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DECOMPOSITION OF BEATTY AND COMPLEMENTARY SEQUENCES

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Abstract

In this paper, we express the difference of two complementary Beatty sequences as the sum of two other closely related Beatty sequences. In the process, we introduce a new algorithm that generalizes the well-known Minimum Excluded algorithm and provides a method to combinatorially generate any pair of complementary Beatty sequences in a natural way.

1. Introduction

Wythoff [31] proved the formula

$$\left[n\phi^2\right] - [n\phi] = n, \tag{1}$$

where ϕ is the golden ratio. Here, $[x]$ denotes the largest integer not exceeding x. Later, other authors (see, for instance, [15] and [19]) used the formula

$$\left[n\alpha^2\right] - [n\beta] = tn,$$

with $\alpha = \frac{2-t+\sqrt{t^2+4}}{2}$, and β satisfying $\alpha^{-1} + \beta^{-1} = 1$, to generalize Wythoff's game. Other generalizations have been made covering Beatty sequences parameterized by limited families of irrational numbers. We obtain a generalization of Identity (1) for all complementary sequences and give a combinatorial interpretation of the corresponding quantity on the right-hand side. For example,

$$\left[n\frac{3+\sqrt{3}}{2}\right] - \left[n\sqrt{3}\right] = \left[n\frac{\sqrt{3}-1}{2}\right] + \left[n(2-\sqrt{3})\right] + 1.$$

In particular, we prove Theorems 2, 3 and 4, which together imply that Equation (1) generalizes to any pair of complementary Beatty sequences A and B with irrational slopes $\beta > 2$ and α as follows:

$$[n\beta] - [n\alpha] = [n(\beta-2)] + [n(2-\alpha)] + 1. \tag{2}$$

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Moreover, as we shall prove later, the two quantities in the brackets on the right-hand side of the equality count the number of integers that belong to A and B, respectively, among those integers lying in the half-open interval $[a_n, b_n)$. Incidentally, the title of this paper alludes to Equation (2), its interpretation, and its generalization in Theorems 2 through Theorem 7.

We start by setting some useful notation. We will call A and B *complementary sets* of natural numbers if $A \cap B = \emptyset$ and $A \cup B = \mathbb{N}$. For any multiset X of integers that is bounded below, we number the elements (with multiplicity) of X as $x_1 \leq x_2 \leq \dots$, linking in this way indexed lowercase letters (the elements in non-decreasing order) and the uppercase letters (the set). We use set subtraction and addition to denote the term-wise operations:

$$X \pm Y = \{x_n \pm y_n : n \in \mathbb{N}\}.$$

A set whose terms are given by $b_n = [n\beta]$ for some $\beta > 0$ is called a *Beatty sequence*. We remind the reader of Beatty's Theorem (see, for instance, [28]):

Theorem 1 (Beatty-Rayleigh, [8])**.** *The sets A and B given by $a_n = [n\alpha]$ and $b_n = [n\beta]$, respectively, are complementary sets of natural numbers if and only if α is a positive irrational and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.*

The rest of the paper is outlined as follows. We end this section by highlighting the importance of the subject. In Section 2 we state the Minimum Excluded (MEX) algorithm, familiar from the theory of impartial games, and generalize it by introducing the novel Minimum Excluded with Skipping (MES) algorithm. The latter will later link to our combinatorial interpretation. In Section 3 we state and prove Theorems 2, 3, and 4, the main results of the paper, and explore further implications of the MES algorithm, such as its combinatorial interpretation.

To prove the results of this paper, we use combinatorial arguments, basic analytic techniques, and standard properties of Beatty and Sturmian sequences. For $\alpha > 1$, the Sturmian sequence with slope $1/\alpha$ encodes the Beatty sequence with slope α [4, Lemma 9.1.3]. There is also a corresponding relationship between non-homogeneous Beatty and Sturmian sequences [11, Lemma 1]. Sturmian sequences in general and Beatty sequences in particular, are the focus of a growing amount of research, as they play a role in various fields of mathematics [25], biology [17], music [24], computer science [13], and physics (see also [4], [22], [9], [10] and the references therein). The name "Sturmian" was introduced by Hendlund and Morse in their influential work in the 1940s [21]; however, the history of these sequences dates back to 1772 when Bernoulli III worked with what is now known as a non-homogeneous Sturmian sequence (see [4]). In pure mathematics, Sturmian and Beatty sequences have been studied in relation to dynamical systems [21], fixed-point morphisms [28], logarithm of irrational numbers [26], prime numbers [7], algebraic numbers [16], arithmetic functions [2], primitive roots, divisor functions, character sums, among others (see [1], [3], [6], [18], [25], [27], [29], and the references therein). More recently, the second and third moments of Beatty sequences and their squares have also been studied [23]. We study differences of Beatty sequences in relation to the well-known MEX algorithm that is widely used in combinatorial game theory, coloring algorithms, and elsewhere (see, for instance, [5], [12], [14], [20], and [30]).

2. Miminum Excluded and Minimum Excluded with Skipping Algorithms

In this section, we present the well-known MEX algorithm, introduce the MES algorithm as its generalization, as well as definitions and notations necessary to state and prove Theorem 2. We start with the following definition.

Definition 1. Given a set $S \subset \mathbb{N}$, the *mex* of the set S is defined as $\text{mex}(S) := \min(S^c \cap \mathbb{N})$, that is, $\text{mex}(S)$ is the minimum natural number that is not in S.

It is well known that complementary sequences can be generated using the mex rule above (see, for instance, [16]). We call this process the MEX algorithm and define it right below. Throughout the remainder of the paper, if A is a sequence, we use A_n to denote the set $A_n = \{a_k \,:\, k \leq n\}$.

Definition 2. Given an input sequence H, the *Minimum Excluded Algorithm* is the following set of algorithmic recursions.

- STEP 1: Take $a_1 = 1$.
- STEP 2: If $n \geq 2$, take $a_n = \text{mex}\{a_i, b_i \mid i < n\} = \text{mex}(A_{n-1} \cup B_{n-1})$.
- STEP 3: If $n \geq 1$, take $b_n = a_n + h_n$.
- STEP 4: Repeat steps 2 and 3.

To our knowledge, the first documented use of the MEX algorithm to generate Beatty sequences combinatorially was by British mathematician Willem A. Wythoff in 1907 [31]. He did so with the purpose of presenting a modification of the combinatorial game Nim. He showed that the winning positions of his game were given by the Beatty pairs (a_n, b_n), defined by the golden ratio. In our context, this corresponds to taking $h_n = n$ in the MEX algorithm above. For the purpose of illustration, let us run the first few rounds of the algorithm with $h_n = n$. Applying STEPS 1, 2, and 3, we get $a_1 = 1$ and $b_1 = a_n + n = 1 + 1 = 2$, so $A_1 = \{1\}$ and $B_1 = \{2\}$. Applying STEP 4, we first obtain $a_2 = \text{mex}(A_1 \cup B_1) = 3$ and $b_2 = a_n + n = 3 + 2 = 5$, so that $A_2 = \{1, 3\}$ and $B_2 = \{2, 5\}$. If we follow this process, the first five iterations give Table 1.

$\text{mex}(A_{n-1} \cup B_{n-1})$	$b_n = a_n + n$	$A_n = \{a_k : k \leq n\}$	$B_n = \{b_k : k \leq n\}$
$a_1 = 1$	$b_1 = 1 + 1 = 2$	$A_1 = \{1\}$	$B_1 = \{2\}$
$a_2 = 3$	$b_2 = 3 + 2 = 5$	$A_2 = \{1, 3\}$	$B_2 = \{2, 5\}$
$a_3 = 4$	$b_3 = 4 + 3 = 7$	$A_3 = \{1, 3, 4\}$	$B_3 = \{2, 5, 7\}$
$a_4 = 6$	$b_4 = 6 + 4 = 10$	$A_4 = \{1, 3, 4, 6\}$	$B_4 = \{2, 5, 7, 10\}$
$a_5 = 8$	$b_5 = 8 + 5 = 13$	$A_5 = \{1, 3, 4, 6, 8\}$	$B_5 = \{2, 5, 7, \ldots\}$

Table 1: First few iterations of the MEX algorithm

Note that the set A_n agrees with the first n terms of the sequence A given by the golden ratio. Similarly, B_n agrees with the first n terms of its complementary Beatty sequence.

We now generalize the MEX algorithm by modifying STEP 3, and call this generalization the Minimum Excluded with Skipping algorithm. To find b_n, we choose the minimum excluded element after skipping a number of integers. We base the MES algorithm on the following generalizations of Definition 1.

Definition 3. Given a set of integers A and a non-negative integer k, we define a function denoted by $\text{mex}_k(A)$ that selects the $(k+1)$st minimum excluded positive integer from the set A. In other words, to find $\text{mex}_k(A)$, we skip the first k excluded integers from A and select the next excluded integer.

We now define the MES algorithm.

Definition 4. Given a sequence of non-negative integers $C = \{c_n\}_{n=1}^{\infty}$, the *Minimun Excluded with Skipping Algorithm* is given by the following set of algorithmic recursions.

- STEP 1: Set $a_1 = 1$, $A_1 = \{a_1\} = \{1\}$, and $B_0 = \emptyset$ (the empty set).
- STEP 2: For $n \geq 2$, set $a_n = \text{mex}(A_n \cup B_n)$.
- STEP 3: For $n \geq 1$, set $b_n = \text{mex}_{c_n}(A_n \cup B_{n-1})$, that is, skip the first c_n excluded positive integers and select the next excluded integer from this union.
- STEP 4: Repeat steps 2 and 3.

Example 1. We use the sequence $C = \{0, 1, 1, 2, 3, 3, 4, 4, 5, 6, 6, 7, 8, 8, \ldots\}$ to illustrate the MES algorithm. Starting with $B_0 = \emptyset$, the first iteration gives $a_1 = 1$, and $b_1 = \text{mex}_{c_1}(A_1 \cup B_0) = \text{mex}_0\{1\} = 2$. So, $A_1 = \{1\}$ and $B_1 = \{2\}$. The second iteration gives $a_2 = \text{mex}(A_1 \cup B_1) = 3$ and $b_2 = \text{mex}_{c_2}(A_2 \cup B_1) = \text{mex}_1\{1, 2, 3\} = 5$, as 5 is the second minimum excluded integer after skipping the first excluded integer 4. Continuing this process, subsequent iterations are shown in Table 2.

a_n	b_n	$A_n = \{a_k : k \leq n\}$	$B_n = \{b_k : k \leq n\}$
$a_1 = 1$	$b_1 = \text{mex}_0 = 2$	$A_1 = \{1\}$	$B_1 = \{2\}$
$a_2 = 3$	$b_2 = \text{mex}_1 = 5$	$A_2 = \{1, 3\}$	$B_2 = \{2, 5\}$
$a_3 = 4$	$b_3 = \text{mex}_1 = 7$	$A_3 = \{1, 3, 4\}$	$B_3 = \{2, 5, 7\}$
$a_4 = 6$	$b_4 = \text{mex}_2 = 10$	$A_4 = \{1, 3, 4, 6\}$	$B_4 = \{2, 5, 7, 10\}$
$a_5 = 8$	$b_5 = \text{mex}_3 = 13$	$A_5 = \{1, 3, 4, 6, 8\}$	$B_5 = \{2, 5, 7, 10, 13\}$
$a_6 = 9$	$b_6 = \text{mex}_3 = 15$	$A_6 = \{1, 3, 4, 6, 8, 9\}$	$B_6 = \{2, 5, 7, 10, 13, 15\}$
$a_7 = 11$	$b_7 = \text{mex}_4 = 18$	$A_7 = \{1, 3, 4, 6, 8, 9, 11\}$	$B_7 = \{2, 5, 7, 10, 13, \ldots\}$

Table 2: MES algorithm: $a_n = \text{mex}(A_{n-1} \cup B_{n-1})$ and $b_n = \text{mex}_{c_n}(A_n \cup B_{n-1})$.

n^{th}	A	B	C
1)	1	2	0
2)			1
3)			1
4)			2

Table 3: MES Steps.

n^{th}	A	B	C
1)	1	2	0
2)	3	5	1
3)	4	7	1
4)			2
5)			3
6)			3
7)			4
8)			4
9)			5

Table 4: MES Steps.

n^{th}	A	B	C
1)	1	2	0
2)	3	5	1
3)	4	7	1
4)	6	10	2
5)	8	13	3
6)	9	15	3
7)	11	18	4
8)	12	20	4
9)	14	23	5
10)	16	26	6
11)	17	28	6
12)	19	31	7

Table 5: MES Steps.

Remark 1. Because of how it is used in STEP 3 of the MES definition above, we call C *the skipping sequence* of the MES algorithm. After a close look at the resulting Table 2, one can see that the sequence C used in the MES above can be defined inductively by the rule: a positive integer that has been assigned by the algorithm to A appears twice in C, otherwise it appears once. In other words, the algorithm can be generated without explicitly listing the sequence C, only using the initial value $c_0 = 0$, and the inductive rule just stated. Tables 3, 4, and 5 illustrate how C is generated inductively in this way. Later we expand on this, showing that the resulting sequences from the MES algorithm (Sequences A and B) are the complementary pair given by the golden ratio. This is Theorem 2 (4a).

We need the following definition.

Definition 5. The *sortjoin* of two integer sequences A and B, denoted by $A \star B$, is the union with repetition of the ordered elements of these two sequences.

We use A^2 for the sortjoin of A with itself, that is, $A^2 = A \star A$, and we represent the sortjoin of k copies of A with itself by A^k. Finally, we use $\mathbb{N}_0$ to denote the non-negative integers. For instance, if A represents the even natural numbers, then $A \star \mathbb{N}_0^2 = \{0, 0, 1, 1, 2, 2, 2, 3, 3, 4, 4, 4, 5, 5, \ldots\}$, because the numbers $2, 4, 6, \ldots$ repeat three times while the others only repeat twice.

We need two more definitions before proving Theorem 2. Part (4) of Theorem 2 mentions the case when a sequence C is given by the sortjoin $C = D \star \mathbb{N}_0^{k-1}$ in the MES algorithm. In this case, the sequence C is completely determined by the sequence D. Since C defines the MES algorithm (and D defines C), it effectively follows that D defines the MES algorithm. We record this in the following definition.

Definition 6. If there is an increasing sequence of natural numbers D such that the sequence C of the MES algorithm is given by $C = D \star \mathbb{N}_0^{k-1}$, we call D the *defining sequence* of the MES algorithm.

We also use the following definition.

Definition 7. For $n \geq 1$, consider the (possibly empty) half-open integer interval $(a_n, b_{n-1}]$, and let r_n be the cardinality of the intersection of this interval with B_{n-1}, i.e., $r_n = \#\{b \mid b \in (a_n, b_{n-1}] \text{ and } b \in B_{n-1}\}$. We define the *auxiliary sequence* R by $R := \{r_n\}_{n=1}^{\infty}$.

Remark 2. Consistent with the use of A_n and B_n, we use the convention $R_n := \{r_k \ : \ k \leq n\}$. We also note here for future reference that the sequence c_n is a counting sequence for some elements of the sequence A. Specifically we have

$$c_n = \#\{a \in A \ \mid \ a_n < a < b_n\}.$$

This formula follows because once an integer is skipped when selecting b_n, that integer will never be selected by the sequence B in future steps, thus that element will be picked up by A. This is true whenever c_n is non-decreasing.

3. Statement and Proof of the Main Theorems

The main results of this paper are Theorems 2, 3, and 4, which together provide a complete picture of the MES algorithm and its relation to decomposing the difference of two Beatty sequences as the sum of two other sequences. Theorem 2, part (a), expresses the difference of two Beatty sequences as the sum of two generalized Beatty sequences, yielding Formula (3). Part (b) of this theorem provides an interpretation of this formula: the MES algorithm. In other words, it states how we can compute a_n and b_n from Equation (3), using the MES algorithm.

Theorems 3 and 4 state that if the resulting complementary sequences in the MES algorithm, A and B, are Beatty sequences, then the skipping sequence C has slope $\beta - 2$, and the counting sequence R has slope $2 - \alpha$, where α and β are the slopes of the sequences A and B, respectively.

Finally, Theorem 7 states necessary and sufficient conditions for the output sequences A and B to be Beatty sequences in the MES algorithm. It also relates their slopes to the slope of the input sequence D, and treats some particular cases of interest.

We now state and prove Theorem 2.

Theorem 2. *Suppose that A and B are any pair of complementary sequences. We have the following statements.*

(a) The sequence formed by the difference $A - B$, term by term, can always be decomposed in terms of the sum of two other sequences C and R as follows:

$$b_n - a_n = c_n + r_n + 1, \tag{3}$$

where c_n counts the number of integers strictly between a_n and b_n that belong to the sequence A, and r_n counts the number of integers strictly between a_n and b_n that are in the sequence B.

(b) There is a combinatorial interpretation of Formula (3)*, that is, an algorithm that we call the Minimum Excluded with Skipping algorithm, that takes as input a sequence D and outputs four sequences A, B and C and R. If a_n, b_n, c_n and r_n are the nth term of A, B, C and R, respectively, then Equation* (3) *holds. Furthermore, the MES algorithm characterizes complementary sequences. Specifically, A and B are complementary sequences if and only if there is a non-negative sequence C that generates A and B through the MES algorithm.*

Proof. Formula (3) of Theorem 2 is evident from the fact that A and B are complementary. Thus, the integers in the interval (a_n, b_n) will belong to either A or B, and hence the stated formula in part (a) must follow. The remainder of part (a) holds by construction as follows. On the one hand, given a non-negative sequence C, the MES can be implemented by skipping c_n excluded integers in the nth step. By construction, this will produce complementary sequences. Conversely, if C is negative, skipping cannot be performed in the natural sense as it is done in the algorithm. On the other hand, if a pair of complementary sequences is given, one can define r_n as in Definition 7. Then c_n can be defined as the number of integers in the following complement: $[(a_n, b_n) \cap B_n]^c$ (in other words, $c_n = \#[(a_n, b_n) \cap B_n]^c$). With this setup, we can implement the algorithm with $C = \{c_n\}_{n=1}^{\infty}$. Then it follows that if $R = \{r_n\}_{n=1}^{\infty}$, then $b_n - a_n = c_n + r_n + 1$. □

We now develop a series of technical lemmas regarding Sturmian and Beatty sequences, and then use them to prove Theorems 3, 4 and 7.

Given a Beatty sequence with slope $0 < \alpha < 1$, each positive integer will occur in the sequence k times or $k+1$ times, for some number $k \geq 1$. This is simply a consequence of the size of α. We have the following lemma.

Lemma 1. *If α belongs to the interval $\frac{1}{k+1} < \alpha < \frac{1}{k}$ for k a positive integer, then all non-negative integers appear in the Beatty sequence C with slope α, each of them repeating k or $k+1$ times. Both occurrences are infinite. In the language of sortjoin, there exists, in this case, a sequence B of non-negative integers such that $C = B \star \mathbb{N}_0^{k-1}$.*

Proof. It is not hard to see that the inequality $\frac{1}{k+1} < \alpha < \frac{1}{k}$ implies each non-negative integer repeats k or $k+1$ times in $\{[\alpha n]\}_{n=1}^{\infty}$. In fact, $k\alpha < 1$ implies that each integer k occurs at least k times, and an integer cannot occur $k+2$ times or more. For if $[n\alpha] = [(n+1)\alpha] = \cdots = [(n+k+1)\alpha] = k$, then $0 = [(n+k+1)\alpha] - [n\alpha] \geq [n\alpha] + [(k+1)\alpha] - [n\alpha] = [(k+1)\alpha] > 1$. Now, some integers n will need to be repeated $k+1$ times. This is so because if all large integers repeat k times, then the sequence would be ultimately periodic, and the following equality would need to hold:

$$\lim_{n\to\infty} \frac{[n\alpha]}{n} = \frac{1}{k},$$

that is, $\alpha = \frac{1}{k}$, which contradicts the irrationality of α. For the same reasons, all large integers cannot be repeated $k+1$ times. Hence, both k and $k+1$ occur infinitely often. □

The following definition will be used in the remaining of the paper.

Definition 8. For a real number α such that $0 < \alpha < 1$, we define the characteristic function of α as

$$f_\alpha(n) = [\alpha(n+1)] - [\alpha n].$$

Clearly, $f_\alpha(n) = 1$ or $f_\alpha(n) = 0$. Since the sum telescopes, we have

$$\sum_{n=1}^{m} f_\alpha(n) = [\alpha(m+1)]. \tag{4}$$

Remark 3. Notice that Equation (4) gives the number of integers $n \leq m$ for which $f_\alpha(n) = 1$.

Another definition we will need is the following one.

Definition 9. For $1 < \beta \in \mathbf{R} \setminus \mathbb{Q}$, define

$$g'_\beta(n) = \begin{cases} 1 & \textit{if } n = [k\beta], \textit{ for } k \in \mathbf{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

It is a well-known fact (see [26] and [4, Lemma 9.1.3]) that for all integers n,

$$g'_\beta(n) = f_{1/\beta}(n).$$

Lemma 2. *For each irrational number α such that $\frac{1}{k+1} < \alpha < \frac{1}{k}$, there exists a number $\beta := \frac{\alpha}{1-\alpha}$, such that $[n\alpha] = t$ for $k+1$ different numbers $n = m, m+1, \ldots, m+k$, if and only if $[n\beta] = t$ for k numbers $n = q, q+1, \ldots, q+k-1$. In other words, if $C = [n\alpha]_{n=1}^\infty$ and $B = \left[\frac{\alpha}{1-\alpha}\cdot n\right]_{n=1}^\infty$, then $C = D \star \mathbb{N}_0^k$ and $B = D \star \mathbb{N}_0^{k-1}$ for some increasing sequence D of non-negative integers.*

Proof. Notice that $\frac{1}{k+1} < \alpha < \frac{1}{k}$ if and only if $k < \frac{1}{\alpha} < k+1$, which is equivalent to $k-1 < \frac{1}{\alpha} - 1 < k$ and $\frac{1}{\alpha} - 1 = \frac{1-\alpha}{\alpha} = \frac{1}{\beta}$. Thus, we claim that it is enough to prove that the sequence $[n\alpha]$ equals t for $k+1$ values of n, if and only if

$$[(t+1)\frac{1}{\alpha}] - [t\frac{1}{\alpha}] = k+1.$$

Indeed, if this is true, then

$$[(t+1)\frac{1}{\beta}] - [t\frac{1}{\beta}] = [(t+1)(\frac{1}{\alpha} - 1)] - [t(\frac{1}{\alpha} - 1)] = [(t+1)\frac{1}{\alpha}] - [t\frac{1}{\alpha}] - 1 = k,$$

it would then follow from this claim that $[n\beta]$ repeats k times. So we just need to prove the double implication in the claim. For that purpose, from (8), we write $f(n) := f_\alpha(n) = [(n+1)\alpha] - [n\alpha]$. And from (3), it follows that $f(n) = 1$ if and only if there exists a t such that $n = [t\frac{1}{\alpha}]$. Thus, from this last sentence we find that $[\alpha(n+m)] = r$ for the first $k+1$ values of m if and only if the following three things happen: first, $f(n-1) = 0$; second, $f(n) = f(n+k+1) = 1$; and third, $f(n+m) = 0$ for all numbers between n and $n+k+1$, i.e., for all m such that $1 \le m < k+1$. For such an n, it follows from (3) that $f(n+m) = 0$ for $1 \le m < k+1$ if and only if there is no j such that $n+m = [j\frac{1}{\alpha}]$ for $1 \le m \le k$. Thus, we see that $[(t+1)\frac{1}{\alpha}] - [t\frac{1}{\alpha}] > k$. Since the difference between two consecutive numbers in the Beatty sequence $[t\frac{1}{\alpha}]$ is either k or $k+1$, we see that $[(t+1)\frac{1}{\alpha}] - [t\frac{1}{\alpha}] = k+1$. Therefore, we have proved that each time the sequence $[n\alpha]$ repeats $k+1$ times, the difference $[(t+1)\frac{1}{\alpha}] - [t\frac{1}{\alpha}]$ equals $k+1$, and thus, the lemma holds. □

Combining Lemma 1 and Lemma 2, we see that if a Beatty sequence is generated by an α such that each non-negative integer repeats k or $k+1$ times, then in the Beatty sequence generated by $\frac{\alpha}{1-\alpha} = -1 + \frac{1}{1-\alpha}$, integers repeat $k-1$ or k times. If we iterate this process, we obtain the following lemma.

Lemma 3. *For each number α with $\frac{1}{k+1} < \alpha < \frac{1}{k}$ for some integer k, there exists a number β with $1 < \beta < 2$ given by*

$$\beta \coloneqq -1 + \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cdots - \cfrac{1}{2 - \cfrac{1}{1-\alpha}}}}}$$

where there are $k-2$ twos in this expansion, such that $[n\alpha] = k$ for $k+1$ different numbers if and only if k appears in the Beatty sequence given by $[n\beta]$.

Proof. In Lemma 2, an irrational α yielding a maximum of $k+1$ repetitions implies that the number

$$\alpha_1 = -1 + \frac{1}{1-\alpha} \tag{5}$$

produces a maximum of k repetitions. We iterate this process a second time and find that

$$\alpha_2 = -1 + \frac{1}{1-\alpha_1} = -1 + \frac{1}{1 - \left(-1 + \frac{1}{1-\alpha_1}\right)} = -1 + \cfrac{1}{2 - \cfrac{1}{1-\alpha}} \tag{6}$$

produces a maximum of $k-1$ repetitions. If we iterate this process a total of k times, we will obtain a sequence with no repetitions, i.e., a Beatty sequence $[n\beta]$, for some $\beta > 1$. Note that in each iteration after the second one, a new digit 2 will continue to appear on the right-hand side of Equation (6). Thus, the slope β of the Beatty sequence is necessarily of the form

$$-1 + \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cdots - \cfrac{1}{2 - \cfrac{1}{1-\alpha}}}}}$$

where the number of times the digit 2 appears in this expression is $k-2$, for the following reasons. We start the process with the number α giving k and $k+1$ repetitions. The next step gives α_1, which does not have the digit 2 in it, as shown in Equation (5). The third number in this process, α_2, is the first one having the digit 2, and thus this digit will appear $k-2$ times in β. □

Corollary 1. *Consider a Beatty sequence with slope* $\alpha < 1$*, and let* k *be the integer such that* $\frac{1}{k+1} < \alpha < \frac{1}{k}$*. Then the sequence of numbers that repeat* k *times and the sequence of numbers that repeat* $k+1$ *times form a partition of the integers into two complementary Beatty sequences.*

Proof. Since $0 < \alpha < 1$, each integer n appears in the Beatty sequence with slope α, and it is repeated k or $k+1$ times. By Lemma 3, those numbers that repeat $k+1$ times form a Beatty sequence with slope $\beta > 1$, hence those that repeat k times are the complementary Beatty sequence with slope $\dfrac{1}{1-\frac{1}{\beta}}$. □

We reinterpret the previous result in terms of the sortjoin of sequences to obtain the following corollary.

Corollary 2. *Let* $\alpha \in [0, 1]$ *be an irrational number, and let* k *be the integer such that* $\frac{1}{k+1} < \alpha < \frac{1}{k}$*. Then the sequence* $C = \{[\alpha n]\}_{n=1}^{\infty}$ *is given by the* k*-fold sortjoin of* $\mathbb{N}_0$ *with a Beatty sequence* D*. Specifically,* $C = D \star \mathbb{N}_0^k$*.*

Lemma 4. *Let* A *and* B *be the complementary Beatty sequences generated by* α *and* β*, respectively, with* $1 < \alpha < 2$*. Let* a_n *and* b_n *be the* n^{th} *elements of* A *and* B*, respectively. Define* r_n *as in Definition 7. Then the difference* $r_{n+1} - r_n$ *equals* 0 *or* 1*, and* $r_{n+1} - r_n = 1$ *if and only if* $a_{n+1} - a_n = 1$*; hence the sequence* $r_n - r_{n-1}$ *is Sturmian.*

Proof. To prove the first part of the lemma, note that by Definition 7, we are given that for an integer n, R_n has r_n elements, all of which belong to B. Let us list all the elements of R_n in descending order:

$$R_n = \{b_{n-1}, b_{n-2}, \ldots, b_{n-r_n}\}.$$

Similarly,

$$R_{n+1} = \{b_n, b_{n-1}, \ldots, b_{n-r_{n+1}}\}.$$

From these two sets, we see that R_{n+1} always contains an element that is not in R_n, namely b_n. Also note that the smallest elements of R_n and R_{n+1}, namely b_{n-r_n} and $b_{n-r_{n+1}}$, respectively, could be the same element. In fact, if a_n and a_{n+1} are consecutive elements, then from Definition 7, we see that $b_{n-r_n} = b_{n-r_{n+1}}$. On the other hand, if a_n and a_{n+1} are not consecutive, then since A and B are complementary Beatty sequences, there must be some element of B between a_n and a_{n+1}. In fact, there can only be one element, since $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and $1 < \alpha < \beta$ implies $\beta > 2$, so $b_{n+1} - b_n \geq 2$. Hence, we see that $r_{n+1} - r_n = 1$ if and only if $a_{n+1} - a_n = 1$, and if $a_{n+1} - a_n \neq 1$, then $r_{n+1} - r_n = 0$. Finally, the above argument, together with the fact that $a_{n+1} - a_n = 1$ or 2, implies that $r_n - r_{n+1} = a_{n+1} - a_n - 2$. Since the sequence given by the right-hand side of this equality is Sturmian, it follows that the sequence given by the left-hand side is also Sturmian. □

In a similar manner, we obtain the following lemma.

Lemma 5. *Let A and B be complementary Beatty sequences with slope $\beta > 2$, and let $q = [\beta]$. Suppose a_n and b_n are the n^{th} terms of A and B, respectively, and let $c_n = \{a \in A \mid a_n < a < b_n\}$. Then the difference $c_{n+1} - c_n$ equals $q - 2$ or $q - 1$, and $c_{n+1} - c_n = q - 2$ if and only if $b_{n+1} - b_n = q$.*

The following corollary is a consequence of Lemma 4.

Corollary 3. *If A and B are complementary Beatty sequences, then the sequence $R = \{r_n\}_{n=1}^{\infty}$ is a Beatty sequence with slope r such that $0 < r < 1$.*

Proof. The difference between two elements of R is 0 or 1. Thus, if R is a Beatty sequence, the slope must be a number r with $0 < r < 1$. Since A is a Beatty sequence, Lemma 4, together with the fact that $r_{n+1} - r_n$ can take only two possible values, implies that R is also a Beatty sequence. Hence, the corollary holds. □

We now state and derive two of the main results of this paper: Theorems 3 and 4.

Theorem 3. *In the MES algorithm, if A and B are complementary Beatty sequences, then the sequence $R = \{r_n\}_{n=1}^{\infty}$ is given by the slope $r := 2 - \alpha$, where α is the slope of the sequence A.*

Proof. Since $0 < 2 - \alpha < 1$, and in light of Lemma 4 and its corollary, it is enough to prove that $[(2-\alpha)(n+1)] - [(2-\alpha)n] = 1$ if and only if $a_{n+1} - a_n = 1$. For this purpose, since $[y + x] = y + [x]$ whenever y is an integer, we have that

$$[(2-\alpha)(n+1)] - [(2-\alpha)n] = 2 + [-\alpha(n+1)] - [-\alpha n].$$

For x non-integral, $[-x] = -1 - [x]$, thus we have

$$2 + [-\alpha(n+1)] - [-\alpha n] = 2 - (a_{n+1} - a_n).$$

This is equal to 1 if and only if $a_{n+1} - a_n = 1$. □

Similarly, we obtain the following theorem, using the definition of C given in Lemma 5.

Theorem 4. *In the MES algorithm, if A and B are complementary Beatty sequences, then the sequence $C = \{c_n\}_{n=1}^{\infty}$ is given by the slope $c := \beta - 2$, where β is the slope of the sequence B.*

We obtain the following theorem, which provides a striking formula involving complementary Beatty sequences. It also has a very natural and straightforward interpretation, which we highlight in the remark that follows the theorem.

Theorem 5. *Let α and β be two irrational numbers generating two complementary Beatty sequences, A and B, respectively, with $1 < \alpha < 2$. Set $a_n = [\alpha n]$ and $b_n = [\beta n]$. Then, for any integer n, we have*

$$[\beta^{-1}(b_{n-1}+1)] - [\beta^{-1}(a_n)] = [(2-\alpha)n]. \tag{7}$$

Similarly, we have

$$[\alpha^{-1}b_n] - [\alpha^{-1}(a_n+1)] = [(\beta-2)n]. \tag{8}$$

Proof. We will prove Identity (7), as Identity (8) follows by a similar argument using Theorem 4. To prove Equation (7) in light of Theorem 3, we need to show that $r_n = [\frac{1}{\beta}(b_{n-1}+1)] - [\frac{1}{\beta}(a_n)]$. To this end, recall that r_n can be written as

$$r_n = \#\{b \in B \mid a_n < b \leq b_{n-1}\} = \sum_{\substack{b \in B \\ a_n < b \leq b_{n-1}}} 1. \tag{9}$$

Now, define $f(k) := f_{\frac{1}{\beta}}(k)$ as in (8). By (3), $f(a_n) = 0$, and thus we find that (9) becomes:

$$\sum_{a_n \leq k \leq b_{n-1}} f(k) = \sum_{1 \leq k \leq b_{n-1}} f(k) - \sum_{1 \leq k < a_n} f(k) = [\beta^{-1}(b_{n-1}+1)] - [\beta^{-1}(a_n)].$$

This last equality follows from (4). □

Remark 4. We will interpret Equation (7); a similar interpretation follows for Equation (8). Note that in Corollary 5, the Sturmian sequence with slope $1/\beta$ is the indicator function of the Beatty sequence with slope β. Hence, in light of Lemmas 4 and 3, the left-hand side of Equation (7) gives the number of integers strictly between a_n and b_n that are in B. As a passing comment, we note that the right-hand side of Equation (7) provides another way to write this quantity. Since these r_n integers are in B, if we want to generate a complementary pair by the MES, we simply need to find a sequence c'_n that consistently gives the number of integers between a_n and b_n that are in A. We would then use these numbers c'_n as the skipping sequence in the algorithm.

Lemma 6. *Let α be an irrational number such that $1 < \alpha < 2$, and let $k \geq 1$ be the unique positive integer for which $\dfrac{2k-1}{k} < \alpha < \dfrac{2k+1}{k+1}$. The skipping sequence C that generates the MES algorithm with output the Beatty sequence given by α, can be generated by a Beatty sequence D with slope δ given as follows.*

1. *If $k > 1$, (i.e., if $\frac{3}{2} < \alpha < 2$), then the slope δ is given by:*

$$\delta := -1 + \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{\cdots}{\cdots - \cfrac{1}{2 - \cfrac{1}{\alpha - 1}}}}}}$$

with $k-1$ two's in this expansion, and the frequency for the skipping sequence is given as follows: Each number n will appear k times in C if $n \in D$, and $k-1$ times otherwise.

2. *If $k = 1$ (the case $1 < \alpha < \frac{3}{2}$), then $\delta = \beta - 2$, and the frequency is given as follows: each number n will appear once in the skipping sequence if $n \in D$ and will not appear in the skipping sequence if $n \notin D$.*

3. *Furthermore, C is a Beatty sequence and is given by the slope $c = \beta - 2$.*

Remark 5. Note that if $\alpha = \phi$, where ϕ is the golden ratio, then

$$c = \frac{2-\phi}{\phi-1} = (2-\phi)\phi = 2\phi - \phi^2 = \phi - 1 = \frac{1}{\phi}.$$

Also, observe that the number δ can be written as

$$\delta = \frac{2-\alpha}{k\alpha - 2k + 1}.$$

This reduces to

$$\delta = \frac{2-\alpha}{\alpha-1}$$

when $k = 1$.

Proof of Lemma 6. We already proved in Theorem 4 that C is generated by $c := \frac{2-\alpha}{\alpha-1}$. Since $1 < \alpha < 2$, there exists a unique positive integer k such that

$$\frac{2k-1}{k} < \alpha < \frac{2k+1}{k+1}. \tag{10}$$

`Case I:` $k > 1$. We will apply Theorem 3 with α equal to $c = \frac{2-\alpha}{\alpha-1} = -1 + \frac{1}{\alpha-1}$ (that is, the slope of the sequence C). We first note that the last portion of the continued fraction of β in Theorem 3 is given by

$$2 - \frac{1}{1-c} = 2 - \cfrac{1}{2 - \cfrac{1}{\alpha - 1}}. \tag{11}$$

Thus, we gain an extra 2 here compared to Lemma 3, so when applying that lemma, we will need to adjust for that extra 2. With that in mind, we see that Equation (10) implies that $\frac{1}{k} < \frac{2-\alpha}{\alpha-1} < \frac{1}{k-1}$. Since there are $k-1$ twos in the expansion of δ, it is equivalent to having $k-2$ twos in Lemma 3 because we need to adjust for the extra 2, as shown in Equation (11). Thus, each positive integer will appear in C k or $k-1$ times. Also, Lemma 3 tells us that the numbers that appear k times are precisely the integers that belong to the Beatty sequence D with the slope we here call δ.

`Case II:` $k = 1$, i.e., $1 < \alpha < \frac{3}{2}$. We see that $c = \beta - 2 = \frac{2-\alpha}{\alpha-1} > 1$. In this case, each positive integer either appears once in C or does not appear at all. Thus, D and C turn out to be the same sequence and thus C can be described as follows: each integer n will appear once in C if $n \in D$, and will not appear in C otherwise. □

By comparing parts (1) and (3) in Lemma 6 with Corollary 2, we are able to express the sequence C in Lemma 6 as a sortjoin.

Corollary 4. *To generate a Beatty sequence with the MES algorithm as in Lemma 6, the skipping sequence is given by the sortjoin of D and k copies of $\mathbb{N}_0$. Specifically, $C = D \star \mathbb{N}_0^k$, and the slope of C is given by $\beta - 2$.*

Theorem 6. *The MES algorithm generates a Beatty sequence if and only if the defining sequence is also a Beatty sequence D given by the slope δ from Lemma 6, and the frequency is given by the following rule: if a number belongs to D, it will appear k times in the skipping sequence; otherwise, it will appear $k-1$ times.*

Proof. The number δ given in Lemma 6 can also be written as

$$\delta = \frac{2-\alpha}{k\alpha - 2k + 1}, \tag{12}$$

or equivalently

$$\alpha = \frac{2 - \delta + 2k\delta}{1 + k\delta}. \tag{13}$$

Equation (12) tells us that to produce the Beatty sequence with slope α, we just need to start with the skipping sequence generated by the number δ given on the right-hand side of the equality, with frequency k. We know that if A is a Beatty sequence, then so is B. So we may assume that both A and B are Beatty sequences. In this case, Lemma 6 tells us that the defining sequence D is also a Beatty sequence given by Equation (12). For the converse, Equation (13) tells us that, given a defining sequence with slope δ, we will generate a different Beatty sequence with slope α for each positive integer k we choose, as given in Equation (13). □

We now state and prove the following corollary, which is part 4 of Theorem 2.

Theorem 7. *The MES algorithm generalizes the MEX algorithm as follows: Suppose that there is an increasing sequence of positive integers D such that $C = D \star \mathbb{N}_0^{k-1}$ for some $k \in \mathbb{N}$. Then we have the following statements.*

1. *If $k = 2$ and D is the Beatty sequence defined by the golden ratio, then the MES and the MEX algorithms coincide, i.e., they both produce A (given by the golden ratio). In other words, the MES produces the sequence A if and only if the skipping sequence C can be defined as follows: any non-negative integer c appears once or twice in C, and c appears twice in C if and only if $c \in A$.*

2. *Let D be the sequence given by*
$$\alpha = \frac{k-1+\sqrt{k^2+1}}{k}.$$
Then the MES algorithm produces the Beatty sequence D and its complement. In other words, the MES produces the sequence D if and only if the skipping sequence C can be defined as follows: any non-negative integer c appears k or $k-1$ times in C, and c appears k times in C if and only if $c \in D$. This coincides with the MEX and the golden ratio when $k = 2$.

Proof. Substitute $\alpha = \delta$ into Equation (13), and solve for δ in the resulting rational expression to obtain a quadratic irrational with parameter k. The case $k = 2$ gives the first part of the corollary. □

Remark 6. As seen in Equations (12) and (13), a given δ can generate infinitely many αs. However, each given α is generated by a unique choice of δ and k.

Example 2. As an illustration of Theorem 6, consider the number $\alpha = \frac{187+2\sqrt{13}}{113}$. The Beatty sequence generated by α is $\{1, 3, 5, 6, 8, 10, 12, 13, 15, 17, 18, \dots\}$. If we run the MES algorithm with frequency $k = 3$ and defining sequence given by $\delta = \frac{\sqrt{13}}{2}$, we obtain Table 6.

Example 3. Example 1 can be rephrased recursively as follows. The skipping sequence C of the MES algorithm that generates the sequence A defined by the golden ratio is given by the condition "if $n \in A$, then it will repeat twice in C. Otherwise, n will appear only once in C". Note that if $\delta = \frac{1+\sqrt{5}}{2}$ and $k = 2$, we obtain the sequence in Example 1, that is, the sequence with the golden ratio. In this case the skipping sequence C and the output sequence A coincide with the defining sequence D.

Table 6, below, shows the application of the MES algorithm when the frequency k is given by $k = 3$ and the defining sequence D is given by $\delta = \sqrt{13}/2$. Note that the output sequences A is given by $\alpha = \dfrac{2-\delta+2l\delta}{1+k\delta} = \dfrac{2-\delta+6\delta}{1+6\delta} = \dfrac{187+2\sqrt{13}}{113}$.

n^{th}	A	B	C	R	D
1)	1	2	0	0	1
2)	3	4	0	0	3
3)	5	7	1	0	5
4)	6	9	1	1	7
5)	8	11	1	1	9
6)	10	14	2	1	10
7)	12	16	2	1	12
8)	13	19	3	2	14
9)	15	21	3	2	16
10)	17	23	3	2	18
11)	18	26	4	3	19
12)	20	28	4	3	21

Table 6: Iterations of the MES for $k = 2$ and $\delta = \sqrt{13}/2$.

4. Concluding Remarks

What does the MES algorithm offer that the MEX algorithm does not? Notice that both algorithms allow us to generate any pair of Beatty sequences (and indeed any pair of complementary sequences). The MEX algorithm uses the function $b_n - a_n = h_n$ to generate these sequences, and any interesting application requires that we be able to easily describe the function h_n independently of a_n and b_n.

To obtain the Beatty sequences generated by α and β, a few cases are straightforward to describe. For example, [31] shows that $h_n = n$ gives the case $\alpha = \frac{1+\sqrt{5}}{2}$. More generally, [15] and [19] show that $h_n = tn$ gives the family $\frac{2-t+\sqrt{t^2+1}}{2}$.

The MES algorithm is more advantageous because it provides a general formula for h_n as $c_n + r_n + 1$, with explicit formulas for c_n and r_n in the case of Beatty sequences given in Theorem 2. It also offers the added benefit of a combinatorial interpretation for c_n and r_n, linking them back to the complementary sequences A and B. We conclude with an open question. Given the many applications of the MEX function or the MEX algorithm as highlighted in Section 1, does the MES algorithm reveal interesting connections that illuminate some aspects of these applications or of complementary sequences in general?

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