n - 6 > 2n - 8$. Thus, we will consider Player 1's options from this point.

If Player 1 effectively moves to u_{n-6}, then, by Lemma 6.14, Player 2 moves to u_{n-5} and Player 1 to u_{n-4}. Player 2 will then move to u_{n-3}. On the other hand, if Player 1 moves from u_{2n-6} to u_{2n-5}, then Player 2 will effectively move to u_{n-5} and then guarantee moving from u_{n-4} to u_{n-3} by Lemma 6.14.

From u_{n-3}, Player 1 can move to u_{n-2} or effectively move to u_{2n-3}. In the former case, Player 2 will effectively move to u_{2n-2} and win at u_0 by Lemma 6.14. In the latter case, Player 2 wins by Table 2 with Player 2 as Player A.

`Case 3:` $n \equiv 5 \pmod 6$. Then by repeated applications of Lemma 6.15 from u_{n+2}, Player 1 guarantees moving from u_{2n-4} to u_{2n-3} since $n+2 \equiv 2n-3 \equiv 1 \pmod 6$. Player 2 can then move to u_{2n-2}, whereby Player 1 effectively moves to u_{n-2} and will win at u_n, or Player 2 can effectively move to u_{n-3}, whereby Player 1 moves to u_{n-2} and will win at u_n.

We now consider $\texttt{MAC}(\mathrm{GP}(2n, n), v_0)$ and show that Player 1 has a winning strategy unless $n \equiv 3 \pmod 6$. We examine each case of n modulo 6 separately.

`Case 1:` $n \equiv 0 \pmod 6$. Then Player 1 will begin by moving from v_0 to v_n and then Player 2 to u_n. Player 1 will then move to u_{n+1} where Player 2 has two choices: they can move to u_{n+2} or effectively move to u_1. If Player 2 moves to u_{n+2}, then Player 1 will move to u_{n+3}. By repeated applications of Lemma 6.15, Player 1 can guarantee moving from u_{2n-4} to u_{2n-3}. By Table 3 we see that Player 1 will win.

Suppose Player 2 effectively moves from u_{n+1} to u_1. Player 1 will move to u_2 and, by Lemma 6.14, Player 2 is forced to move to u_3. Player 1 will then move to u_4 so that repeated applications of Lemma 6.15 will guarantee them moving from u_{n-3} to u_{n-2}. Then Player 1 will win at u_n if Player 2 moves to u_{n-1}, or Player 1 will win at v_0 if Player 2 effectively moves to u_{2n-2} since Player 2 is forced to move to u_0 by Lemma 6.14.

Player 2 moves	Player 1 moves	Player 1 wins by
$u_{2n-3} \xrightarrow{2} u_{2n-2}$	$u_{2n-2} \xrightarrow{1} u_{2n-1}$	$u_{2n-1} \xrightarrow{2} u_0 \xrightarrow{1} v_0$ or, $u_{2n-1} \overset{2}{\twoheadrightarrow} u_{n-1} \xrightarrow{1} u_n$
$u_{2n-3} \overset{2}{\twoheadrightarrow} u_{n-3}$	$u_{n-3} \xrightarrow{1} u_{n-2}$	$u_{n-2} \xrightarrow{2} u_{n-1} \xrightarrow{1} u_n$ or, $u_{n-2} \overset{2}{\twoheadrightarrow} u_{2n-2} \xrightarrow{1} u_{2n-3}$

Table 3: Moves in the case of $n \equiv 0 \pmod 6$ for $\texttt{MAC}(\mathrm{GP}(2n, n), v_0)$.

`Case 2:` $n \equiv 1 \pmod 6$. Player 1 will move from v_0 to u_0 on their first turn. Without loss of generality, Player 2 will move to u_1. Player 1 will effectively move to u_{n+1}. From here, Player 2 can move to u_{n+2} or to u_n. If Player 2 moves to u_{n+2}, then Player 1 will move to u_{n+3} and by repeated applications of Lemma 6.15, Player 1 can guarantee moving from u_{2n-5} to u_{2n-4}. By Table 4, Player 1 wins.

Instead, if Player 2 moves from u_{n+1} to u_n, then Player 1 is forced to move to u_{n-1}, else Player 2 wins at v_0. Since u_0 has been previously visited, Player 2 cannot effectively move to u_{2n-1}. Hence, Player 2 moves to u_{n-2}. Player 1 will then move to u_{n-3} so that repeated applications of Lemma 6.15 (in reverse) will guarantee Player 1 moving from u_5 to u_4. If Player 2 effectively moves to u_{n+4} then Player 1 will win at u_{n+1} since Player 2 is forced to move to u_{n+2} by Lemma 6.14. If Player 2 moves to u_3, then Player 1 will effectively move to u_{n+3} so that, by Lemma 6.14, they win at u_{n+1}.

`Case 3:` $n \equiv 2 \pmod 6$. Similar to the previous case, Player 1 will start by moving to u_0 and, without loss of generality, we can assume Player 1 moves from u_1 to u_2

Player 2 moves	Player 1 moves	Player 1 wins by
$u_{2n-4} \xrightarrow{2} u_{2n-3}$	$u_{2n-3} \xrightarrow{1} u_{2n-2}$	$u_{2n-2} \xrightarrow{2} u_{2n-1} \xrightarrow{1} u_0$ or, $u_{2n-2} \overset{2}{\twoheadrightarrow} u_{n-2} \xrightarrow{1} u_{n-1},$ $u_{n-1} \xrightarrow{2} u_n \xrightarrow{1} u_{n+1}$
$u_{2n-4} \overset{2}{\twoheadrightarrow} u_{n-4}$	$u_{n-4} \xrightarrow{1} u_{n-3}$	$u_{n-3} \xrightarrow{2} u_{n-2} \xrightarrow{1} u_{n-1},$ $u_{n-1} \xrightarrow{2} u_n \xrightarrow{1} u_{n+1}$ or, $u_{n-3} \xrightarrow{2} u_{n-2} \xrightarrow{1} u_{n-1},$ $u_{n-1} \overset{2}{\twoheadrightarrow} u_{2n-1} \xrightarrow{1} u_0$

Table 4: Moves in the case where $n \equiv 1 \pmod 6$ for $\texttt{MAC}(\mathrm{GP}(2n, n), v_0)$.

on their next turn. By repeated applications of Lemma 6.15, Player 1 guarantees moving from u_{n-1} to u_n. Hence, Player 1 will win at v_0 if Player 2 moves to v_n, or Player 1 will win at u_1 if Player 2 moves to u_{n+1}.

`Case 4:` $n \equiv 3 \pmod 6$. This is the lone case that Player 2 wins. If Player 1 begins by moving to u_0, then Player 2 will move to u_1. Player 1 can then move to u_2 or to u_{n+1}. If Player 1 moves to u_2, then Player 2 will move to u_3 and then, by repeated applications of Lemma 6.15, guarantee moving from u_{n-1} to u_n. If Player 1 then moves to v_n, then Player 2 will win at v_0. If Player 1 instead moves to u_{n+1}, then Player 2 will win by effectively moving to u_1.

On the other hand, if Player 1 effectively moves from u_1 to u_{n+1}, then Player 2 will move to u_n. Player 1 must move to u_{n-1} since Player 2 wins at v_0 should Player 1 move to v_n. Player 2 will then move to u_{n-2}. Note that Lemma 6.15 can also be applied "in reverse," just like previous lemmas in this section (see Remark 6.5). Doing so here repeatedly, Player 2 can guarantee moving from u_2 to u_1 to win.

Now suppose Player 1 begins by moving to v_n and Player 2 follows by moving to u_n. Then, without loss of generality, Player 1 moves to u_{n+1}. Player 2 will then move to u_{n+2}, and, by repeated applications of Lemma 6.15, can guarantee moving from u_{2n-2} to u_{2n-1}. If Player 1 moves to $u_{2n} = u_0$, then Player 2 wins at v_0. If Player 1 effectively moves to u_{n-1}, then Player 2 wins at u_n.

`Case 5:` $n \equiv 4 \pmod 6$. Player 1 will begin by moving to v_n. Player 2 is forced to move to u_n, and Player 1 will then choose to move to u_{n+1}. Player 2 can then move to u_{n+2} or effectively move to u_1. If they move to u_{n+2}, Player 1 will move to u_{n+3}. Then, by repeated applications of Lemma 6.15, Player 1 guarantees moving from u_{2n-2} to u_{2n-1}. Then as in the previous case, Player 1 will win at v_0 or at u_n.

Suppose instead that Player 2 effectively moves from u_{n+1} to u_1. Then Player 1 will move to u_2 and Player 2 is forced to move to u_3 by Lemma 6.14. Player 1 will then move to u_4 and then, by repeated applications of Lemma 6.15, Player 1 can

guarantee moving from u_{n-1} to u_n to win.
`Case 6:` $n \equiv 5 \pmod 6$. Player 1 will begin by moving to u_0. Without loss of generality, assume Player 2 moves to u_1. Then Player 1 effectively moves from u_1 to u_{n+1}. Player 2 can then move to u_{n+2} or to u_n. If Player 2 moves to u_{n+2}, then Player 1 will move to u_{n+3}. By repeated applications of Lemma 6.15, Player 1 can guarantee moving from u_{2n-3} to u_{2n-2}. Player 2 is then forced to effectively move to u_{n-2}. Player 1 will then move to u_{n-1}. From here, if Player 2 moves to u_n, Player 1 wins by moving to u_{n+1} . If Player 2 effectively moves to u_{2n-1}, then Player 1 wins by moving to u_0.

On the other hand, if Player 2 moves from u_{n+1} to u_n, then Player 1 will move to u_{n-1}. By repeated applications of Lemma 6.15 in reverse, Player 1 can guarantee moving from u_5 to u_4 and hence will win as they did in the $n \equiv 1 \pmod 6$ case. □

7. Game Code

This work was greatly aided by the use of a computer program written by co-author Lillis. Some results were first conjectured after using a computer to determine which player should win a given game. Proofs were then able to be constructed once a pattern was observed. This approach is being applied in ongoing research such as determining winning strategies for the Make-A-Cycle game on the family of graphs $\mathrm{GP}(n, 3)$ (see Section 8). The code (written in C++) can be found at this github repository.

Here is a fun example of how we can use our game code to analyze a unique graph. Note that if a graph Γ has trivial automorphism group, then one necessarily needs to examine $\mathtt{MAC}(\Gamma, v)$ and $\mathtt{AAC}(\Gamma, v)$ for every vertex v. The *Frucht graph* is a 3-regular graph with a trivial automorphism group (see Figure 20) and twelve vertices. Hence, analyzing the Make-A-Cycle game by hand here is challenging. With the aid of the game code, we find that Player 2 wins `MAC` if and only if the starting vertex is the one colored red, as seen in Figure 20. For the Avoid-A-Cycle game, one can see that the Frucht graph has a perfect matching and therefore Player 1 has a winning strategy for every vertex.

8. Open Questions

This work has been a natural extension of the Relator Achievement and Relator Avoidance games defined in [7]. As was done in that paper, we close with some open questions.

1. In Theorem 2.8 we were able to characterize winning strategies for the `AAC` game due to its equivalence with the Undirected Geography Game. This result

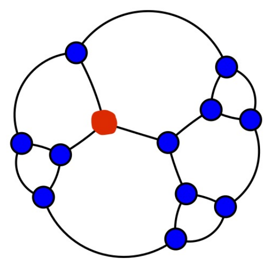

Figure 20: The Frucht Graph. Image Source: Wikicommons.

highlights how `AAC` is connected to the graph theory concept of maximum matchings. We ask: are there any properties of graphs that could provide a result akin to Theorem 2.8 for the Make-A-Cycle game? For instance, can anything be said within the class of vertex-transitive graphs as a starting point?

2. One can see that the Avoid-A-Cycle game is solved for cubic, bridgeless graphs by Petersen's Theorem which states that all such graphs have a perfect matching. Hence, we wonder: what can be determined in general for cubic, bridgeless graphs for the Make-A-Cycle game?

3. Jahangir graphs [9] are generalizations of wheel graphs. How do the results of Section 4 generalize? This question is already being considered by undergraduate students at The College of Wooster.

4. For the general case of Make-A-Cycle on stacked prism graphs $\mathrm{SP}(n,m)$, we have some extensions as stated previously in Remark 5.3 and Remark 5.5. For the general case of $\mathrm{SP}(n,m)$ with n even, we have results for only a few special cases, which were achieved using the following generalization of Lemma 5.8.

   Lemma 8.1. *For the game* `MAC`$(\mathrm{SP}(n,m))$*, with* $B \in \{1,2\}$*, suppose Player* B *moves from* $(i-1,0)$ *to* $(i,0)$ *such that no vertices* (p,q) *with* $i-1 < p < n-1$ *and* $0 \le q \le m-1$ *have been previously visited. Then Player* B *can guarantee moving from* $(i+m, m-1)$ *to* $(i+m+1, m-1)$.

   Can this lemma (or possibly other observations) yield a general statement regarding the Make-A-Cycle game for the $\mathrm{SP}(n,m)$ family of graphs?

5. In Section 6 we proved results for MAC on the family $\mathrm{GP}(n, 2)$. What can be said about graphs $\mathrm{GP}(n, k)$ for $k > 2$? The case of $k = 3$ is currently being examined by undergraduate students at The College of Wooster.

6. As we noted in Section 1 and Section 2, the AAC game is equivalent to undirected geography, which the authors of [5] call undirected vertex geography (to distinguish it from the edge variant they primarily discuss in their paper). In [5], it is noted that there exist polynomial-time algorithms for finding maximum matchings of any graph. Hence, by Theorem 2.8, determining which player wins an AAC game can be done in polynomial time. Now it is natural to ask: what is the computational complexity for the MAC game?

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A. Appendix

Below is the proof of Player 2's winning strategy for MAC(GP$(n,2)$) with $n \equiv 0 \pmod 5$ with starting vertex u_0 from Theorem 6.6.

Proof. We have two cases based off Player 1's first move: Player 1 moves from u_0 to u_1 and Player 1 moves from u_0 to v_0.

Case 1: Player 1 moves from u_0 to u_1. Player 2 will then move to v_1. Player 1 then has two choices: they can move to v_3 or to v_{n-1}.

Subcase 1a: Player 1 moves to v_3. Then we can apply Lemma 6.1 statement (1), with Player 2 as Player A and $i = 1$, guaranteeing the following moves:

$$\begin{aligned} u_j &\to v_j \text{ for } j \equiv 1 \pmod 5, \\ v_j &\to u_j \text{ for } j \equiv 3 \pmod 5, \\ u_j &\to u_{j+1} \text{ for } j \equiv 4 \pmod 5, \end{aligned}$$

for $1 \leq j < n$. In particular, Player 2 is guaranteed to move from v_{n-2} to u_{n-2} since $n-2 \equiv 3 \pmod 5$. From there Player 2 will win on their next move at either u_0 or u_{n-4}.

Subcase 1b: Player 1 moves instead to v_{n-1}. Then Player 2 will move to v_{n-3}. If $n = 5$, then Player 2 will win on their next move at either u_0 or u_1. If $n > 5$, then we must consider more carefully Player 1's next move. If Player 1 moves from v_{n-3} to u_{n-3}, then Player 2 will move to u_{n-2}.

Now, we wish to apply Lemma 6.3 statement (2) in reverse, with Player A as Player 2, but the index $n-3$ is too large for the hypotheses. However, the sequence of moves detailed in Lemma 6.3 statement (2) are still forced in this case. Specifically, Player 1 is forced to move to v_{n-2} since moving to u_{n-1} results in a loss on the following turn (at either v_{n-1} or u_0). Player 2 then moves to v_{n-4}, forcing Player 1 to move to v_{n-6}, or else Player 2 wins at u_{n-3}. Thus, Player 2 is able to move from v_{n-6} to v_{n-8}, as Lemma 6.3 statement (2) would guarantee were its hypotheses satisfied.

Note that, if $n = 10$, then Player 2 wins on their next turn, at either u_0 or u_1. Hence we suppose $n > 10$. If Player 1 continues to move from v_j to u_j where $j \equiv 2 \pmod 5$, then repeated applications of Lemma 6.3 statement (2) in reverse ensure Player 2 moves from v_{j-3} to v_{j-5}. Eventually, Player 2 will move from v_4 to v_2 (this being an application of Lemma 6.3 statement (2) with an index of 7) and will win on their next turn at either u_0 or u_1. Thus, Player 1 must move from v_j to v_{j-2} for some $7 < j \leq n-3$ and where $j \equiv 2 \pmod 5$. Player 2 will then move from v_{j-2} to u_{j-2}.

If Player 1 moves from u_{j-2} to u_{j-1}, then we can apply Lemma 6.3 statement (1) in reverse, with Player A as Player 2 and $i = j-2$, ensuring that Player 2 moves from v_{j-7} to u_{j-7}. If Player 1 continues to move from u_k to u_{k+1} with $k \equiv 0$

(mod 5) and $10 \le k \le j-2$, then repeated applications of Lemma 6.3 statement (1) will guarantee that Player 2 moves from v_{k-5} to u_{k-5}. In this case, Player 2 will eventually move from v_5 to u_5. Player 1, by our assumption, then moves to u_6. Note that we cannot technically apply Lemma 6.3 statement (1) in reverse because the vertices u_1 and v_1 have been previously visited (this is why the j above was strictly greater than 7). However, the same reasoning as given in the proof of Lemma 6.3 forces the same moves, specifically that Player 2 is guaranteed to move from v_4 to v_2 as one of the "preceding moves". From there Player 1 moves to v_0 or u_2, whereby Player 2 wins at u_0 or u_1, respectively.

To summarize, we have that Player 1 must move from v_j to v_{j-2} for some $7 < j \le n-3$ with $j \equiv 2 \pmod 5$ and following this, if they continue to make a move of the form u_k to u_{k+1} with $k \equiv 0 \pmod 5$ and $10 \le k \le j-2$, then Player 2 will win. Hence, it must be the case that Player 1 moves from u_k to u_{k-1} instead, for some $10 \le k \le j-2$. However, this means Player 2 can apply Lemma 6.1 statement (2) in reverse, where here Player A is again Player 2 and $i = k$, thus ensuring the moves

$$\begin{aligned}
&u_m \to u_{m-1} \text{ for } m \equiv 4 \pmod 5,\\
&u_m \to v_m \text{ for } m \equiv 2 \pmod 5,\\
&v_m \to u_m \text{ for } m \equiv 0 \pmod 5,
\end{aligned}$$

for all $5 \le m \le n-5$. Thus, Player 2 guarantees moving from v_5 to u_5. Player 1 must move to u_4, or else Player 2 wins at u_7. Player 2 then moves to u_3. From here, Player 1 can move to v_3 or u_2, whereby Player 2 will win at v_5 or u_1, respectively.
`Case 2:` Player 1 begins by moving from u_0 to v_0. We show, as in the proof of Theorem 6.6 given above, that Player 1 loses if they apply the strategy of Lemma 6.1 from this point, which implies that Player 1 must at some point deviate from this strategy. We then show that, given any deviation from the strategy by Player 1, Player 2 will have a winning a strategy.

If Player 1 begins by moving from u_0 to v_0 and Player 2 moves from v_0 to v_2, then Lemma 6.1 statement (1) applies with Player 1 as Player A. This guarantees Player 1 the following moves:

$$\begin{aligned}
&u_i \to v_i \text{ for } i \equiv 0 \pmod 5,\\
&v_i \to u_i \text{ for } i \equiv 2 \pmod 5,\\
&u_i \to u_{i+1} \text{ for } i \equiv 3 \pmod 5,
\end{aligned}$$

for all $0 \le i < n$. Should Player 1 execute this strategy, they will eventually move from u_{n-2} to u_{n-1} and lose when Player 2 moves to u_0. Thus, we may assume that Player 1 applies the strategy of Lemma 6.1 statement (1) until a certain point in the game and then deviates from the strategy. There are three cases, one for each of the move types listed above.

`Subcase 2a:` For some $0 < i \leq n-5$ with $i \equiv 0 \pmod 5$, Player 1 moves from u_i to u_{i+1} instead of u_i to v_i. Player 2 will move from u_{i+1} to v_{i+1} and Player 1 is forced to move from v_{i+1} to v_{i+3}; otherwise Player 2 will win at u_{i-1}. Now we apply Lemma 6.1 statement (1) with Player 2 as Player A (and where the starting index is $i+1$). This guarantees that Player 2 can move from u_j to v_j for $j \equiv 1 \pmod 5$ with $i+1 \leq j \leq n-4$. Once Player 2 moves from u_{n-4} to v_{n-4}, they win on their next move at either v_0 or u_{n-6}.

`Subcase 2b:` Player 1 moves from v_i to v_{i+2}, instead of to u_i, for some $2 \leq i \leq n-3$ where $i \equiv 2 \pmod 5$. Player 2 will then move from v_{i+2} to v_{i+4}. Note that, if $i = n-3$, then Player 2 wins on their next turn since, in this case $v_{i+4} = v_1$, and hence regardless of Player 1's next move, Player 2 can win at either u_0 or u_3.

Suppose then that $2 \leq i < n-8$ and Player 1 moves from v_{i+4} to u_{i+4}. Then Player 2 will move from u_{i+4} to u_{i+3}. Hence, Lemma 6.3 statement (2) can be applied, with Player A as Player 2 and starting parameter equal to $i+4$. This guarantees Player 2 moves from v_{i+7} to v_{i+9}, with the preceding move being v_{i+5} to v_{i+7}. Note that, if $i = n-8$, then we cannot technically apply Lemma 6.3 statement (2), because the vertices u_0 and v_0 have been previously visited. However, the reasoning given in the proof of Lemma 6.3 forces the same moves. Player 2 will still guarantee moving from v_{n-1} to v_1 and, just as above, will win on their next turn at either u_0 or u_3.

Now, if Player 1 continues to move from v_j to u_j for $j \equiv 1 \pmod 5$, then Lemma 6.3 statement (2) will guarantee Player 2 moves from v_{j+3} to v_{j+5}. Suppose Player 1 moves from v_j to u_j for all $j \equiv 1 \pmod 5$ with $j \geq i+4$. Then repeated applications of Lemma 6.3 statement (2) guarantee that Player 2 can move from v_{n-1} to v_1, where the last application with $j = n-4$ technically cannot be done but still forces the same moves similar to the case in the above paragraph. From v_1, Player 2 will win on their next turn at either u_0 or u_3.

Thus, Player 1 must move from v_j to v_{j+2} for some $i+4 \leq j \leq n-9$ where $j \equiv 1 \pmod 5$. Then Player 2 will move from v_{j+2} to u_{j+2}. If Player 1 moves from u_{j+2} to u_{j+1}, then we can apply Lemma 6.3 statement (1) with Player A as Player 2. Hence, if Player 1 continues to move from u_k to u_{k-1} for $k \geq j+2$ and $k \equiv 3 \pmod 5$, then repeated applications of Lemma 6.3 statement (1) guarantee that Player 2 can move from v_{k+5} to u_{k+5}. Eventually, Player 2 moves from v_{n-2} to u_{n-2}. By assumption, Player 1 will move to u_{n-3}. Then although the exact hypotheses of Lemma 6.3 statement (1) are not satisfied, the same moves are forced and Player 2 can guarantee moving from v_{n-1} to v_1. They will then win at either u_0 or u_3 on their next turn.

Since we have shown that Player 2 wins if Player 1 continues to move from u_k to u_{k-1}, Player 1 must move from u_k to u_{k+1} for some $j+2 \leq k \leq n-7$ where $k \equiv 3 \pmod 5$. Since the previous move by Player 2 was from v_k to u_k, we can apply Lemma 6.1 statement (2) with Player 2 as Player A. In particular, Player 2 guarantees moving from u_ℓ to v_ℓ for $\ell \equiv 1 \pmod 5$. Eventually Player 2 will move from u_{n-4} to v_{n-4}. From there Player 2 wins on their next turn at either v_0 or u_{n-6}.

Subcase 2c: Player 1 moves from u_i to v_i, instead of to u_{i+1}, for some $3 \leq i \leq n-2$ where $i \equiv 3 \pmod 5$. If $i = n-2$, then Player 2 will move to v_0 and win. If $i < n-2$ and $i \neq 3$, then Player 2 will move from v_i to v_{i-2}. Then, by our assumption of Player 1's prior strategy, Player 2 will win on their next turn regardless of Player 1's move. Should Player 1 move further back to v_{i-4}, then Player 2 will win at u_{i-4}. If Player 1 moves to u_{i-2}, then Player 2 wins at either u_{i-1} or u_{i-3}.

Now consider the remaining case that $i = 3$ and Player 1 moves from u_3 to v_3. Then Player 2 will move to v_1 and the next two moves are forced: Player 1 must move from v_1 to v_{n-1}, followed by Player 2 from v_{n-1} to v_{n-3}. If Player 1 moves to u_{n-3}, then we wish to apply Lemma 6.3 statement (2) in reverse, with Player A as Player 2, but the index $n-3$ is too large for the hypotheses. However, the sequence of moves detailed in Lemma 6.3 statement (2) are still forced in this case. Specifically, Player 2 moves from u_{n-3} to u_{n-2}, forcing Player 1 to move to v_{n-2} since moving to u_{n-1} results in a loss on the following turn because v_{n-1} has been previously visited. Player 2 then moves to v_{n-4}, forcing Player 1 to move to v_{n-6}, or else Player 2 wins at u_{n-3}. Thus, Player 2 is able to move from v_{n-6} to v_{n-8}, as Lemma 6.3 statement (2) would guarantee were its hypotheses satisfied.

If Player 1 continues to move from v_j to u_j, for $j \equiv 2 \pmod 5$ and $7 \leq j \leq n-3$, then Lemma 6.3 statement (2) in reverse will guarantee that Player 2 moves from v_{j-3} to v_{j-5}. Suppose Player 1 moves from v_j to u_j for all $j \equiv 2 \pmod 5$ with $7 \leq j \leq n-3$. Then repeated applications of Lemma 6.3 statement (2) in reverse ensure Player 2 moves from v_9 to v_7. By hypothesis, Player 1 will then move to u_7. As in the previous paragraph, we cannot apply Lemma 6.3 statement (2) in reverse here, because the vertices u_3 and v_3 have been previously visited. However, as in the last paragraph, the moves of Lemma 6.3 statement (2) are still forced. Hence, we can guarantee that Player 2 wins by moving from v_4 to v_2.

Now we must assume that Player 1 moves from v_k to v_{k-2} for some $7 \leq k \leq n-3$ with $k \equiv 2 \pmod 5$. Then Player 2 will move to u_{k-2}. If Player 1 moves from u_{k-2} to u_{k-1}, then we can apply Lemma 6.3 statement (1) in reverse and guarantee Player 2 moves from v_{k-7} to u_{k-7}. If Player 1 continues to move from u_ℓ to $u_{\ell+1}$ for all $5 \leq \ell \leq n-5$ with $\ell \equiv 0 \pmod 5$, then repeated applications of Lemma 6.3 statement (1) in reverse will guarantee Player 2 eventually moves from v_5 to u_5. Then, by assumption, Player 1 moves to u_6. Player 2 will move from u_6 to v_6 and then Player 1 is forced to move to v_4 (otherwise, Player 2 wins back at v_{10}). Thus, Player 2 wins by moving from v_4 to v_2.

Finally, we have exhausted Player 1's possible lines of play and they must move from u_ℓ to $u_{\ell-1}$, for some $5 \leq \ell \leq n-5$, with $\ell \equiv 0 \pmod 5$. If $\ell = 5$, then Player 2 wins by moving from u_4 to u_3. Otherwise, since Player 2's last move was from v_ℓ

to u_ℓ, we can apply Lemma 6.1 statement (2) in reverse with Player A as Player 2 and $i = \ell$ in the lemma. Thus, Player 2 is guaranteed to move from v_5 to u_5. Regardless of Player 1's next move, Player 2 will win at either u_7 or at u_3. $\square$