MAXIMUM NIM AND THE JOSEPHUS PROBLEM

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Abstract

This study examines the relation between the Grundy numbers of a Maximum Nim and the Josephus problem. Let $f(x) = \lfloor \frac{x}{k} \rfloor$, where $\lfloor \ \rfloor$ is a floor function and k is a positive integer such that $k \geq 2$. We prove that there is a simple relation between a Maximum Nim with the rule function f and the Josephus problem, in which every k-th number is to be removed from $(1, 2, 3, \dots, n)$ for some natural number n. Based on this relation, we propose a new method for solving the Josephus problem.

1. Introduction

Let $\mathbb{Z}_{\geq 0}$ and $\mathbb{N}$ represent sets of non-negative and positive integers, respectively. For any real number x, the floor of x denoted by $\lfloor x \rfloor$ represents the greatest integer less than or equal to x.

The classic game of Nim is played using stone piles. Players can remove any number of stones from any pile during their turn, and the player who removes the last stone is considered the winner. See [2]. Several variants of the classical Nim game exist. For the Maximum Nim, we place an upper bound $f(n)$ on the number

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of stones that can be removed in terms of the number n of stones in the pile (see [4]). For other articles on Maximum Nim, see [5] and [6].

This study explores the relation between a Maximum Nim and the Josephus problem. Let $f(x) = \left\lfloor \frac{x}{k} \right\rfloor$, where k is a positive integer such that $k \geq 2$. We prove that there is a simple relation between a Maximum Nim with the rule function f and the Josephus problem, in which every k-th number is to be removed. This is remarkable because the games for Nim and Josephus problems are considered entirely different.

2. Maximum Nim with the Rule Function $f(x) = \left\lfloor \frac{x}{k} \right\rfloor$ for a Positive Integer Such That $k \geq 2$ and the Josephus Problem

Let $k \in \mathbb{N}$ such that $k \geq 2$ and $f(x) = \left\lfloor \frac{x}{k} \right\rfloor$. We consider Maximum Nim as follows. Suppose there is a pile of n stones and two players take turns removing stones from the pile. At each turn, the player is allowed to remove at least one and at most $f(m)$ stones if the number of stones is m. The player who cannot make another move loses the game.

Definition 2.1. We denote the pile of m stones as (m), which we refer to as the *position* of the game.

We briefly review some necessary concepts in combinatorial game theory (see [1] for more details). We consider impartial games with no draws, and therefore, there are only two outcome classes.
(a) A position is called a $\mathcal{P}$-*position*, if it is a winning position for the previous player, as long as he/she plays correctly at every stage.
(b) A position is called an $\mathcal{N}$-*position*, if it is a winning position for the next player, as long as he/she plays correctly at every stage.

The Grundy number is one of the most important tools in research on combinatorial game theory, and we define it in the following definition.

Definition 2.2. (i) For any position (x), there exists a set of positions that can be reached in precisely one move in this game, which we denote as *move*(x).
(ii) The *minimum excluded value* (*mex*) of a set S of non-negative integers is the smallest non-negative integer that is not in S.
(iii) Let (x) be a position in the game. The associated *Grundy number* is denoted by $\mathcal{G}(x)$ and is recursively defined as follows: $\mathcal{G}(x) = mex(\{\mathcal{G}(u) : (u) \in move(x)\})$.

For the maximum Nim of x stones with rule function $f(x)$,
$move(x) = \{x - u : 1 \leq u \leq f(x) \text{ and } u \in \mathbb{N}\}$.

We assume that $\mathcal{G}$ is the Grundy number of the Maximum Nim with rule function $f(x) = \left\lfloor \frac{x}{k} \right\rfloor$. The next result demonstrates the usefulness of the Sprague–Grundy theorem for impartial games.

Theorem 1. *For any position (x), $\mathcal{G}(x) = 0$ if and only if (x) is the $\mathcal{P}$-position.*

See [1] for the proof of this theorem.

Lemma 1. *Let $\mathcal{G}$ represent the Grundy number of the maximum Nim with rule function $f(x)$. Then, we have the following properties:*
(i) If $f(x) > f(x-1)$, then $\mathcal{G}(x) = f(x)$.
(ii) If $f(x) = f(x-1)$, $\mathcal{G}(x) = \mathcal{G}(x - f(x) - 1)$.

These properties are proven in Lemma 2.1 of [4].

Lemma 2. *Let $x \in \mathbb{N}$. If x is not a multiple of k, then we have the following equation:*

$$x - \left\lfloor \frac{x}{k} \right\rfloor - 1 = \left\lfloor \frac{(k-1)x}{k} \right\rfloor.$$

Proof. Because x is not a multiple of k, there exist $t, u \in \mathbb{Z}_{\geq 0}$ such that $x = kt + u$ and $0 < u < k$. Therefore,

$$\begin{aligned} \left\lfloor \frac{(k-1)x}{k} \right\rfloor &= \left\lfloor \frac{kx - kt - u}{k} \right\rfloor \\ &= \left\lfloor x - t - \frac{u}{k} \right\rfloor \\ &= x - t - 1 \\ &= x - \left\lfloor \frac{x}{k} \right\rfloor - 1. \end{aligned}$$

□

We use the following function F in the remainder of this article.

Definition 2.3. Let

$$F(x) = \left\lfloor \frac{(k-1)x}{k} \right\rfloor$$

for $x \in \mathbb{Z}_{\geq 0}$.

Lemma 3. *For the Grundy number $\mathcal{G}$, the following hold:*
(i) $\mathcal{G}(x) = \frac{x}{k}$ for any positive integer x that is a multiple of k;
(ii) $\mathcal{G}(x) = \mathcal{G}(F(x))$ for any positive integer x that is not a multiple of k;
(iii)

$$\mathcal{G}(k - j) = 0$$

for $j = 1, 2, 3, \ldots, k$.

Proof. When a positive integer x is a multiple of k, $f(x) = \left\lfloor \frac{x}{k} \right\rfloor > \left\lfloor \frac{x-1}{k} \right\rfloor = f(x-1)$. Therefore, by (i) of Lemma 1, $\mathcal{G}(x) = f(x) = \left\lfloor \frac{x}{k} \right\rfloor = \frac{x}{k}$. When a positive integer x is not a multiple of k, $f(x) = \left\lfloor \frac{x}{k} \right\rfloor = \left\lfloor \frac{x-1}{k} \right\rfloor = f(x-1)$. Therefore, by Lemma 2 and (ii) of Lemma 1,

$$\begin{aligned}
\mathcal{G}(x) &= \mathcal{G}(x - f(x) - 1 \\
&= \mathcal{G}(x - \left\lfloor \frac{x}{k} \right\rfloor - 1) \\
&= \mathcal{G}(\left\lfloor \frac{(k-1)x}{k} \right\rfloor) \\
&= \mathcal{G}(F(x)).
\end{aligned}$$

(iii) For $i = 1, 2, 3, \ldots, k-1$, the number $k-i$ is not a multiple of k. Hence by (ii),

$$\begin{aligned}
\mathcal{G}(k-i) &= \mathcal{G}(F(k-i)) \\
&= \mathcal{G}(\left\lfloor \frac{(k-i)(k-1)}{k} \right\rfloor) \\
&= \mathcal{G}(k - (i+1) + \left\lfloor \frac{i}{k} \right\rfloor) \\
&= \mathcal{G}(k - (i+1)),
\end{aligned}$$

and hence we have

$$\mathcal{G}(k-1) = \mathcal{G}(k-2) = \cdots = \mathcal{G}(1) = \mathcal{G}(0) = 0.$$

□

We define the Josephus problem. For the details of the Josephus problem, see [3].

Definition 2.4. Let $n, k \in \mathbb{N}$. We have a finite sequence of positive integers $1, 2, 3, \cdots, n-1, n$ arranged in a circle, and we remove every k-th number until only one remains. This is the well-known Josephus problem. For m such that $1 \leq m \leq n$ and $1 \leq i \leq n-1$, we define

$$o_n(m) = i$$

where m is the i-th number to be removed, and

$$o_n(m) = n$$

where m is the number that remains after removing other $n-1$ numbers.

Lemma 4. *Suppose that $k = nt + h$ such that $n, t, h \in \mathbb{N}$ and $h \leq n$. Then, the following hold:*
(i) *If $s \in \mathbb{Z}_{\geq 0}$ and $s < t$, then $n(k-s) - i$ is not a multiple of k for any $i \in \mathbb{N}$ such*

that $1 \leq i \leq n$*;*
(ii) *For* $i \in \mathbb{N}$ *such that* $1 \leq i \leq n$ *and* $i \neq h$*, the number* $n(k-t)-i$ *is not a multiple of* k*.*

Proof. (i) For any $i \in \mathbb{N}$ and $s \in \mathbb{Z}_{\geq 0}$ such that $s < t$ and $i \leq n$, we have

$$n(k-s)-i = nk-(ns+i). \tag{1}$$

Since $s < t$ and $h, i \geq 1$, we have $0 < ns+i \leq nt < nt+h = k$. Hence, $nk-(ns+i)$ is not a multiple of k. Therefore, by Equation (1), $h(k-s)$ is not a multiple of k.
(ii) For any $i \in \mathbb{N}$ such that $1 \leq i \leq n$, we have

$$n(k-t)-i = nk-(nt+h)-(i-h) = nk-k-(i-h). \tag{2}$$

As $1 \leq i, h \leq n < k$, the number $nk-k-(i-h)$ is not a multiple of k for $i \neq h$. Therefore, by Equation (2), $n(k-t)-i$ is not a multiple of k. □

Lemma 5. *Let* F *be the function of Definition 2.3.*
(i) *Suppose that* $n = kt+u$ *for* $t \in \mathbb{N}$ *and* $u \in \mathbb{Z}_{\geq 0}$ *such that* $u \leq k-1$*. For* $i \in \mathbb{N}$ *such that* $1 \leq i \leq n$ *and* i *is not a multiple of* k*, we have*

$$\mathcal{G}(nk-i) = \mathcal{G}((n-t)k - \left(u+i-\left\lfloor \frac{i}{k} \right\rfloor\right)).$$

(ii) *Suppose that* $k = nt+h$ *such that* $n, t, h \in \mathbb{N}$ *and* $h \leq n$*.*
$(ii.1)$ *For any* $i \in \mathbb{N}$ *such that* $1 \leq i \leq n$ *we have*

$$\mathcal{G}(nk-i) = \mathcal{G}(F(nk-i)) = \mathcal{G}(n(k-1)-i). \tag{3}$$

$(ii.2)$ *For any* $i \in \mathbb{N}$ *such that* $1 \leq i \leq n$*, we have*

$$\mathcal{G}(nk-i) = \mathcal{G}(F^t(nk-i)) = \mathcal{G}(n(k-t)-i), \tag{4}$$

where F^t *is the* t*-th functional power of* F*.*
$(ii.3)$ *For* $i \in \mathbb{N}$ *such that* $i \leq h-1$*, we have*

$$\mathcal{G}(nk-i) = \mathcal{G}(F^{t+1}(nk-i)) = \mathcal{G}((n-1)k-(n-h+i)). \tag{5}$$

Proof. (i) Since i is not a multiple of k, by (ii) of Lemma 3,

$$\begin{aligned}
\mathcal{G}(nk-i) &= \mathcal{G}\left(\left\lfloor \frac{(nk-i)(k-1)}{k} \right\rfloor\right) \\
&= \mathcal{G}\left(nk-i-n+\left\lfloor \frac{i}{k} \right\rfloor\right) \\
&= \mathcal{G}\left(nk-i-(kt+u)+\left\lfloor \frac{i}{k} \right\rfloor\right) \\
&= \mathcal{G}((n-t)k - \left(u+i-\left\lfloor \frac{i}{k} \right\rfloor\right)).
\end{aligned}$$

(ii.1) By (i) of Lemma 4, $nk-i$ is not a multiple of k. Hence, by (ii) of Lemma 3,

$$\begin{aligned}\mathcal{G}(nk-i) &= \mathcal{G}(F(nk-i))\\ &= \mathcal{G}(\left\lfloor \frac{(nk-i)(k-1)}{k} \right\rfloor)\\ &= \mathcal{G}(\left\lfloor nk-i-\frac{nk-i}{k} \right\rfloor)\\ &= \mathcal{G}(nk-i-n+\left\lfloor \frac{i}{k} \right\rfloor)\\ &= \mathcal{G}(nk-i-n)\\ &= \mathcal{G}(n(k-1)-i).\end{aligned}$$

(ii.2) We prove this via mathematical induction. By Equation (3), Equation (4) is valid for $t=1$. For $v \in \mathbb{N}$ such that $1 \leq v < t$, we assume that

$$\mathcal{G}(nk-i) = \mathcal{G}(F^v(nk-i)) = \mathcal{G}(n(k-v)-i). \tag{6}$$

Here, we have

$$nv+i \leq nt < nt+h = k. \tag{7}$$

By (i) of Lemma 4, $n(k-v)-i$ is not a multiple of k. Hence, by (ii) of Lemma 3 and Inequality (7),

$$\begin{aligned}\mathcal{G}(n(k-v)-i) &= \mathcal{G}(F(n(k-v)-i))\\ &= \mathcal{G}(\left\lfloor \frac{n(k-v)(k-1)-i(k-1)}{k} \right\rfloor)\\ &= \mathcal{G}(\left\lfloor \frac{nk^2-nkv-nk-ik+nv+i}{k} \right\rfloor)\\ &= \mathcal{G}(nk-nv-n-i+\left\lfloor \frac{nv+i}{k} \right\rfloor)\\ &= \mathcal{G}(nk-nv-n-i)\\ &= \mathcal{G}(n(k-(v+1))-i).\end{aligned} \tag{8}$$

Therefore, by Equations (6) and (8) we obtain Equation (4).
(ii.3) Since $i \leq h-1$, we have

$$nt+i < nt+h = k. \tag{9}$$

By $(ii.2)$ we have

$$\mathcal{G}(nk-i) = \mathcal{G}(F^t(nk-i)) = \mathcal{G}(n(k-t)-i). \tag{10}$$

Since $i \leq h-1$, by (ii) of Lemma 4, $n(k-t)-i$ is not a multiple of k. Hence by (ii) of Lemma 3 and Inequality (9), we have

$$\begin{aligned}
\mathcal{G}(n(k-t)-i) &= \mathcal{G}(F(n(k-t)-i)) \\
&= \mathcal{G}(\left\lfloor \frac{(n(k-t)-i)(k-1)}{k} \right\rfloor) \\
&= \mathcal{G}(\left\lfloor \frac{(n(k-t)-i)k-(n(k-t)-i)}{k} \right\rfloor) \\
&= \mathcal{G}(\left\lfloor \frac{(n(k-t)-i-n)k+(nt+i)}{k} \right\rfloor) \\
&= \mathcal{G}(n(k-t-1)-i+\left\lfloor \frac{nt+i}{k} \right\rfloor) \\
&= \mathcal{G}(n(k-t-1)-i) \\
&= \mathcal{G}(nk-nt-n-i) \\
&= \mathcal{G}(nk-(k-h)-n-i) \\
&= \mathcal{G}((n-1)k-(n-h+i)). \qquad (11)
\end{aligned}$$

Therefore, by Equations (10) and (11), we have Equation (5). □

The following theorem presents a simple relation between the Maximum Nim and the Josephus problem.

Theorem 2. *For any $n, m \in \mathbb{N}$ such that $1 \leq m \leq n$, we have*

$$n - o_n(m) = \mathcal{G}(nk-m). \qquad (12)$$

Proof. We prove this utilizing mathematical induction. If $n = 1$, then $m = 1$. Hence, by Definition 2.4 $o_n(m) = o_1(1) = 1$. Thus, $\mathcal{G}(nk-m) = \mathcal{G}(k-1)$. By (iii) of Lemma 3, $\mathcal{G}(k-1) = 0$. Hence, the base case of the induction is established. Therefore, we assume that $n \geq 2$ and

$$n' - o_n(m) = \mathcal{G}(n'k-m) \qquad (13)$$

for $n', m \in \mathbb{N}$ such that $n' < n$ and $1 \leq m \leq n'$. We will separately consider the cases when $n \geq k$ and $n < k$.

[I] We assume that $n = kt+u$ for some $t \in \mathbb{N}$ and $u \in \mathbb{Z}_{\geq 0}$ with $u \leq k-1$. Since we remove every k-th number, $o_n(ik) = i$ for $1 \leq i \leq t$. Hence,

$$n - o_n(ik) = n - i. \qquad (14)$$

By (ii) of Lemma 3,

$$n - i = \mathcal{G}((n-i)k) = \mathcal{G}(nk-ik). \qquad (15)$$

By Equations (14) and (15), Equation (12) holds for multiples of k that are less than n.

Let A denote the remaining numbers $\{i \in \mathbb{N} : i \leq n \text{ and } i \text{ is not a multiple of } k\}$, and consider the sequence $(a_i)_{i=1}^{n-t}$ of remaining numbers in the order starting with $kt+1$, which equals $(kt+1, \ldots, kt+u, 1, \ldots, k-1, k+1, \ldots, 2k-1, 2k+1, \ldots, kt-1)$. Then, we have

$$(kt+1, kt+2, \ldots, kt+u) = (a_1, a_2, \ldots, a_u), \tag{16}$$

$$(1, 2, \ldots, k-1) = (a_{u+1}, a_{u+2}, \ldots, a_{u+k-1}), \tag{17}$$

$$(k+1, k+2, \ldots, 2k-1) = (a_{u+k}, a_{u+k+1}, \ldots, a_{u+2k-2}), \tag{18}$$

$$\vdots$$

$$((l-1)k+1, \ldots, lk-1) = (a_{u+(l-1)k-(l-2)}, \ldots, a_{u+lk-l}), \tag{19}$$

$$\vdots$$

$$((t-1)k+1, \ldots, tk-1) = (a_{u+(t-1)k-(t-2)}, \ldots, a_{u+tk-t}), \tag{20}$$

where $3 \leq l < t$. By Equation (16), for $kt+1 \leq i \leq kt+u$, we have

$$i = a_{i-kt}.$$

For $1 \leq i \leq k-1$, we have $\left\lfloor \frac{i}{k} \right\rfloor = 0$, and hence by Equation (17),

$$i = a_{u+i} = a_{u+i-\left\lfloor \frac{i}{k} \right\rfloor}. \tag{21}$$

For $k+1 \leq i \leq 2k-1$, we have $\left\lfloor \frac{i}{k} \right\rfloor = 1$, and hence by Equation (18),

$$i = a_{u+i-1} = a_{u+i-\left\lfloor \frac{i}{k} \right\rfloor}. \tag{22}$$

Let $l \in \mathbb{N}$ such that $3 \leq l < t$. For $(l-1)k+1 \leq i \leq lk-1$, we have $\left\lfloor \frac{i}{k} \right\rfloor = l-1$, and hence by Equation (19),

$$i = a_{u+i-(l-1)} = a_{u+i-\left\lfloor \frac{i}{k} \right\rfloor}. \tag{23}$$

Finally, for $(t-1)k+1 \leq i \leq tk-1$, we have $\left\lfloor \frac{i}{k} \right\rfloor = t-1$, and hence by Equation (20),

$$i = a_{u+i-(t-1)} = a_{u+i-\left\lfloor \frac{i}{k} \right\rfloor}. \tag{24}$$

Therefore, by Equations (21), (22), (23), and (24), for $1 \leq i \leq tk-1$,we have

$$i = a_{u+i-\left\lfloor \frac{i}{k} \right\rfloor}.$$

This defines a map $\alpha : A \to \{1, \ldots, n-t\}$ so that $\alpha(i) = i - kt$ if $i > kt$ and $\alpha(i) = u + i - \left\lfloor \frac{i}{k} \right\rfloor$ otherwise. Since the first t removals were multiples of k, so

$$o_n(i) = t + o_{n-t}(\alpha(i)). \tag{25}$$

By the induction hypothesis in Equation (13),

$$(n-t)-o_{n-t}(j)=\mathcal{G}((n-t)k-j) \tag{26}$$

for $1 \leq j \leq n-t$. For i such that $1 \leq i \leq u$, by Equations (25) and (26),

$$\begin{aligned}\mathcal{G}(nk-(kt+i)) &= \mathcal{G}((n-t)k-i)\\ &= (n-t)-o_{n-t}(i)\\ &= (n-t)-o_{n-t}(\alpha(kt+i))\\ &= n-(t+o_{n-t}(\alpha(kt+i)))\\ &= n-o_n(kt+i).\end{aligned}$$

Hence, the result holds for $kt+1,\ldots,kt+u=n$.

For i such that $1 \leq i \leq kt-1$ that are not multiples of k, by (i) of Lemma 5 and the definition of the map α,

$$\begin{aligned}\mathcal{G}(nk-i) &= \mathcal{G}((n-t)k-\left(u+i-\left\lfloor\frac{i}{k}\right\rfloor\right))\\ &= \mathcal{G}((n-t)k-\alpha(i)).\end{aligned} \tag{27}$$

By Equation (26),

$$\begin{aligned}\mathcal{G}((n-t)k-\alpha(i)) &= (n-t)-o_{n-t}(\alpha(i))\\ &= n-(t+o_{n-t}(\alpha(i)))\\ &= n-o_n(i).\end{aligned} \tag{28}$$

Thus, by Equations (27) and (28), Equation (12) holds if $k \leq n$.

$[II]$ We now assume that $k > n$ so that $k = nt+h$ for some $t \in \mathbb{N}$ and $h \in \mathbb{N}$ with $h \leq n$. After skipping all of the numbers t times, the first number that is removed is h. Then

$$n-o_n(h)=n-1=\mathcal{G}((n-1)k)=\mathcal{G}(nk-k)=\mathcal{G}(nk-(nt+h))=\mathcal{G}(n(k-t)-h). \tag{29}$$

By $(ii.2)$ of Lemma 5,

$$\mathcal{G}(n(k-t)-h)=\mathcal{G}(F^t(nk-h))=\mathcal{G}(nk-h). \tag{30}$$

By Equations (29) and (30), we have Equation (12) for $m=h$.

We now prove that the result holds for $i \neq h$. We use a method similar to the one in $[I]$. We will use the induction hypothesis on $n-1$ remaining numbers.

$[II.1]$ Suppose that $h \leq n-1$. Let B denote the remaining numbers $\{i \in \mathbb{N} : i \leq n$ and $i \neq h\}$, and consider the sequence $(b_i)_{i=1}^{n-1}$ of remaining numbers in order but starting with $h+1$, which equals $(h+1,h+2,\ldots,n,1,\ldots,h-1)$. Then, we have

$$\begin{aligned}(h+1,h+2,\ldots,n) &= (b_1,b_2,\ldots,b_{n-h}),\\ (1,2,\ldots,h-1) &= (b_{n-h+1},\ldots,b_{n-1}).\end{aligned}$$

This defines a map $\beta : B \to \{1, \ldots, n-1\}$ so that $\beta(i) = i - h$ if $i > h$ and $\beta(i) = n - h + i$ otherwise. Since the first number to be removed is h, it follows that

$$o_n(i) = 1 + o_{n-1}(\beta(i)). \tag{31}$$

For i such that $1 \leq i \leq n-h$, we have $h+1 \leq h+i \leq n$. Hence, by $(ii.2)$ of Lemma 5,

$$\mathcal{G}(nk - (h+i)) = \mathcal{G}(F^t(nk - (h+i))) = \mathcal{G}(n(k-t) - (h+i)). \tag{32}$$

By the definition of the map β,

$$\begin{aligned}
\mathcal{G}(n(k-t) - (h+i)) &= \mathcal{G}(nk - nt - h - i) \\
&= \mathcal{G}(nk - (nt+h) - i) \\
&= \mathcal{G}(nk - k - i) \\
&= \mathcal{G}((n-1)k - i) \\
&= \mathcal{G}((n-1)k - \beta(h+i)),
\end{aligned} \tag{33}$$

and by the induction hypothesis and Equation (31),

$$\begin{aligned}
\mathcal{G}((n-1)k - \beta(h+i)) &= (n-1) - o_{n-1}(\beta(h+i)) \\
&= n - (1 + o_{n-1}(\beta(h+i)) \\
&= n - o_n(h+i).
\end{aligned} \tag{34}$$

By Equations (32), (33) and (34), Equation (12) holds for $m = h+1, h+2, \ldots, n$.

For i such that $1 \leq i \leq h-1$, by $(ii.3)$ of Lemma 5 and the definition of the map β,

$$\begin{aligned}
\mathcal{G}(nk - i) &= \mathcal{G}(F^{t+1}(nk - i)) \\
&= \mathcal{G}((n-1)k - (n-h+i)) \\
&= \mathcal{G}((n-1)k - \beta(i)),
\end{aligned} \tag{35}$$

and by the induction hypothesis and Equation (31),

$$\begin{aligned}
\mathcal{G}((n-1)k - \beta(i)) &= (n-1) - o_{n-1}(\beta(i)) \\
&= n - (1 + o_{n-1}(\beta(i))) \\
&= n - o_n(i).
\end{aligned} \tag{36}$$

Therefore, by Equations (35) and (36), Equation (12) holds for i such that $1 \leq i \leq h-1$.

$[II.2]$ Suppose that $h = n$. Let

$$(1, 2, \ldots, h-1) = (b_1, \ldots, b_{n-1}).$$

We define $\beta(i) = i$ for $i = 1, 2, \ldots, n-1$. Since the first number to be removed is n, it follows that

$$o_n(i) = 1 + o_{n-1}(\beta(i)). \tag{37}$$

For i such that $1 \leq i \leq n-1$, by (ii.3) of Lemma 5 and the definition of the map β,

$$\begin{aligned}
\mathcal{G}(nk - i) &= \mathcal{G}(F^{t+1}(nk - i)) \\
&= \mathcal{G}((n-1)k - (n - h + i)) \\
&= \mathcal{G}((n-1)k - (n - n + i)) \\
&= \mathcal{G}((n-1)k - i) \\
&= \mathcal{G}((n-1)k - \beta(i)),
\end{aligned} \tag{38}$$

and by the induction hypothesis and Equation (37),

$$\begin{aligned}
\mathcal{G}((n-1)k - \beta(i)) &= (n-1) - o_{n-1}(\beta(i)) \\
&= n - (1 + o_{n-1}(\beta(i))) \\
&= n - o_n(i).
\end{aligned} \tag{39}$$

Therefore, by Equations (38) and (39), Equation (12) holds for i such that $1 \leq i \leq n-1$. □

3. Maximum Nim and the Josephus Problem Algorithm

In this section, we study an algorithm for the Josephus problem based on Theorem 2.

Definition 3.1. For $k \in \mathbb{N}$ such that $k \geq 2$, we define a function $h_k(x)$ for a non-negative integer x as

$$h_k(x) = x + \left\lfloor \frac{x}{k-1} \right\rfloor + 1.$$

Lemma 6. *For the function $h_k(x)$ in Definition 3.1, the following hold:*
(i) $h_k(x) < h_k(x')$ for any $x, x' \in \mathbb{Z}_{\geq 0}$ such that $x < x'$;
(ii) $x < x + 1 \leq h_k(x)$ for any $x \in \mathbb{Z}_{\geq 0}$;
(iii) $\lim_{p \to \infty} h_k^p(x_0) = \infty$, where h_k^p is the p-th functional power of h_k.

We obtain (i), (ii) and (iii) directly from Definition 3.1.

Proposition 1. *For h_k in Definition 3.1, the following hold:*
(i) $h_k(x) = \left\lfloor \frac{h_k(x)}{k} \right\rfloor + x + 1$ for $x \in \mathbb{Z}_{\geq 0}$;

(ii) $\mathcal{G}(h_k(x)) = \mathcal{G}(x)$ *for* $x \in \mathbb{Z}_{\geq 0}$*;*
(iii) For any $m \in \mathbb{Z}_{\geq 0}$*, there exists* $x_0 \in \mathbb{Z}_{\geq 0}$ *such that*

$$\{x \in \mathbb{Z}_{\geq 0} : \mathcal{G}(x) = m\} = \{h_k^p(x_0) : p \in \mathbb{Z}_{\geq 0}\}.$$

Proof. Let $x \in \mathbb{Z}_{\geq 0}$ and

$$x = (k-1)t + s \tag{40}$$

for $t, s \in \mathbb{Z}_{\geq 0}$ such that $s < k-1$. Then, we have

$$\left\lfloor \frac{x}{k-1} \right\rfloor = t. \tag{41}$$

(i) Based on Definition 3.1 and Equation (41), we have

$$h_k(x) = x + t + 1, \tag{42}$$

and hence, by Equation (40), we obtain

$$\frac{h_k(x)}{k} = \frac{x+t+1}{k} = \frac{(k-1)t+s+t+1}{k} = t + \frac{s+1}{k} \tag{43}$$

and

$$\frac{h_k(x)-1}{k} = t + \frac{s}{k}. \tag{44}$$

As $0 \leq s < s+1 < k$, based on Equations (43) and (44),

$$\left\lfloor \frac{h_k(x)-1}{k} \right\rfloor = \left\lfloor \frac{h_k(x)}{k} \right\rfloor = t. \tag{45}$$

According to Equations (42) and (45),

$$h_k(x) - \left\lfloor \frac{h_k(x)}{k} \right\rfloor = x + t + 1 - t = x + 1, \tag{46}$$

and hence, we have (i) of this proposition.
(ii) Based on Equation (45),

$$f(h_k(x) - 1) = f(h_k(x)),$$

and hence, by (ii) of Lemma 1,

$$\mathcal{G}(h_k(x)) = \mathcal{G}(h_k(x) - f(h_k(x)) - 1) = \mathcal{G}(h_k(x) - \left\lfloor \frac{h_k(x)}{k} \right\rfloor - 1), \tag{47}$$

and by Equation (46),

$$\mathcal{G}(h_k(x) - \left\lfloor \frac{h_k(x)}{k} \right\rfloor - 1) = \mathcal{G}(x + 1 - 1) = \mathcal{G}(x). \tag{48}$$

Then, based on Equations (47) and (48), we have (ii) of this proposition.
(iii) Let $m \in \mathbb{Z}_{\geq 0}$. By (i) of Lemma 3, $\mathcal{G}(mk) = m$. Hence, the set $\{x \in \mathbb{Z}_{\geq 0} : \mathcal{G}(x) = m\}$ is not empty. Therefore, there is the smallest $x \in \mathbb{Z}_{\geq 0}$ such that $\mathcal{G}(x) = m$. We denote this number by x_0. By (ii), $\{h_k^p(x_0) : p \in \mathbb{Z}_{\geq 0}\} \subset \{x \in \mathbb{Z}_{\geq 0} : \mathcal{G}(x) = m\}$, where $h_k^0(x_0) = x_0$.

For any $x' \notin \{h_k^p(x_0) : p \in \mathbb{Z}_{\geq 0}\}$ such that $x' > x_0$, by (ii) and (iii) of Lemma 6,

$$h_k^p(x_0) < x' < h_k^{p+1}(x_0) \tag{49}$$

for some $p \in \mathbb{Z}_{\geq 0}$. From (i),

$$h_k^{p+1}(x_0) = h_k(h_k^p(x_0)) = \left\lfloor \frac{h_k(h_k^p(x_0))}{k} \right\rfloor + h_k^p(x_0) + 1,$$

and hence, by the definition of *move* in Definition 2.2,

$$\begin{aligned} move(h_k^{p+1}(x_0)) &= \{h_k^{p+1}(x_0) - u : 1 \leq u \leq f(h_k^{p+1}(x_0)) \text{ and } u \in \mathbb{N}\} \\ &= \{h_k^{p+1}(x_0) - 1, h_k^{p+1}(x_0) - 2, \cdots, h_k^{p+1}(x_0) - \left\lfloor \frac{h_k^{p+1}(x_0)}{k} \right\rfloor\} \\ &= \{h_k^{p+1}(x_0) - 1, h_k^{p+1}(x_0) - 2, \cdots, h_k^p(x_0) + 1\}. \end{aligned} \tag{50}$$

From Inequality (49) and Equation (50), $x' \in move(h_k^{p+1}(x_0))$. Hence, from the definition of the Grundy number in Definition 2.2, $\mathcal{G}(x') \neq \mathcal{G}(h_k^{p+1}(x_0)) = m$. Thus, we have $\{h_k^p(x_0) : p \in \mathbb{Z}_{\geq 0}\} = \{x : \mathcal{G}(x) = m\}$. □

Theorem 3. *Let n be a natural number. Then, there exists $p \in \mathbb{N}$ such that $h_k^{p-1}(0) < n(k-1) \leq h_k^p(0)$, and the last number that remains is $nk - h_k^p(0)$ in the Josephus problem of n numbers, where every k-th number is removed.*

Proof. Let m be the number that remains in the Josephus problem of n numbers, where the k-th number is removed. Then, based on Definition 2.4 and Theorem 2,

$$\mathcal{G}(nk - m) = 0. \tag{51}$$

Since $h_k^0(0) = 0$, by (ii) and (iii) of Lemma 6, there exists $p \in \mathbb{N}$ such that

$$h_k^{p-1}(0) < n(k-1) \leq h_k^p(0). \tag{52}$$

Since $\mathcal{G}(0) = 0$, by Proposition 1,

$$\mathcal{G}(h_k^{p-1}(0)) = \mathcal{G}(h_k^p(0)) = \mathcal{G}(h_k^{p+1}(0)) = \mathcal{G}(0) = 0.$$

As

$$h_k(n(k-1)) = n(k-1) + n + 1 = nk + 1,$$

by Inequality (52) and (i) of Lemma 6,

$$\begin{aligned} h_k^{p+1}(0) &= h_k(h_k^p(0)) \\ &\geq h_k(n(k-1)) \\ &= nk+1. \end{aligned}$$

Hence, by Inequality (52),

$$h_k^{p-1}(0) < n(k-1) \leq nk-m < nk+1 \leq h_k^{p+1}(0). \tag{53}$$

By (ii) of Lemma 6,

$$h_k^{p-1}(0) < h_k^p(0) < h_k^{p+1}(0),$$

and hence by Equation (51), Inequality (53) and (iii) of Proposition 1, $nk - m = h_k^p(0)$. Hence, $m = nk - h_k^p(0)$. □

The following corollary presents a more efficient way to calculate the last number that remains in the Josephus problem.

Corollary 1. *Let n be a natural number. Then, there exists $p \in \mathbb{N}$ such that $h_k^{p-1}(k-1) < n(k-1) \leq h_k^p(k-1)$, and the last number that remains is $nk - h_k^p(k-1)$ in the Josephus problem of n numbers, where every k-th number is removed.*

Proof. Since $h_k^0(k-1) = k-1 < n(k-1)$, by (ii) and (iii) of Lemma 6, there exists $p \in \mathbb{N}$ such that

$$h_k^{p-1}(k-1) < n(k-1) \leq h_k^p(k-1). \tag{54}$$

As $h_k^{k-1}(0) = k-1$, by Equation (54),

$$h_k^{p-1+k-1}(0) < n(k-1) \leq h_k^{p+k-1}(0).$$

Hence, based on Theorem 3, the last number that remains is

$$nk - h_k^{p+k-1}(0) = nk - h_k^p(k-1).$$

□

Definition 3.2. We define two functions $h_k^-(x)$ and $h_k^+(x)$ for a non-negative integer x as

$$h_k^-(x) = x + \frac{x}{k-1}$$

and

$$h_k^+(x) = x + \frac{x}{k-1} + 1.$$

Lemma 7. *For functions $h_k^+(x)$ and $h_k^-(x)$ in Definition 3.2, the following hold:*
(i) $(h_k^-)^p(x) = (\frac{k}{k-1})^p x$;
(ii) $(h_k^+)^p(x) = (\frac{k}{k-1})^p(x+k-1) - (k-1)$;
(iii) $(h_k^-)(x) \leq (h_k)(x) \leq (h_k^+)(x)$.

Proof. (i) In Definition 3.2, $h_k^-(x) = \frac{k}{k-1}x$. Hence, we have (i).
(ii) In Definition 3.2, $h_k^+(x) = \frac{k}{k-1}x + 1$. Hence,

$$h_k^+(x) + k - 1 = \frac{k}{k-1}(x + (k-1)). \tag{55}$$

By using Equation (55) p times, we have (ii).
(iii) This is directly obtained from Definitions 3.1 and 3.2. □

Lemma 8. *For $n \in \mathbb{N}$, let*

$$v = \frac{\log n}{\log k - \log(k-1)} \tag{56}$$

and

$$w = \frac{\log(n+1) - \log 2}{\log k - \log(k-1)}. \tag{57}$$

Then,

$$h_k^{\lfloor w \rfloor}(k-1) \leq n(k-1) \leq h_k^{\lceil v \rceil}(k-1). \tag{58}$$

Proof. Let

$$(\frac{k}{k-1})^v(k-1) = n(k-1)$$

for a positive real number v. Then, we have Equation (56), and by (i) of Lemma 7,

$$n(k-1) \leq (h_k^-)^{\lceil v \rceil}(k-1), \tag{59}$$

where $\lceil \quad \rceil$ is the ceiling function. Let

$$(\frac{k}{k-1})^w(k-1+k-1) - (k-1) = n(k-1)$$

for a positive real number w. Then,

$$(h_k^+)^{\lfloor w \rfloor}(k-1) + (k-1) \leq (\frac{k}{k-1})^w(k-1+k-1) = (n+1)(k-1),$$

which motivates the definition in Equation (57). Then, by (ii) of Lemma 7,

$$(h_k^+)^{\lfloor w \rfloor}(k-1) \leq n(k-1), \tag{60}$$

and based on the inequalities in (59), (60), and (iii) of Lemma 7,

$$h_k^{\lfloor w \rfloor}(k-1) \leq (h_k^+)^{\lfloor w \rfloor}(k-1) \leq n(k-1) \leq (h_k^-)^{\lceil v \rceil}(k-1) \leq h_k^{\lceil v \rceil}(k-1).$$

Therefore, we have Inequality (58). □

Theorem 4. *Let*

$$v = \frac{\log n}{\log k - \log(k-1)}$$

and

$$w = \frac{\log(n+1) - \log 2}{\log k - \log(k-1)}.$$

Then, there exists $p \in \mathbb{N}$ such that $\lfloor w \rfloor \leq p \leq \lceil v \rceil$ and $nk - h_k^p(k-1)$ is the last number that remains in the Josephus problem of n numbers, where every k-th number is removed.

Proof. By Corollary 1 and Lemma 8, the last number that remains belongs to the set $\{nk - h_k^p(k-1) : \lfloor w \rfloor \leq p \leq \lceil v \rceil\}$ in the Josephus problem of n numbers, where every k-th number is removed. □

Corollary 1 and Theorem 4 present a new way to calculate the last number that remains in the Josephus problem.

4. Prospect for Future Research

Theorem 2 presents a new way to calculate the number that remains in the Josephus problem; thus, the authors attempt to build a good algorithm for the Josephus problem.

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