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NONEXISTENCE OF CONSECUTIVE POWERFUL TRIPLETS AROUND CUBES WITH PRIME-SQUARE FACTORS

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Abstract

The Erdős–Mollin–Walsh conjecture, asserting the nonexistence of three consecutive powerful integers, remains a celebrated open problem in number theory. A natural line of inquiry, following recent work by Chan (2025), is to investigate potential counterexamples centered around perfect cubes, which are themselves powerful. This paper establishes a new non-existence result for a family of such integer triplets with distinct structural constraints, combining techniques from modular arithmetic, p-adic valuation, Thue equations, and the theory of elliptic curves.

1. Introduction

A positive integer is called a *powerful number* if each of its prime factors appears with an exponent of at least two. Every powerful number n admits a *unique* representation as

$$n = a^2 b^3, \tag{1}$$

where $a, b \in \mathbb{Z}$ and b is square-free (meaning that it is not divisible by any perfect square other than 1) (see [5] for example). The Erdős–Mollin–Walsh conjecture [3, 9] asserts that no three consecutive integers are all powerful.

A natural and interesting case arises when the triplet is centered at a perfect cube x^3, itself always powerful. That is, does the triplet $(x^3 - 1, x^3, x^3 + 1)$ ever consist entirely of powerful numbers? Recently, Chan [2] resolved a subcase by proving no such triplets exist when

$$x^3 - 1 = p^3 y^2, \quad x^3 + 1 = q^3 z^2, \tag{2}$$

for primes p, q and integers $x, y, z > 0$.

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In this paper, we prove the non-existence of a new family of triplets in this setting, extending Chan's result while addressing distinct constraints. Our main result is as follows.

Theorem 1. *There exist no consecutive powerful numbers of the form*

$$x^3 - 1 = p^2\, a^3, \qquad x^3, \qquad x^3 + 1 = q^2\, b^3.$$

where p, q are primes and a, b, x are integers.

Notice that the powers in Theorem 1 differ from those in (2). Furthermore, unlike [2], we do not need to impose the restriction that variables x, a, and b are positive. This result establishes the non-existence of a class of consecutive powerful triplets not covered by previous literature.

Corollary 1. *For any primes p, q and any integers x, a with $a \neq 0$, the equation $x^6 - 1 = p^2 q^2 a^3$ has no solution.*

2. Proof of the Main Result

First, let us introduce some lemmas.

Lemma 1. *Let p be a prime and R, S, C be integers that satisfy*

$$R\, S = p^2\, C^3,$$

and set $g = \gcd(R, S)$. If $g = 1$, then one of R, S is a perfect cube and the other is p^2 times a perfect cube. If g is a prime, then there exist $C_1, C_2 \in \mathbb{Z}$ such that either $(R, S) = (gC_1^3, gC_2^3)$, $\{R, S\} = \{gC_1^3, g^2p^2C_2^3\}$, or $\{R, S\} = \{gp^2C_1^3, g^2C_2^3\}$.

Proof. For $g = 1$, if $p \nmid C$, any prime factor $r \neq p$ of C divides exactly one of R, S, and if $p \mid C$, then all p factors in p^2C^3 must be contained entirely within either R or C, but not both. Assume g is a prime. If $g = p$, then both R/p and S/p are perfect cubes. Otherwise, the two factors can be written as gC_1^3 and $g^2p^2C_2^3$, or $gp^2C_1^3$ and $g^2C_2^3$, up to ordering. The conclusion follows. □

Lemma 2. *The Diophantine equation $u^2 + u + 1 = 3v^3$ has exactly two integer solutions $(u, v) \in \{(-2, 1), (1, 1)\}$. The Diophantine equation $u^2 - u + 1 = 3v^3$ has exactly two integer solutions $(u, v) \in \{(2, 1), (-1, 1)\}$.*

Proof. The core of the proof is to convert the original equations into the standard form of a Mordell curve through proper transformations. Taking the second equation as an example, we first multiply the equation by 3^2 and substitute $u' = 3u$ and $v' = 3v$, which yields $(u')^2 - 3u' + 9 = (v')^3$. Multiplying by 4 and completing the square on the left side allows for the substitution $u'' = 2u' - 3$ and $v'' = v'$, leading to $(u'')^2 + 27 = 4(v'')^3$. Finally, multiplying both sides by 2^4 and letting $x = 4v''$ and $y = 4u''$ reduces the equation to the canonical Mordell form: $y^2 = x^3 - 432$. The integer solutions to this well-known curve are $(x, y) = (12, \pm 36)$ (see [4] for example). Working backwards gives the two solutions for (u, v). The other equation can be handled by substituting $u \mapsto -u$. □

Lemma 3. *The integer solutions for u to the equation $u^2 + u + 1 = v^3$ are given by $u = -19, -1, 0, 18$. Similarly, the integer solutions for u to $u^2 - u + 1 = v^3$ are $-18, 0, 1, 19$.*

Proof. The first equation can be handled using the transformations and Mordell curve techniques in Lemma 2, or directly by citing the corollary of [10]. The second equation then follows by the substitution $u \mapsto -u$. □

Lemma 4. *For any integer x, we have $\gcd\big(x - 1, x^2 + x + 1\big) = \gcd(x - 1, 3)$ and $\gcd\big(x + 1, x^2 - x + 1\big) = \gcd(x + 1, 3)$.*

Proof. Writing $x^2 + x + 1 = (x - 1)(x + 2) + 3$ and $x^2 - x + 1 = (x + 1)(x - 2) + 3$, the Euclidean algorithm on polynomials yields the conclusion. □

Lemma 5. *The Diophantine equation $u^3 - v^3 = 2$ has the unique integer pair solution $(1, -1)$, and the Diophantine equation $u^3 - v^3 = 1$ has the integer solutions $(1, 0)$ and $(0, -1)$.*

Proof. We may assume u and v are both nonzero. Consider $u^3 - v^3 = 2$. Since $u > v$, both factors $u - v$ and $u^2 + uv + v^2$ are positive integers. Therefore, we have two possibilities:

$$u - v = 2, u^2 + uv + v^2 = 1,$$

or

$$u - v = 1, u^2 + uv + v^2 = 2.$$

Simple calculation yields the unique solution $(1, -1)$. The other equation can be treated similarly. □

We are now prepared to establish the main result. Suppose for contradiction that

$$x^3 - 1 = p^2\, a^3, \quad x^3 + 1 = q^2\, b^3, \tag{3}$$

with integers x, a, b and primes p, q. Set $g_- = \gcd(x - 1, x^2 + x + 1)$ and $g_+ = \gcd(x + 1, x^2 - x + 1)$. By Lemma 4,

$$g_- = \begin{cases} 3 & x \equiv 1 \pmod 3, \\ 1 & \text{otherwise}, \end{cases} \qquad g_+ = \begin{cases} 3 & x \equiv 2 \pmod 3, \\ 1 & \text{otherwise}. \end{cases}$$

We proceed by casework since the pair (g_-, g_+) can be $(1,1)$, $(1,3)$, or $(3,1)$ only.

`Case 1:` $(g_-, g_+) = (1,1)$, $x \equiv 0 \pmod 3$. By Lemma 1, we have the following possibilities

$$\begin{array}{ll} \text{(i)} & x-1 = u^3, x^2+x+1 = p^2v^3, \\ \text{(ii)} & x-1 = p^2u^3, x^2+x+1 = v^3, \end{array} \quad \text{and} \quad \begin{array}{ll} \text{(a)} & x+1 = s^3, x^2-x+1 = q^2t^3, \\ \text{(b)} & x+1 = q^2s^3, x^2-x+1 = t^3, \end{array}$$

where u, v, s, and t are integers. We explore each subcase below.

- $(i)+(a)$. We obtain $x-1 = u^3, x+1 = s^3$, or $s^3 - u^3 = 2$. By Lemma 5, $s = 1$ and $u = -1$. But then $x = 0$ and there exists no prime p satisfying (3).
- $(i)+(b)$. Applying Lemma 3 to $x^2-x+1 = t^3$ yields $x = -18, 0, 1, 19$. However, none of these values satisfies $x+1 = q^2s^3$ under the given constraints (for example, taking $x = 19$ leads to $q^2s^3 = 20$ and thus $q = 2$, but no integer s exists).
- $(ii)+(a)$. This subcase can be treated similarly by applying Lemma 3 to the equation $x^2+x+1 = v^3$.
- $(ii)+(b)$. In this subcase, x^2-x+1 and x^2+x+1 are perfect cubes. Lemma 3 forces $x = 0$, but then no prime q exists satisfying (3).

`Case 2:` $(g_-, g_+) = (1,3)$, $x \equiv 2 \pmod 3$. Let $v_p(n)$ denote the p-adic valuation of n.

First, applying the Lifting-the-Exponent lemma (see, e.g., Theorem 1.37 of [7]) to $v_3(x^3+1)$ gives $v_3(x+1)+1 = v_3(x^3+1) = v_3(x+1) + v_3(x^2-x+1)$, or

$$v_3(x^2-x+1) = 1. \tag{4}$$

We split into two subcases based on q.

- $q = 3$: For $r \neq 3$ as an arbitrary prime factor of x^2-x+1, since $g_+ = 3$, we have

  $$v_r(x^2-x+1) = v_r((x+1)(x^2-x+1)) = v_r(9b^3) = 3v_r(b).$$

  Combined with (4), x^2-x+1 can be written as $3\prod_{i=1}^{n} r_i^{3\alpha_i}$, or $3s^3$ for some $s \in \mathbb{Z}$. Applying Lemma 2 gives $x = -1$ or 2. Yet neither yields a valid solution for (3) with prime p.

- $q \neq 3$: Here, we have $v_3(x^3+1) = v_3(q^2b^3) = 3v_3(b)$. Combined with (4), $v_3(x^3+1) \geq 3$, from which it follows that

  $$v_3(x+1) = v_3((x^3-1)/(x^2-x+1)) \geq 3-1 = 2,$$

and thus

$$x \equiv -1 \pmod 9. \tag{5}$$

Next, applying Lemma 1 to $(x-1)(x^2+x+1) = p^2a^3$, we have two possibilities: (i) $x - 1 = u^3$, $x^2 + x + 1 = p^2v^3$, and (ii) $x - 1 = p^2u^3$, $x^2 + x + 1 = v^3$. Combining (i) and (5) yields $u^3 \equiv -2 \pmod 9$, which is impossible. For (ii), (5) and Lemma 3 force $x = -19$ or -1. Neither yields a valid solution for (3) with prime p.

`Case 3:` $(g_-, g_+) = (3, 1)$, $x \equiv 1 \pmod 3$. This case is analogous to the previous one, with x replaced by $-x$. The result follows from a combination of p-adic analysis and Lemmas 1, 2, and 3.

3. Proof of the Corollary

The proof of Corollary 1 combines the arguments for our main theorem with that of Corollary 1 of [2]. We begin with a few preparatory lemmas.

Lemma 6. *If* $3 \mid x-1$*, then* $v_3(x^2+x+1) = 1$*; if* $3 \mid x+1$*, then* $v_3(x^2-x+1) = 1$.

Proof. The argument is analogous to the one used in the main theorem's proof. □

Lemma 7. *The only integer solutions of the Diophantine equation* $u^3 - 2v^3 = 1$ *are* $(1, 0)$ *and* $(-1, -1)$.

Proof. Both $(1, 0)$ and $(-1, -1)$ satisfy $u^3 - 2v^3 = 1$. By the Delone–Nagell theorem (see, e.g., [1, Theorem V, §72]), there is at most one solution in addition to $(1, 0)$. The conclusion follows. □

Although the following result is likely known, we were unable to locate a specific reference for these parameters. For the sake of completeness, we provide a brief, self-contained proof based on Lemma 2.

Lemma 8. *For* $d \in \{4, 18, 36\}$*, the Diophantine equation* $u^3 - dv^3 = 1$ *has the unique integer solution* $(1, 0)$.

Proof. First, for $d = 4$, we consider the equation $u^3 - 4v^3 = 1$. Reducing this modulo 9 implies that $u^3 \equiv 1 \pmod 9$ and $v^3 \equiv 0 \pmod 9$, and so $u \equiv 1 \pmod 3$ and $v \equiv 0 \pmod 3$. Let $v = 3v_0$ for some integer v_0. The equation becomes

$$(u-1)(u^2+u+1) = 4 \cdot 3^3 \cdot v_0^3.$$

Lemma 6 gives $v_3(u^2+u+1)=1$, from which it follows that $\gcd(u-1,u^2+u+1)=3$. We can therefore set $u-1=3a$ and $u^2+u+1=3b$ with $\gcd(a,b)=1$, and obtain $ab=12v_0^3$. Since u^2+u+1 is odd, b must also be odd. As $v_3(3b)=1$, $3\nmid b$. It follows that $b=s^3$, and $u^2+u+1=3s^3$. By Lemma 2, $u=1,-2$. Noting u must be odd, the conclusion follows.

Similarly, for $d=18$, the equation $u^3-18v^3=1$ taken modulo 9 gives $u^3\equiv 1 \pmod 9$, which implies $u\equiv 1 \pmod 3$. By Lemma 6, $v_3(u^2+u+1)=1$, and thus $\gcd(u-1,u^2+u+1)=3$. Setting $u-1=3a$ and $u^2+u+1=3b$ with $\gcd(a,b)=1$ results in $ab=2v^3$. Since b must be odd and is coprime to a, we have $b=s^3$, which yields $u^2+u+1=3s^3$. The unique integer solution $(1,0)$ again follows by Lemma 2. The case for $d=36$ follows from an identical argument, applying a modulo 9 reduction along with Lemma 6 and Lemma 2. □

We now prove Corollary 1 by contradiction. The equation in the corollary can be written as

$$\big(x^3-1\big)\big(x^3+1\big)=p^2q^2a^3. \tag{6}$$

Since $\gcd(x^3-1,x^3+1)\mid x^3+1-(x^3-1)=2$, we have $\gcd(x^3-1,x^3+1)=1$ or 2.

For $\gcd(x^3-1,x^3+1)=1$, there are two possibilities for the factors on the left-hand side of (6). Assume p^2q^2 divides one of the factors. This implies that the other factor must be a perfect cube. If $x^3+1=a_1^3$ or $x^3-1=a_1^3$, then x has no valid solution by Lemma 5. Therefore, p^2 divides one factor and q^2 divides the other; without loss of generality, assume $x^3-1=p^2a_1^3$ and $x^3+1=q^2a_2^3$. But this contradicts our main theorem.

Next, consider $\gcd(x^3-1,x^3+1)=2$. If one of the primes is 2, say $p=2$, then we have two possible systems of equations:

$$\begin{cases} x^3-1=2q^2u^3\\ x^3+1=2v^3, \end{cases} \quad \text{or} \quad \begin{cases} x^3-1=2u^3\\ x^3+1=2q^2v^3. \end{cases} \tag{7}$$

By Lemma 7, $x=\pm 1$, which violates $a\neq 0$.

For the remainder of the proof, assume $p\neq 2$, $q\neq 2$, and (6) becomes $(x^3-1)(x^3+1)=8p^2q^2b^3$. One possibility is that p^2q^2 divides one of the factors on the left. This leads to four subcases, each of which yields a contradiction by applying either Lemma 7 or Lemma 8:

- $x^3-1=2p^2q^2u^3, x^3+1=4v^3$. By Lemma 8, $x=-1$, violating $a\neq 0$.
- $x^3-1=4p^2q^2u^3, x^3+1=2v^3$. By Lemma 7, $x=\pm 1$, violating $a\neq 0$.
- $x^3-1=2u^3, x^3+1=4p^2q^2v^3$. By Lemma 7, $x=\pm 1$, violating $a\neq 0$.
- $x^3-1=4u^3, x^3+1=2p^2q^2v^3$. By Lemma 8, $x=1$, violating $a\neq 0$.

Therefore, p^2 must divide one of x^3-1 and x^3+1, and q^2 must divide the other.

It remains to study the system

$$x^3-1=2p^2u^3, \quad x^3+1=4q^2v^3. \tag{8}$$

Indeed, when the factors of 2 and 4 are swapped, $x^3-1=4p^2u^3$ and $x^3+1=2q^2v^3$ reduce to the form in (8) under the change of variables $x\mapsto -x, u\mapsto -v, v\mapsto -u, p\mapsto q$, and $q\mapsto p$. The equations in (8) resemble those in (3), but are distinct: for example, x^3-1 here is not necessarily a powerful number. Nevertheless, the same proof strategy used in the previous section can be applied.

Recall the definitions of g_- and g_+ in the proof of the main theorem. We proceed in a similar manner.

`Case 1:` $(g_-,g_+)=(1,1)$. The factorization of the first equation in (8) yields four possibilities: (i) $x-1=2t_1^3, x^2+x+1=p^2t_2^3$, (ii) $x-1=2p^2t_1^3, x^2+x+1=t_2^3$, (iii) $x-1=t_1^3, x^2+x+1=2p^2t_2^3$, and (iv) $x-1=p^2t_1^3, x^2+x+1=2t_2^3$, and similarly, the second equation gives: (a) $x+1=4t_3^3, x^2-x+1=q^2t_4^3$, (b) $x+1=4q^2t_3^3, x^2-x+1=t_4^3$, (c) $x+1=t_3^3, x^2-x+1=4q^2t_4^3$, and (d) $x+1=q^2t_3^3, x^2-x+1=4t_4^3$, where t_i are integers.

Since $x^2\pm x+1=x(x\pm1)+1$ is always odd, (iii), (iv), (c), and (d) are impossible. Among the remaining combinations, those involving (ii) or (b) can be excluded by Lemma 3. For example, under (i) + (b), we have x odd, and Lemma 3 forces $x\in\{1,19\}$; $x=1$ gives $a=0$ (a contradiction), while $x=19$ yields $x+1=20$, so no prime q exists satisfying (8).

Thus only (i) + (a) remains. In this case, $2t_3^3-t_1^3=1$. By Lemma 7, $t_1=\pm1$, hence $x\in\{3,-1\}$, but neither value produces a prime p satisfying (8).

`Case 2:` $(g_-,g_+)=(3,1)$. If $p=3$, Lemma 8 implies $x=1$, violating $a\neq 0$. Assume $p\neq 3$. The case condition requires $x\equiv 1 \pmod 3$, and applying Lemma 6 gives $v_3(x^2+x+1)=1$. Combining this with the parity argument from Case 1, it suffices to analyze the factorizations: (i) $x-1=18t_1^3, x^2+x+1=3p^2t_2^3$, or (ii) $x-1=18p^2t_1^3, x^2+x+1=3t_2^3$ for the first equation in (8), and (a) $x+1=4t_3^3, x^2-x+1=q^2t_4^3$, or (b) $x+1=4q^2t_3^3, x^2-x+1=t_4^3$ for the second equation. By a similar line of reasoning, applying Lemma 2 and Lemma 3 eliminates all but one nontrivial combination, (i) + (a), from which it follows that $2t_3^3-9t_1^3=1$. Reducing the equation modulo 9 gives $2t_3^3\equiv 1 \pmod 9$, a contradiction.

`Case 3:` $(g_-,g_+)=(1,3)$. If $q=3$, Lemma 8 implies $x=-1$, violating $a\neq 0$. Assume $q\neq 3$. Applying Lemma 6 gives $v_3(x^2-x+1)=1$. Combining the 3-adic valuation and the parity argument as in Case 2, the first equation in (8) leads to two possibilities: (i) $x-1=2t_1^3, x^2+x+1=p^2t_2^3$, or (ii) $x-1=2p^2t_1^3, x^2+x+1=t_2^3$, and the second equation leads to two possibilities: (a) $x+1=36t_3^3, x^2-x+1=3q^2t_4^3$, or (b) $x+1=36q^2t_3^3, x^2-x+1=3t_4^3$. Applying Lemma 2 and Lemma 3 leaves only one nontrivial case, $x-1=2t_1^3, x^2+x+1=p^2t_2^3, x+1=36t_3^3$, and $x^2-x+1=3qt_4^3$. This combination implies $18t_3^3-t_1^3=1$. Then Lemma 8 gives $t_1=-1$, and thus

$x = -1$, violating $a \neq 0$.

The proof is complete.

4. Conclusion

We have proved that no three consecutive integers centered at a perfect cube can all be powerful under the structural constraints studied here. This result extends recent advances on the Erdős–Mollin–Walsh conjecture by eliminating a notable family of potential counterexamples through modular and elliptic curve methods.

One natural extension is to examine whether similar non-existence holds when centered around higher powers or other special integers, and what further constraints might eliminate all consecutive powerful numbers entirely. A related and more fundamental question that arose during our research is the following: we conjecture that for every integer $x > 1$ and every integer $n > 2$, the number $x^n - 1$ is not powerful. This is a stronger claim than Mihăilescu's Theorem (formerly Catalan's Conjecture) [8]. A proof of this general assertion would have significant implications; the specific case for $n = 3$ would resolve the question addressed in Theorem 1 and provide deeper insight into the structure of powerful numbers.

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