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FROM SCHUR'S THEOREM TO THE PAIRWISE SUM THEOREM

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Abstract
We show that by only assuming Schur's Theorem and the existence of a non-principal ultrafilter, one can directly prove that in every finite coloring of $\mathbb{N}$ there exist infinite disjoint sets A, B such that all elements of $A \cup B \cup (A+B)$ are monochromatic. This gives a partial answer to a question posed by N. Hindman, I. Leader, and D. Strauss in 2003. In the last section we propose a formalization of that open question in purely topological terms.

1. Introduction

In their 2003 article [7] entitled "Open problems in partition regularity," N. Hindman, I. Leader, and D. Strauss compiled a list of 13 open problems considered to be the most relevant in that field at the time. After more than twenty years, questions enumerated there as Questions 5, 6, 7, 8, 9, 10, and 13, have now been solved. Specifically, Questions 5 and 6 have been settled with positive answers in [1] by B. Barber, N. Hindman, and I. Leader, and Questions 7, 8, and 9 have been settled with negative answers in [2] by B. Barber and I. Leader. Besides, I. Leader and P.A. Russell gave a negative answer to Question 10 in the paper [9];[1] and finally, the conjecture of Question 13 was recently proved to be true by Z. Zelenyk in [10].

Here we focus on Open Question 12 (see below), which is about the strength of a reduced version of Hindman's Theorem where only pairwise sums from an infinite sequence are considered.

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[1] This paper [9] is actually referenced in the open problems paper [7] as "manuscript", but before publication the authors found a counterexample and included it in the final version.

Before proceeding, let us recall some basic terminology. A *finite coloring* of a set X is a finite partition $X = C_1 \cup \ldots \cup C_r$ where the pieces C_i are called *colors*. A set A is *monochromatic* with respect to a given finite coloring if all elements of A belong to the same color, *i.e.*, $A \subseteq C_i$ for some i. A family $\mathfrak{P} \subseteq \mathcal{P}(X)$ is called *weakly partition regular* on X if for every finite coloring of X there exists $A \in \mathfrak{P}$ which is monochromatic. For example, Schur's Theorem states that the family $\{\{a, b, a+b\} \mid a, b \in \mathbb{N}\}$ is weakly partition regular on $\mathbb{N}$.

Let us recall the Finite Sums Theorem, a cornerstone of combinatorics proved by N. Hindman in 1974.

Theorem 1.1 (Finite Sums Theorem [6]). *For every finite coloring of the natural numbers there exists an infinite set A such that its set of finite sums*

$$FS(A) := \left\{ \sum_{a \in F} a \mid \emptyset \neq F \subset A \text{ finite} \right\}$$

is monochromatic.

Shortly after Hindman's article was published, Galvin and Glazer found a different proof using "idempotent ultrafilters"; this proof paved the way for an entire field of research based on the algebraic properties of the space of ultrafilters $\beta\mathbb{N}$ (see the comprehensive monograph [8]).

The following problem was posed in [7] as Open Question 12, with the comment: "It seems truly remarkable that this can be unknown".

(Q12) *Is there a proof that whenever $\mathbb{N}$ is finitely coloured there is a 1-1 sequence $x_1, x_2, \ldots$ such that all x_i and all $x_i + x_j$ $(i \neq j)$ have the same colour, that does not also prove the Finite Sums Theorem?*

For convenience, let us give a name to the property considered in the above question.

Theorem 1.2 (Pairwise Sum Theorem). *For every finite coloring of the natural numbers there exists an infinite A such that the set of pairwise sums*

$$FS_{\leq 2}(A) = A \cup \{a + a' \mid a, a' \in A, a \neq a'\}$$

is monochromatic.

We will show that by only using Schur's Theorem and the existence of a non-principal ultrafilter on $\mathbb{N}$, one can prove the following weaker version of the Pairwise Sum Theorem, thus providing a partial answer to (Q12).

Theorem 2.2. *For every finite coloring of the natural numbers there exist infinite disjoint sets A, B such that $A \cup B \cup (A + B)$ is monochromatic.*

We observe that (Q12), as formulated in [7], is somewhat vague and ambiguous. In the last section we propose a possible rigorous formalization as a purely topological property of the space $\beta\mathbb{N}^2$.

2. The Proof

Let us first recall a few notions and facts about ultrafilters. The *tensor product* $\mathcal{U} \otimes \mathcal{V}$ between ultrafilters on $\mathbb{N}$ is defined by setting, for every $X \subseteq \mathbb{N} \times \mathbb{N}$,

$$X \in \mathcal{U} \otimes \mathcal{V} \text{ if and only if } \{n \in \mathbb{N} \mid X_n \in \mathcal{V}\} \in \mathcal{U}$$

where $X_n = \{m \in \mathbb{N} \mid (n, m) \in X\}$ is the vertical n-fiber of X. The *pseudo-sum* $\mathcal{U} \oplus \mathcal{V}$ is the ultrafilter on $\mathbb{N}$ defined by setting, for every $A \subseteq \mathbb{N}$,

$$A \in \mathcal{U} \oplus \mathcal{V} \text{ if and only if } \{n \in \mathbb{N} \mid A - n \in \mathcal{V}\} \in \mathcal{U}$$

where $A - n := \{m \in \mathbb{N} \mid m + n \in A\}$. Observe that $\mathcal{U} \oplus \mathcal{V} = \mathrm{Sum}(\mathcal{U} \otimes \mathcal{V})$ is the image ultrafilter of the tensor product under the sum function $\mathrm{Sum}(n, m) = n + m$. Recall that if $\mathcal{U}$ is an ultrafilter on a set I and $f : I \to J$, the *image ultrafilter* $f(\mathcal{U})$ on J is defined by setting $A \in f(\mathcal{U})$ if and only if $f^{-1}(A) \in \mathcal{U}$ for all $A \subseteq J$.[2]

In our proof, we will use the following combinatorial property of tensor products. It is a particular case of a general result presented in [3]. However, for completeness, we give here a self-contained proof.

Lemma 2.1. *For every $X \subseteq \mathbb{N} \times \mathbb{N}$, the following conditions are equivalent.*

1. *There exist disjoint increasing sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that*

$$\{(a_i, b_j) \mid i \le j\} \cup \{(b_j, a_i) \mid j < i\} \subseteq X.$$

2. *There exist non-principal ultrafilters $\mathcal{V}_1, \mathcal{V}_2$ on $\mathbb{N}$ such that*

$$X \in (\mathcal{V}_1 \otimes \mathcal{V}_2) \cap (\mathcal{V}_2 \otimes \mathcal{V}_1).$$

Proof. (1) $\Rightarrow$ (2). Let $A = \{a_n \mid n \in \mathbb{N}\}$ and $B = \{b_n \mid n \in \mathbb{N}\}$. We observe that the family $\mathcal{F}_1 := \{A\} \cup \{X_{b_j} \mid j \in \mathbb{N}\} \cup \{[n, \infty] \mid n \in \mathbb{N}\}$ has the finite intersection property; indeed, $a_i \in A \cap X_{b_1} \cap \ldots \cap X_{b_k} \cap [n, \infty]$ for every $i > k, n$. Similarly, the family $\mathcal{F}_2 := \{B\} \cup \{X_{a_i} \mid i \in \mathbb{N}\} \cup \{[n, \infty] \mid n \in \mathbb{N}\}$ also has the finite intersection property. So, we can pick non-principal ultrafilters $\mathcal{V}_1 \supseteq \mathcal{F}_1$ and $\mathcal{V}_2 \supseteq \mathcal{F}_2$. Note that the set $Y_1 := \{(a_i, b_j) \mid i \le j\} \in \mathcal{V}_1 \otimes \mathcal{V}_2$ because for every $a_s \in A$ the vertical fiber $(Y_1)_{a_s} = \{b_j \mid j \ge s\} = B \cap [a_s, \infty] \in \mathcal{V}_2$. Similarly, $Y_2 := \{(b_j, a_i) \mid i \le j\} \in \mathcal{V}_2 \otimes \mathcal{V}_1$ because for every $b_t \in B$ the vertical fiber $(Y_2)_{b_t} \in \mathcal{V}_1$. We conclude that $X \supseteq Y_1 \cup Y_2 \in (\mathcal{V}_1 \otimes \mathcal{V}_2) \cap (\mathcal{V}_2 \otimes \mathcal{V}_1)$, as desired.

(2) $\Rightarrow$ (1). Let $X^{(1)} := \{n \mid X_n \in \mathcal{V}_1\}$ and $X^{(2)} := \{n \mid X_n \in \mathcal{V}_2\}$. By definition, $X \in \mathcal{V}_1 \otimes \mathcal{V}_2 \Leftrightarrow X^{(2)} \in \mathcal{V}_1$ and $X \in \mathcal{V}_2 \otimes \mathcal{V}_1 \Leftrightarrow X^{(1)} \in \mathcal{V}_2$. Pick

[2] The standard reference for pseudo-sums of ultrafilters, and more generally for the algebra on the space of ultrafilters on a set S as determined by an arbitrary associative operation on S, is Hindman-Strauss' book [8]. For more information on tensor products, see [8, §11.1]; see also [3] where their combinatorial properties are investigated.

$a_1 \in X^{(2)}$; then $X_{a_1} \in \mathcal{V}_2$. Pick $b_1 \in X^{(1)} \cap X_{a_1} \in \mathcal{V}_2$ with $b_1 > a_1$ (this is possible because $\mathcal{V}_2$ is non-principal); then $(a_1, b_1) \in X$ and $X_{b_1} \in \mathcal{V}_1$. Inductively, pick $a_{n+1} \in X^{(2)} \cap \bigcap_{i=1}^{n} X_{b_i} \in \mathcal{V}_1$ with $a_{n+1} > b_n$, so that $(b_i, a_{n+1}) \in X$ for all $i = 1, \ldots, n$ and $X_{a_{n+1}} \in \mathcal{V}_2$. Then pick $b_{n+1} \in X^{(1)} \cap \bigcap_{i=1}^{n+1} X_{a_i} \in \mathcal{V}_2$ with $b_{n+1} > a_{n+1}$, so that $(a_i, b_{n+1}) \in X$ for all $i = 1, \ldots, n+1$ and $X_{b_{n+1}} \in \mathcal{V}_1$. It is easily verified that the increasing sequences (a_n) and (b_n) satisfy the desired properties. □

We are finally ready to give a proof of the Pairwise Sum Theorem.

Theorem 2.2. *For every finite coloring of the natural numbers there exist infinite disjoint sets A, B such that $A \cup B \cup (A + B)$ is monochromatic.*

Proof. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. For $n \in \mathbb{N}$, denote by $\mathcal{U}^{n\oplus}$ the iterated sum of the ultrafilter $\mathcal{U}$ with itself, *i.e.*, $\mathcal{U}^{1\oplus} = \mathcal{U}$, and inductively $\mathcal{U}^{(n+1)\oplus} = \mathcal{U}^{n\oplus} \oplus \mathcal{U}$. We observe that $\mathcal{U}^{n\oplus} \oplus \mathcal{U}^{m\oplus} = \mathcal{U}^{(n+m)\oplus}$.

Given a finite coloring $\mathbb{N} = C_1 \cup \ldots \cup C_r$, consider the coloring $\mathbb{N} = D_1 \cup \ldots \cup D_r$ where we put $n \in D_i$ if and only if $C_i \in \mathcal{U}^{n\oplus}$. By Schur's Theorem there exists a color D_i and a pair $a \neq b$ such that $a, b, a + b \in D_i$. This means that the color $C = C_i$ is a member of $\mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{W}$, where $\mathcal{V}_1 := \mathcal{U}^{a\oplus}$, $\mathcal{V}_2 := \mathcal{U}^{b\oplus}$, and $\mathcal{W} := \mathcal{U}^{(a+b)\oplus}$.

We now observe that the image ultrafilters $\pi_1(\mathcal{V}_1 \otimes \mathcal{V}_2) = \pi_2(\mathcal{V}_2 \otimes \mathcal{V}_1) = \mathcal{V}_1$ and $\pi_2(\mathcal{V}_1 \otimes \mathcal{V}_2) = \pi_1(\mathcal{V}_2 \otimes \mathcal{V}_1) = \mathcal{V}_2$ where $\pi_1, \pi_2 : \mathbb{N} \times \mathbb{N}$ are the canonical projections. We also observe that $\mathcal{W} = \mathcal{V}_1 \oplus \mathcal{V}_2 = \mathcal{V}_2 \oplus \mathcal{V}_1 = \mathrm{Sum}(\mathcal{V}_1 \otimes \mathcal{V}_2) = \mathrm{Sum}(\mathcal{V}_2 \otimes \mathcal{V}_1)$, and so,

$$X := \pi_1^{-1}(C) \cap \pi_2^{-1}(C) \cap \mathrm{Sum}^{-1}(C) = \\ = \{(a, b) \in C \times C \mid a + b \in C\} \in (\mathcal{V}_1 \otimes \mathcal{V}_2) \cap (\mathcal{V}_2 \otimes \mathcal{V}_1).$$

Now apply Lemma 2.1 to X, and obtain the existence of disjoint increasing sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that

$$\Gamma := \{(a_i, b_j) \mid i \leq j\} \cup \{(b_j, a_i) \mid j < i\} \subseteq X.$$

Let $A := \{a_n \mid n \in \mathbb{N}\}$ and $B := \{b_n \mid n \in \mathbb{N}\}$. Then $A = \pi_1(\Gamma) \subseteq \pi_1(X) \subseteq C$, $B = \pi_2(\Gamma) \subseteq \pi_2(X) \subseteq C$, and $A + B = \mathrm{Sum}(\Gamma) \subseteq \mathrm{Sum}(X) \subseteq C$, as desired. □

3. A Topological Formalization

As already remarked in the introduction, Open Question (Q12) as formulated in [7] is somewhat vague and ambiguous. Here we propose a formalization in topological terms.

As it is well-known, there is a close connection between weak partition regularity and ultrafilters, grounding on the following fact.

Proposition 3.1 ([8, Thm. 5.7])**.** *Let $\mathcal{F} \subseteq \mathcal{P}(S)$. The following properties are equivalent:*

1. *$\mathcal{F}$ is weakly partition regular on S.*
2. *There exists an ultrafilter $\mathcal{U}$ on S such that for every $A \in \mathcal{U}$ there exists $F \in \mathcal{F}$ with $F \subseteq A$.*

An ultrafilter $\mathcal{U}$ as above is called a PR-*witness* of the family $\mathcal{F}$.

Notation 3.2. Following a common use, given sets of natural numbers

$$A = \{a_n \mid n \in \mathbb{N}\} \text{ and } B = \{b_n \mid n \in \mathbb{N}\}$$

where the sequences (a_n) and (b_n) are increasing, we write

$$A \oplus B = \{a_i + b_j \mid i < j\}.$$

In particular, $A \oplus A$ contains all pairwise sums of distinct elements of A.[3]

Notice that Hindman's Theorem states that $\mathcal{H} := \{\mathrm{FS}(X) \mid X \subseteq \mathbb{N} \text{ infinite}\}$ is a weakly partition regular family; and similarly, the Pairwise Sum Theorem states that $\mathcal{S}_2 = \{B \cup (B \oplus B) \mid B \subseteq \mathbb{N} \text{ infinite}\}$ is a weakly partition regular family.

One can formalize the Open Question (Q12) as a precise mathematical statement in terms of witness ultrafilters, as follows:

(†) *Is there an ultrafilter $\mathcal{U}$ that is a PR-witness of $\mathcal{S}_2$ but not a PR-witness of $\mathcal{H}$?*

As we will show below, property (†) can be reformulated in purely topological terms within the Stone–Čech compactification $\beta\mathbb{N}^2$ of the discrete space $\mathbb{N}^2$. Recall that $\beta\mathbb{N}^2$ is usually represented as the space of ultrafilters on $\mathbb{N}^2$ where a base of (cl)open sets is given by the family of sets of the form $\mathcal{O}_X := \{\mathcal{W} \in \beta\mathbb{N}^2 \mid X \in \mathcal{W}\}$ (see [8, Section 3.2]). Recall the following fact.

Proposition 3.3 ([8, Lemma 5.19])**.** *An ultrafilter is a PR-witness of the family $\mathcal{H}$ if and only if it belongs to the topological closure of the set of idempotents:*

$$\mathcal{H} = \overline{\{\mathcal{U} \in \beta\mathbb{N} \mid \mathcal{U} \oplus \mathcal{U} = \mathcal{U}\}}.$$

In the recent paper [4], the class of witnesses of the PR property given by Ramsey's Theorem on pairs was introduced and studied.

[3] We observe that the same $\oplus$ symbol is also used for the "pseudo-sum" operation between ultrafilters on $\mathbb{N}$; however, here this should not cause misunderstandings.

Definition 3.4. An ultrafilter $\mathcal{W}$ on $\mathbb{N}^2$ is a *Ramsey's witness* if for every $X \in \mathcal{W}$ there exists an infinite H such that

$$[H]^2 = \{(h, h') \in H \times H \mid h < h'\} \subseteq X.$$

The following fact was already pointed out in [4]. For completeness, we give a proof here.

Proposition 3.5. *An ultrafilter $\mathcal{W} \in \beta\mathbb{N}^2$ is a Ramsey's witness if and only if it belongs to the closure $\overline{\mathbb{T}}$ of the set of tensor powers $\mathbb{T} := \{\mathcal{U} \otimes \mathcal{U} \mid \mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}\}$.*

Proof. Let $\mathcal{W} \in \overline{\mathbb{T}}$, and let $X \in \mathcal{W}$. Then $X \in \mathcal{U} \otimes \mathcal{U}$ for some non-principal $\mathcal{U}$. Now we observe that every such tensor power is a Ramsey's witness. Indeed, $X \in \mathcal{U} \otimes \mathcal{U}$ if and only if $X_{\mathcal{U}} := \{n \mid X_n \in \mathcal{U}\} \in \mathcal{U}$. Pick $h_1 \in X_{\mathcal{U}}$ and, inductively, pick $h_{n+1} \in X_{\mathcal{U}} \cap X_{h_1} \cap \ldots \cap X_{h_n} \in \mathcal{U}$. Since $\mathcal{U}$ is non-principal we can pick $h_{n+1} > h_n$. Then $H = \{h_n \mid n \in \mathbb{N}\}$ is the desired infinite homogeneous set $[H]^2 \subseteq X$.

Conversely, let $H = \{h_1 < \ldots < h_n < \ldots\}$ be an infinite set with $[H]^2 \subseteq X$. The family $\mathcal{F} := \{\{h_m \mid m \geq n\} \mid n \in \mathbb{N}\}$ has the finite intersection property, and it is easily verified that every (necessarily non-principal) ultrafilter $\mathcal{U} \supseteq \mathcal{F}$ is such that $[H]^2 \in \mathcal{U} \otimes \mathcal{U}$, and hence $X \in \mathcal{U} \otimes \mathcal{U}$. □

We now observe the following property that connects Ramsey's witnesses with the Pairwise Sum Theorem.

Proposition 3.6. *$\mathcal{U}$ is a PR-witness of the family $\mathcal{S}_2$ if and only if there exists a Ramsey's witness $\mathcal{W}$ such that $\mathcal{U} = \pi_1(\mathcal{W}) = \pi_2(\mathcal{W}) = Sum(\mathcal{W})$.*

Proof. Let $A \in \mathcal{U} = \pi_1(\mathcal{W}) = \pi_2(\mathcal{W}) = \mathrm{Sum}(\mathcal{W})$. Then

$$X_A := \pi_1^{-1}(A) \cap \pi_2^{-1}(A) \cap \mathrm{Sum}^{-1}(A) = \{(a, a') \in A \times A \mid a + a' \in A\} \in \mathcal{W}.$$

Pick an infinite $B = \{b_1 < \ldots < b_n < \ldots\}$ with $[B]^2 \subseteq X_A$. Then it is easily verified from the definitions that $B \cup (B \oplus B) \subseteq A$.

For the converse implication, observe that an ultrafilter $\mathcal{W}$ on $\mathbb{N}^2$ has the property $\pi_1(\mathcal{W}) = \pi_2(\mathcal{W}) = \mathrm{Sum}(\mathcal{W}) = \mathcal{U}$ if and only if the preimages $\pi_1^{-1}(A)$, $\pi_2^{-1}(A)$, and $\mathrm{Sum}^{-1}(A)$ belong to $\mathcal{W}$ for every $A \in \mathcal{U}$; equivalently, if and only if $\Gamma_A := (A \times A) \cap \mathrm{Sum}^{-1}(A) \in \mathcal{W}$ for every $A \in \mathcal{U}$. Observe also that an ultrafilter $\mathcal{W}$ on $\mathbb{N}^2$ is a Ramsey's witness if and only if $\mathcal{W}$ extends the following family:

$$\mathcal{R} = \{X \subseteq \mathbb{N}^2 \mid [H]^2 \nsubseteq X^c \text{ for every infinite } H\}.$$

Thus we reach the thesis if we show that there exists an ultrafilter $\mathcal{W}$ that extends the family $\mathcal{R} \cup \{\Gamma_A \mid A \in \mathcal{U}\}$, and this is equivalent to having $\mathcal{R} \cup \{\Gamma_A \mid A \in \mathcal{U}\}$ satisfy the finite intersection property. Assume towards a contradiction that there are $X_1, \ldots, X_s \in \mathcal{R}$ and $A_1, \ldots, A_t \in \mathcal{U}$ such that $\bigcap_{i=1}^s X_i \cap \bigcap_{j=1}^t \Gamma_{A_j} = \emptyset$. If $A := A_1 \cap \ldots \cap A_t$ then we have

$$(A \times A) \cap \mathrm{Sum}^{-1}(A) = \bigcap_{j=1}^{t} \Gamma_{A_j} \subseteq (X_1)^c \cup \ldots \cup (X_s)^c.$$

Since $A \in \mathcal{U}$ we can pick an infinite $B = \{b_1 < \ldots < b_n < \ldots\}$ such that $B \cup (B \oplus B) \subseteq A$. This means that $[B]^2 \subseteq (A \times A) \cap \mathrm{Sum}^{-1}(A) \subseteq (X_1)^c \cup \ldots \cup (X_s)^c$. Finally, consider the finite coloring $[\mathbb{N}]^2 = C_1 \cup \ldots \cup C_s$ where $(n, m) \in C_i \Leftrightarrow i$ is the least index such that $(b_n, b_m) \in (X_i)^c$. By Ramsey's Theorem there exists an infinite H such that its pairs $[H]^2 \subseteq C_i$ are monochromatic. But then we would have $[\{b_h \mid h \in H\}]^2 \subseteq (X_i)^c$, a contradiction. $\square$

Putting all of the above together, we can finally give a precise formalization of Open Question 12 from [7] as the following property of the topological space $\beta\mathbb{N}^2$, obtained as a reformulation of the previous (†).

(Q12) *Consider the following subspaces of* $\beta\mathbb{N}^2$*:*

- $\mathbb{T} := \{\mathcal{U} \otimes \mathcal{U} \mid \mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}\}$.
- $\mathbb{S} := \{\mathcal{W} \in \beta\mathbb{N}^2 \mid \pi_1(\mathcal{W}) = \pi_2(\mathcal{W}) = \mathrm{Sum}(\mathcal{W})\}$,

Is it true that $\mathit{Sum}\left(\overline{\mathbb{T}} \cap \mathbb{S}\right) = \overline{\mathit{Sum}\,(\mathbb{T} \cap \mathbb{S})}$*?*

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