ON p-FROBENIUS NUMBERS FOR THE NUMERICAL SEMIGROUPS GENERATED BY THREE CONSECUTIVE STAR NUMBERS

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Abstract

The Frobenius coin problem involves computing the largest integer, known as the Frobenius number, that cannot be expressed as a non-negative integral linear combination of given relatively prime positive integers. A more generalized version of this problem, termed as the p-Frobenius number, aims to find the largest integer that has at most p representations in terms of linear combinations, where p is any non-negative integer. In this article, we give the closed-form expressions of the p-Frobenius numbers for the numerical semigroups generated by the triplets of the consecutive star numbers for the cases $p = 0, 1$, and 2. Also, we present explicit formulas for their p-Sylvester numbers (which count the positive integers having no more than p representations) where $p = 0, 1$, and 2.

1. Introduction

The *Star numbers*, denoted by S_n, are commonly referred to as the centered 12-gonal numbers or centered dodecagonal numbers. The formula expressing the n^{th} star number is given by $S_n = 6n(n-1)+1$, where $n \geq 1$. The first few star numbers

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[21, A.131] are as follows

$$\{S_n\}_{n\geq 1} = 1, 13, 37, 73, 121, 181, 253, 337, 433, 541, 661, 793, 937, 1093, 1261, 1441, 1633, 1837, 2053, \ldots.$$

These numbers appear in many number theoretic problems. Some well-known formulas include [27, OEIS A306980]

$$\sum_{n=1}^{\infty} \frac{1}{S_n} = \frac{\pi \tan(\pi/2\sqrt{3})}{2\sqrt{3}}, \quad \sum_{n=0}^{\infty} \frac{S_n}{n!} = 7e, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{S_n}{2^n} = 25.$$

One can also easily prove the first identity above using the Cauchy's residue theorem.

Geometrically, the n^{th} star number consists of a central point along with 12 copies of the $(n-1)^{th}$ triangular number t_{n-1}. A notable observation is that infinitely many star numbers are also triangular numbers and among the initial instances, we have $S_1 = 1 = t_1$, $S_7 = 253 = t_{22}$, $S_{91} = 49141 = t_{313}$, and $S_{1261} = 9533161 = t_{4366}$ on the OEIS entry A156712. The star numbers are used for a new set of vector-valued Teichmüller modular forms, defined on the Teichmüller space, strictly related to the Mumford forms, which are holomorphic global sections of the vector bundle [19].

Let $A = \{a_1, a_2, \ldots, a_k\}$ be a set of relatively prime positive integers, where $k \geq 2$, and let p be any non-negative integer. The *p- numerical semigroup* $S_p(A)$ is defined as the set of integers whose non-negative integral linear combinations in terms of given positive integers $a_1, a_2, \ldots, a_k$ can be expressed in more than p ways [18]. For some background on the number of representations, refer, e.g., [4, 6, 10, 28]. For the set of non-negative integers $\mathbb{N}_0$, the set $G_p(A) := \mathbb{N}_0 \backslash S_p(A)$ is finite if and only if gcd $(a_1, a_2, \ldots, a_k) = 1$. Then, the maximum element of the set $G_p(A)$, denoted by $g_p(A)$ is called the *p-Frobenius number*. The cardinality of the set $G_p(A)$ is called the *p-Sylvester number* (or the p-genus) and is denoted by $n_p(A)$. This kind of concept is a generalization of the famous Diophantine problem of Frobenius [2, 25], since $p = 0$ is the classical case, where the original Frobenius number is denoted by $g(A) = g_0(A)$ and the genus as $n(A) = n_0(A)$. Here, the set A is called the *system of generators* of the p-numerical semigroup $S_p(A)$.

When $k = 2$, there exists an explicit closed formula of the p-Frobenius number for any non-negative integer p [3]. However, for $k = 3$, the p-Frobenius number cannot be given by any set of closed formulas that can be reduced to a finite set of certain polynomials [9]. Since it is very difficult to give a closed explicit formula of any general sequence for three or more variables, many researchers have tried to find the Frobenius numbers for some special cases (see, e.g., [14, 22, 23, 24] for more details). Recently, in [7, 8], the Frobenius numbers for the triplets of successive centered triangular, centered square, centered pentagonal, and centered hexagonal numbers were studied. Though it is even more difficult when $p > 0$ (see, e.g., [12, 15, 16, 17]), in [11], the p-Frobenius numbers of three consecutive triangular numbers were studied. In this paper, the p-Frobenius numbers of the three consecutive star numbers are examined. Initially, the focus is on understanding the structure of the p-Apéry set for the classical case, followed by other positive integral values of $p = 1, 2$. Additionally, we present explicit formulas for their p-Sylvester numbers for $p = 0, 1, 2$.

2. Preliminaries

In this section, we recall the notion of the p-Apéry set [1] and some results to compute the p-Frobenius number and p-Sylvester number which will play a crucial role in proving the main theorem. Let us define the p-Apéry set.

Definition 1. Consider a set of positive integers $A = \{a_1, a_2, \dots, a_k\}$ $(k \ge 2)$ with $\gcd(A) = 1$. Without loss of generality, assume that $a_1 = \min(A)$ and let p be any non-negative integer. Then, the *p-Apéry set of* A, denoted by $\mathrm{Ap}_p(A)$, is defined as

$$\mathrm{Ap}_p(A) = \mathrm{Ap}(a_1, a_2, \dots, a_k) = \{m_0^{(p)}, m_1^{(p)}, \dots, m_{a_1-1}^{(p)}\},$$

where $m_i^{(p)}$ is the least positive integer of $S_p(A)$, and satisfies the congruence $m_i^{(p)} \equiv i \pmod{a_1}$, for $0 \le i \le a_1 - 1$. This definition is equivalent to saying that $m_i^{(p)} \in S_p(A)$, $m_i^{(p)} - a_1 \notin S_p(A)$, and $m_i^{(p)} \equiv i \pmod{a_1}$. Note that $m_0^{(0)}$ is defined to be 0. It follows that for given p,

$$\mathrm{Ap}_p(A) \equiv \{0, 1, \dots, a_1 - 1\} \pmod{a_1}.$$

In other words, $\mathrm{Ap}_p(A)$ forms a complete residue system modulo a_1.

One of the convenient formulas to obtain the p-Frobenius number and the p-Sylvester number (or p-genus) is via the elements in the corresponding p-Apéry set. The lemma given below describes the relationship between the Frobenius number and the Sylvester number with the associated Apéry set [13].

Lemma 1. *Let* $\gcd(a_1, \dots, a_k) = 1$ *with* $a_1 = \min\{a_1, \dots, a_k\}$. *Then, we have*

$$g_p(a_1, \dots, a_k) = \left(\max_{0 \le j \le a_1 - 1} m_j^{(p)} \right) - a_1,$$

$$n_p(a_1, \dots, a_k) = \frac{1}{a_1} \left(\sum_{j=0}^{a_1-1} m_j^{(p)} \right) - \frac{a_1 - 1}{2}.$$

Remark 1. When $p = 0$ (classical case), the Frobenius number is essentially due to Brauer and Shockley [5], and the classical Sylvester number is due to Selmer [26]. More general formulas, including the p-power sum and the p-weighted sum, can also be seen in [13].

In order to discuss the Frobenius number for triples of successive star numbers, we must ensure that they are relatively prime. Here, we give a lemma that articulates this requirement.

Lemma 2. *For any three consecutive star numbers $S_n, S_{n+1},$ and S_{n+2},*

$$\gcd(S_n, S_{n+1}, S_{n+2}) = 1.$$

Proof. We know that

$$\gcd(S_n, S_{n+1}, S_{n+2}) = \gcd(\gcd(S_n, S_{n+1}), \gcd(S_{n+1}, S_{n+2})).$$

Let $d_n = \gcd(S_n, S_{n+1})$ and observe that $(n+1)S_n - (n-1)S_{n+1} = 2$. This implies $d_n \mid 2$ and since star numbers are odd, we conclude that $d_n = 1$. Thus, $\gcd(S_n, S_{n+1}) = 1$ and hence, $\gcd(S_n, S_{n+1}, S_{n+2}) = 1$. □

In addition, in a later section we use a classical identity known as Bézout's Lemma, which is stated as follows.

Lemma 3 ([20])**.** *Let a and b be integers with greatest common divisor d. Then there exist integers x and y such that $ax + by = d$. Moreover, the integers of the form $az + bt$ are exactly the multiples of d.*

3. Main Results

Now, we derive the explicit expressions for the p-Frobenius numbers and the p-Sylvester numbers for the triples consisting of successive star numbers, discussed in Section 3.1 and Section 3.2, respectively.

3.1. p-Frobenius Numbers

The p-Frobenius numbers of the numerical semigroups generated by three consecutive star numbers are given as follows.

Theorem 1. *For $p = 0, 1, 2,$ we have*

$$g_p(S_n, S_{n+1}, S_{n+2}) = \begin{cases} 2nS_{n+1} + (p+2)nS_{n+2} - S_n, & \text{if } 6 \le n \le 9; \\ (2n-11)S_{n+1} + (p+3)nS_{n+2} - S_n, & \text{if } n \ge 10. \end{cases}$$

Remark 2. More explicitly, we can write

$$g_p(S_n, S_{n+1}, S_{n+2}) = \begin{cases} 24n^3 + 42n^2 + 34n - 1 + (6n^3 + 18n^2 + 13)p, & \text{if } 6 \le n \le 9; \\ 30n^3 - 6n^2 - 19n - 12 + (6n^3 + 18n^2 + 13)p, & \text{if } n \ge 10. \end{cases}$$

When $p \ge 3$, the situation becomes complex, and no explicit formula has been derived so far. The cases where $n = 2, 3, 4, 5$ are discussed later.

Proof. Our main goal is to find the p-Apéry set and establish the validity of the theorem case by case for $n \geq 6$.
`Case 1:` $p = 0$. For convenience, substitute $t_{y,z} := yS_{n+1} + zS_{n+2}$ for non-negative integers y and z. Then, we can show that the elements of the 0-Apéry set are given as in Table 1.

$t_{0,0}$	$\cdots$	$t_{2n-11,0}$	$t_{2n-10,0}$	$\cdots$	$\cdots$	$t_{2n,0}$
$\vdots$		$\vdots$	$\vdots$			$\vdots$
$t_{0,2n}$	$\cdots$	$t_{2n-11,2n}$	$t_{2n-10,2n}$	$\cdots$	$\cdots$	$t_{2n,2n}$
$t_{0,2n+1}$	$\cdots$	$t_{2n-11,2n+1}$				
$\vdots$		$\vdots$				
$\vdots$		$\vdots$				
$t_{0,3n}$	$\cdots$	$t_{2n-11,3n}$				

Table 1: $\mathrm{Ap}_0(S_n, S_{n+1}, S_{n+2})$

Firstly, we prove that the elements $t_{y,z}$ in Table 1 form a complete residue system modulo S_n. To prove this we show that any such two elements in the table are incongruent modulo S_n. Assume for a contradiction that there exist ordered pairs (y_1, z_1) and (y_2, z_2) such that $t_{y_1,z_1} \neq t_{y_2,z_2}$ in Table 1 with

$$t_{y_1,z_1} \equiv t_{y_2,z_2} \pmod{S_n}. \tag{1}$$

Substituting the values of $t_{y,z}$ in Equation (1), we have

$$y_1 S_{n+1} + z_1 S_{n+2} \equiv y_2 S_{n+1} + z_2 S_{n+2} \pmod{S_n}.$$

Re-arranging the above equation, we get

$$(y_1 - y_2)S_{n+1} + (z_1 - z_2)S_{n+2} \equiv 0 \pmod{S_n}.$$

This implies

$$(y_1 - y_2)(S_{n+1} - S_n) + (z_1 - z_2)(S_{n+2} - S_n) \equiv 0 \pmod{S_n}.$$

Consequently,

$$12n(y_1 - y_2) + 12(2n+1)(z_1 - z_2) \equiv 0 \pmod{S_n}.$$

Furthermore, $\gcd(12, S_n) = 1$, and hence

$$n(y_1 - y_2) + (2n+1)(z_1 - z_2) \equiv 0 \pmod{S_n}. \tag{2}$$

For y_1, y_2, z_1, z_2 as in Table 1, we have

$$|y_1 - y_2| \leq 2n \quad \text{and} \quad |z_1 - z_2| \leq 3n. \tag{3}$$

From Equation (2), $n(y_1 - y_2) + (2n+1)(z_1 - z_2)$ is a multiple of S_n, and under the constraints of Equation (3), the only possible values are 0, S_n, or $-S_n$. If $n(y_1 - y_2) + (2n+1)(z_1 - z_2) = 0$, it follows that $t_{y_1,z_1} = t_{y_2,z_2}$ which contradicts our initial assumption. Now, consider the case when

$$n(y_1 - y_2) + (2n+1)(z_1 - z_2) = S_n = 6n^2 - 6n + 1. \tag{4}$$

Using Bézout's lemma, the extended Euclidean algorithm, and the inequalities in Equation (3), the only integral solution to Equation (4) is

$$(y_1 - y_2, z_1 - z_2) = (2n - 10, 2n + 1).$$

As a result,

$$\begin{aligned} y_1 &= 2n - 10 + y_2, \\ z_1 &= 2n + 1 + z_2. \end{aligned}$$

Since y_2 and z_2 are non-negative integers, it follows that t_{y_1,z_1} lies outside Table 1. Hence, our initial assumption was wrong. On a similar line of reasoning, we can argue that $n(y_1 - y_2) + (2n+1)(z_1 - z_2) \neq -S_n$. Thus, no two elements in Table 1 are congruent modulo S_n. Additionally, note that the count of elements in Table 1 is equal to $(2n+1)^2 + n(2n-10) = S_n$. Therefore, we obtain that the set $\{t_{y,z} : y, z \in \text{Table } 1\}$ constitutes a complete residue system modulo S_n. In other words, we show that for each $i \in \{0, 1, \ldots, S_{n-1}\}$, there exists a unique element $t_{y,z}$ in Table 1 such that $t_{y,z} \equiv i \pmod{S_n}$.

We now proceed to prove that the elements in Table 1 are indeed the smallest ones in their corresponding residue classes. Observe that

$$(2n - 10)S_{n+1} + (2n+1)S_{n+2} = (4n+3)S_n, \tag{5}$$

$$(2n+1)S_{n+1} - nS_{n+2} = (n+1)S_n, \tag{6}$$

$$(3n+1)S_{n+2} - 11S_{n+1} = (3n+2)S_n. \tag{7}$$

Consequently,

$$\begin{aligned} t_{2n-10+y,z} &\equiv t_{y,z-2n-1} \pmod{S_n} \text{ and } t_{2n-10+y,z} > t_{y,z-2n-1} \\ &\quad (2n+1 \leq z \leq 3n, 0 \leq y \leq 10), \\ t_{2n+1+y,z} &\equiv t_{y,n+z} \pmod{S_n} \text{ and } t_{2n+1+y,z} > t_{y,n+z} \\ &\quad (0 \leq y \leq 2n-11, 0 \leq z \leq 2n), \\ t_{4n-9+y,z} &\equiv t_{2n-10+y,n+z} \pmod{S_n} \text{ and } t_{4n-9+y,z} > t_{2n-10+y,n+z} \\ &\quad (0 \leq y \leq 10, 0 \leq z \leq n), \\ t_{y,z} &\equiv t_{y-11,z+3n+1} \pmod{S_n} \text{ and } t_{y,z} > t_{y-11,z+3n+1} \\ &\quad (0 \leq z \leq n-1, 11 \leq y \leq 2n). \end{aligned}$$

Therefore, the elements in Table 1 form the 0-Apéry set $\mathrm{Ap}_0(S_n, S_{n+1}, S_{n+2})$.

Now, from Table 1, there are two possibilities for the largest value of the set $\mathrm{Ap}_0(S_n, S_{n+1}, S_{n+2})$: $t_{2n,2n}$ or $t_{2n-11,3n}$. Note that, $t_{2n,2n} < t_{2n-11,3n}$ if and only if $nS_{n+2} > 11S_{n+1}$, which is equivalent to $6n^3 - 48n^2 - 53n - 11 > 0$. Observe that the roots of the equation $6n^3 - 48n^2 - 53n - 11 = 0$ are -0.7214, -0.2822, and 9.0036. Therefore, for $6 \le n \le 9$, $t_{2n,2n}$ is the largest element of the Apéry set, so by Lemma 1, we have that

$$g_0(S_n, S_{n+1}, S_{n+2}) = 2nS_{n+1} + 2nS_{n+2} - S_n.$$

While for $n \ge 10$, we have $t_{2n-11,3n}$ is the maximum among all the elements, so using Lemma 1

$$g_0(S_n, S_{n+1}, S_{n+2}) = (2n - 11)S_{n+1} + 3nS_{n+2} - S_n.$$

`Case 2:` $p = 1$. The elements of the 1-Apéry set can be determined from those of the 0-Apéry set as follows.

							$t_{2n+1,0}$	$\cdots$		$\cdots$	$t_{4n+1,0}$
							$\vdots$				$\vdots$
							$\vdots$				$\vdots$
							$t_{2n+1,n}$	$\cdots$		$\cdots$	$t_{4n+1,n}$
							$t_{2n+1,n+1}$	$\cdots$	$t_{4n-10,n+1}$		
							$\vdots$		$\vdots$		
							$t_{2n+1,2n}$	$\cdots$	$t_{4n-10,2n}$		
			$t_{2n-10,2n+1}$	$\cdots$	$\cdots$	$t_{2n,2n+1}$					
			$\vdots$			$\vdots$					
			$t_{2n-10,3n}$	$\cdots$	$\cdots$	$t_{2n,3n}$					
$t_{0,3n+1}$	$\cdots$	$t_{2n-11,3n+1}$									
$\vdots$		$\vdots$									
$t_{0,4n}$	$\cdots$	$t_{2n-11,4n}$									

Table 2: $\mathrm{Ap}_1(S_n, S_{n+1}, S_{n+2})$

From the set of Equations (5), (6), and (7) and the relation

$$(2n + 12)S_{n+1} + (2n + 1)S_n = (4n + 1)S_{n+2},$$

we have the following one-to-one correspondence between the elements of the 0-Apéry set (on the left-hand side of congruences) and that of the 1-Apéry set (on the right-hand side of congruences):

$$\begin{aligned}
&t_{y,z} \equiv t_{y+2n+1,z-n} \pmod{S_n}\\
&\quad (0 \le y \le 2n,\ n \le z \le 2n;\quad 0 \le y \le 2n-11,\ 2n+1 \le z \le 3n),\\
&t_{y,z} \equiv t_{y+2n-10,z+2n+1} \pmod{S_n} \quad (0 \le y \le 10,\ 0 \le z \le n-1),\\
&t_{y,z} \equiv t_{y-11,z+3n+1} \pmod{S_n} \quad (11 \le y \le 2n,\ 0 \le z \le n-1).
\end{aligned}$$

The elements in the first n rows of Table 1 are divided into two parts. One part is simply moved below the 0-Apéry set to fill in the gap as shown in Table 2. However, the remaining portion is moved to the lower left of the 0-Apéry set. Elements other than the first n rows of Table 1 are shifted to the right side of the 0-Apéry set by shifting up by n rows.

Set $\tau_{x,y,z} := xS_n + yS_{n+1} + zS_{n+2}$. We show that all the elements of Table 2 have at least two different representations. In fact, for $0 \le y \le 2n$, $n \le z \le 2n$ and $0 \le y \le 2n-11$, $2n+1 \le z \le 3n$, we have

$$\tau_{n+1,y,z} = \tau_{0,y+2n+1,z-n}.$$

Similarly, we get

$$\tau_{4n+3,y,z} = \tau_{0,y+2n-10,z+2n+1} \quad (\text{for } 0 \le \text{y} \le 10,\ 0 \le \text{z} \le \text{n}-1),$$
$$\tau_{3n+2,y,z} = \tau_{0,y-11,z+3n+1} \quad (\text{for } 11 \le \text{y} \le 2\text{n},\ 0 \le \text{z} \le \text{n}-1).$$

From Table 2, there are four possible choices for the largest value of $\mathrm{Ap}_1(S_n, S_{n+1}, S_{n+2})$:

$$t_{4n+1,n},\ t_{4n-10,2n},\ t_{2n,3n},\ \text{and } t_{2n-11,4n}\,.$$

Since $18n^2 + 18n - 1 > 0$, we have $t_{4n+1,n} < t_{2n,3n}$ and $t_{4n-10,2n} < t_{2n-11,4n}$. Analogous to the case $p = 0$, we obtain $t_{2n,3n} < t_{2n-11,4n}$ if and only if $n > 9$. Therefore, for $6 \le n \le 9$, the maximum element is $t_{2n,3n}$, and by Lemma 1,

$$g_1(S_n, S_{n+1}, S_{n+2}) = 2nS_{n+1} + 3nS_{n+2} - S_n.$$

When $n \ge 10$, $t_{2n-11,4n}$ is the largest of all the elements, and hence using Lemma 1,

$$g_1(S_n, S_{n+1}, S_{n+2}) = (2n-11)S_{n+1} + 4nS_{n+2} - S_n.$$

`Case 3:` $p = 2$. The elements of the 2-Apéry set can be derived from those of the 1-Apéry set in the following manner (see Table 3).

				$t_{4n+2,0}\ \cdots\ t_{6n-9,0}$	$t_{6n-8,0}\ \cdots\ t_{6n+2,0}$
				$t_{4n+2,1}\ \cdots\ t_{6n-9,1}$ $\vdots \qquad \vdots$ $t_{4n+2,n}\ \cdots\ t_{6n-9,n}$	
			$t_{4n-9,n+1}\ \cdots\ t_{4n+1,n+1}$ $\vdots \qquad \vdots$ $t_{4n-9,2n}\ \cdots\ t_{4n+1,2n}$		
		$t_{2n+1,2n+1}\ \cdots\ t_{4n-10,2n+1}$ $\vdots \qquad \vdots$ $t_{2n+1,3n}\ \cdots\ t_{4n-10,3n}$			
	$t_{2n-10,3n+1}\ \cdots\ t_{2n,3n+1}$ $\vdots \qquad \vdots$ $t_{2n-10,4n}\ \cdots\ t_{2n,4n}$				
$t_{0,4n+1}\ \cdots\ t_{2n-11,4n+1}$ $\vdots \qquad \vdots$ $t_{0,5n}\ \cdots\ t_{2n-11,5n}$					

Table 3: $\mathrm{Ap}_2(S_n, S_{n+1}, S_{n+2})$

Similar to the case when $p = 1$, the elements in the first n rows of the main part of the 1-Apéry set are subdivided into two parts and then relocated beneath the two staircase sections of the 1-Apéry set. Elements other than the first n rows of the main portion undergo a shift to the right side of the 1-Apéry set which is achieved by moving up by n rows. The other two staircase parts are shifted to the right by $(2n + 1)$ steps and upward by n steps. More precisely,

$$\begin{aligned}
&t_{y,z} \equiv t_{y+2n+1,z-n} \pmod{S_n} \\
&\quad (2n+1 \le y \le 4n+1,\ z = n; \quad 2n+1 \le y \le 4n-10,\ n+1 \le z \le 2n), \\
&t_{y,z} \equiv t_{y-11,z+3n+1} \pmod{S_n} \quad (2n+1 \le y \le 2n+11,\ 0 \le z \le n-1), \\
&t_{y,z} \equiv t_{y-2n-12,z+4n+1} \pmod{S_n} \quad (2n+12 \le y \le 4n+1,\ 0 \le z \le n-1), \\
&t_{y,z} \equiv t_{y+2n+1,z-n} \pmod{S_n} \\
&\quad (0 \le y \le 2n-11,\ 3n+1 \le z \le 4n; \quad 2n-10 \le y \le 2n,\ 2n+1 \le z \le 3n).
\end{aligned}$$

Using Equations (5), (6), and (7), we can demonstrate that the elements of the 2-Apéry set possess atleast three distinct representations. For $0 \le y \le 2n$, $z = 2n$ and $0 \le y \le 2n - 11$, $2n + 1 \le z \le 3n$, we have

$$\tau_{2n+2,y,z} = \tau_{n+1,y+2n+1,z-n} = \tau_{0,y+4n+2,z-2n}.$$

Similarly, we have

$$\begin{aligned}
&\tau_{4n+3,y,z} = \tau_{3n+2,y+2n+1,z-n} = \tau_{0,y+2n-10,z+2n+1} \\
&\quad (0 \le y \le 10,\ n \le z \le 2n-1), \\
&\tau_{3n+2,y,z} = \tau_{2n+1,y+2n+1,z-n} = \tau_{0,y-11,z+3n+1} \\
&\quad (11 \le y \le 2n,\ n \le z \le 2n-1), \\
&\tau_{5n+4,y,z} = \tau_{n+1,y+2n-10,z+2n+1} = \tau_{0,y+4n-9,z+n+1} \\
&\quad (0 \le y \le 10,\ 0 \le z \le n-1), \\
&\tau_{4n+3,y,z} = \tau_{n+1,y-11,z+3n+1} = \tau_{0,y+2n-10,z+2n+1} \\
&\quad (11 \le y \le 2n,\ 0 \le z \le n-1)\,.
\end{aligned}$$

In Table 3, by comparing the six candidates

$$t_{2n-11,5n},\ t_{2n,4n},\ t_{4n-10,3n},\ t_{4n+1,2n},\ t_{6n-9,n},\ \text{and } t_{6n+2,0},$$

we find that $t_{2n,4n}$ is the largest element of the 2-Apéry set, when $6 \le n \le 9$, and $t_{2n-11,5n}$ is the largest element of the 2-Apéry set, when $n \ge 10$. Therefore,

$$g_2(S_n, S_{n+1}, S_{n+2}) = \begin{cases} 2nS_{n+1} + 4nS_{n+2} - S_n, & \text{if } 6 \le n \le 9; \\ (2n-11)S_{n+1} + 5nS_{n+2} - S_n, & \text{if } n \ge 10. \end{cases}$$

This completes the proof. $\square$

Remark 3. For $p \geq 3$, the elements of the 3-Apéry set can also be computed from that of the 2-Apéry set. However, the uniform pattern observed in previous cases does not persist, as illustrated by Table 4.

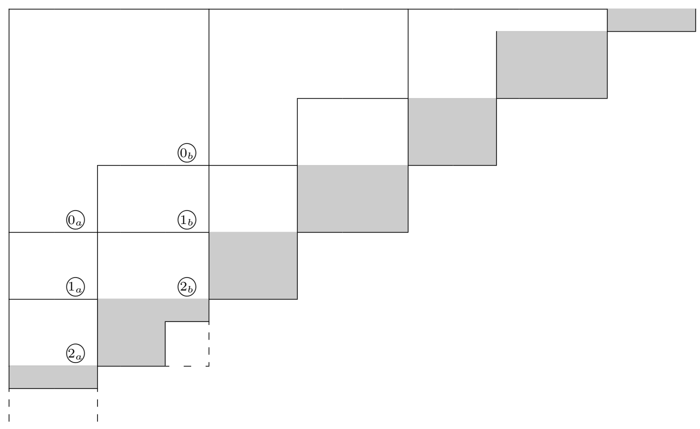

Table 4: $\mathrm{Ap}_3(S_n, S_{n+1}, S_{n+2})$

When $n \geq 10$, elements 0_a, 1_a, and 2_a indicate the position of the largest elements of the 0-, 1-, and 2-Apéry sets, respectively, and when $6 \leq n \leq 9$, elements 0_b, 1_b, and 2_b represent the corresponding largest elements. We can see that no element of the 3-Apéry set can exist at the expected location for the largest value. Thus, for $p \geq 3$, the application of the same formula becomes impractical and the situation becomes more and more complicated, though the p-Frobenius numbers should exist.

Remark 4. For $n = 2, 3, 4, 5$, the Frobenius number $g_p(S_n, S_{n+1}, S_{n+2})$ is computed explicitly and the elements of the 0-Apéry set are given in Table 5, Table 6, Table 7, and Table 8, respectively.

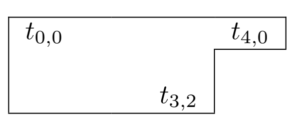

Table 5: $\mathrm{Ap}_0(S_2, S_3, S_4)$

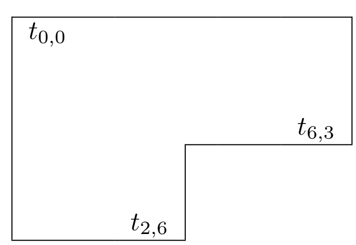

Table 6: $\mathrm{Ap}_0(S_3, S_4, S_5)$

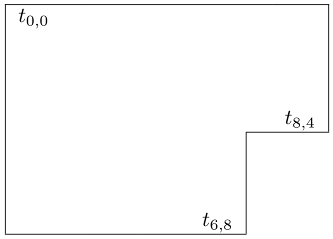

Table 7: $\mathrm{Ap}_0(S_4, S_5, S_6)$

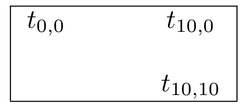

Table 8: $\mathrm{Ap}_0(S_5, S_6, S_7)$

In each case, comparing the possible two candidates, we can find that $t_{3,2}$, $t_{2,6}$, $t_{6,8}$, and $t_{10,10}$ take the largest values of the 0-Apéry sets when $n = 2, 3, 4, 5$, respectively. Therefore, we have

$$\begin{aligned} g_0(S_2, S_3, S_4) &= 3S_3 + 2S_4 - S_2 = 244, \\ g_0(S_3, S_4, S_5) &= 2S_4 + 6S_5 - S_3 = 835, \\ g_0(S_4, S_5, S_6) &= 6S_5 + 8S_6 - S_4 = 2101, \\ g_0(S_5, S_6, S_7) &= 10S_6 + 10S_7 - S_5 = 4219. \end{aligned}$$

When $p \geq 0$, we can find that

$$\begin{aligned} &\{g_p(S_2, S_3, S_4)\}_{p=0}^{10} = 244, 353, 426, 499, 538, 608, 647, 686, 725, 764, 801, \\ &\{g_p(S_3, S_4, S_5)\}_{p=0}^{10} = 835, 1198, 1417, 1565, 1780, 1928, 2076, 2224, 2345, 2466, 2587, \\ &\{g_p(S_4, S_5, S_6)\}_{p=0}^{10} = 2101, 2825, 3307, 3672, 4037, 4402, 4761, 5126, 5418, 5660, 5902\,. \end{aligned}$$

However, identifying any consistent pattern for the maximum element of the p-Apéry set appears to be challenging. For example, the corresponding elements of the p-Apéry set for $\{S_4, S_5, S_6\}$ are given by

$$\begin{aligned} &t_{6,8} \to t_{6,12} \to t_{4,16} \to t_{13,12} \to t_{22,8} \to t_{31,4} \\ &\quad \to t_{22,12} \to t_{31,8} \to t_{2,29} \to t_{4,29} \to t_{6,29}. \end{aligned}$$

3.2. *p*-Sylvester Numbers

The p-Sylvester numbers for the numerical semigroups generated by the triplets of successive star numbers are given below.

Theorem 2. *For $n \geq 6$ and $p = 0, 1, 2$, we have*

$$n_p(S_n, S_{n+1}, S_{n+2}) = \begin{cases} n(15n^2 + 8n + 7), & \text{if } p = 0; \\ 25n^3 + 23n^2 + 14n + 1, & \text{if } p = 1; \\ 33n^3 + 35n^2 + 20n + 2, & \text{if } p = 2. \end{cases}$$

Proof. We deal with each case separately.
`Case 1:` $p = 0$. By using Table 1, we have

$$
\begin{aligned}
\sum_{j=0}^{S_n-1} m_j^{(0)} &= \sum_{y=0}^{10}\sum_{z=0}^{n-1} t_{y,z} + \sum_{y=11}^{2n}\sum_{z=0}^{n-1} t_{y,z} + \sum_{y=0}^{2n}\sum_{z=n}^{2n} t_{y,z} + \sum_{y=0}^{2n-11}\sum_{z=2n+1}^{3n} t_{y,z} \\
&= n^2(2n+1)S_{n+1} + \left[\frac{n(n-1)(2n+1)}{2}\right] S_{n+2} \\
&\quad + n(n+1)(2n+1)S_{n+1} + \left[\frac{3n(2n+1)(n+1)}{2}\right] S_{n+2} \\
&\quad + n(n-5)(2n-11)S_{n+1} + n(5n+1)(n-5)S_{n+2} \\
&= n(6n^2 - 17n + 56)S_{n+1} + n(9n^2 - 20n - 4)S_{n+2} \\
&= n(15n^2 + 11n + 4)S_n.
\end{aligned}
$$

Using Lemma 1, the 0-Sylvester number is given as

$$
\begin{aligned}
n(S_n, S_{n+1}, S_{n+2}) &= n(15n^2 + 11n + 4) - \frac{S_n - 1}{2} \\
&= n(15n^2 + 8n + 7).
\end{aligned}
$$

`Case 2:` $p = 1$. Summing up the elements of the 1-Apéry set (see Table 2), we obtain

$$
\begin{aligned}
\sum_{j=0}^{S_n-1} m_j^{(1)} &= \sum_{y=0}^{10}\sum_{z=0}^{n-1} t_{y+2n-10,z+2n+1} + \sum_{y=11}^{2n}\sum_{z=0}^{n-1} t_{y-11,z+3n+1} \\
&\quad + \sum_{y=0}^{2n}\sum_{z=n}^{2n} t_{y+2n+1,z-n} + \sum_{y=0}^{2n-11}\sum_{z=2n+1}^{3n} t_{y+2n+1,z-n} \\
&= \sum_{j=0}^{S_n-1} m_j^{(0)} + 11n(4n+3)S_n + n(2n-10)(3n+2)S_n \\
&\quad + (n+1)^2(2n+1)S_n + n(n+1)(2n-10)S_n \\
&= n(15n^2 + 11n + 4)S_n + (2n+1)(5n^2 + 5n + 1)S_n \\
&= (25n^3 + 26n^2 + 11n + 1)S_n.
\end{aligned}
$$

Using Lemma 1, the 1-Sylvester number is given by

$$
\begin{aligned}
n_1(S_n, S_{n+1}, S_{n+2}) &= (25n^3 + 26n^2 + 11n + 1) - \frac{S_n - 1}{2} \\
&= 25n^3 + 23n^2 + 14n + 1.
\end{aligned}
$$

Case 3: $p = 2$. Using Table 3, we get

$$\begin{aligned}
\sum_{j=0}^{S_n-1} m_j^{(2)} &= \sum_{y=0}^{10}\sum_{z=0}^{n-1} t_{y+2n-10,z+3n+1} + \sum_{y=11}^{2n}\sum_{z=0}^{n-1} t_{y-11,z+4n+1} \\
&\quad + \sum_{y=0}^{2n}\sum_{z=2n} t_{y+2(2n+1),z-2n} + \sum_{y=0}^{2n-11}\sum_{z=2n+1}^{3n} t_{y+2(2n+1),z-2n} \\
&\quad + \sum_{y=0}^{2n-11}\sum_{z=n}^{2n-1} t_{y+2n+1,z+n+1} + \sum_{y=2n-10}^{2n}\sum_{z=n}^{2n-1} t_{y+2n+1,z+1} \\
&= \Big[\ 11n(4n+3) + n(2n-10)(3n+2) + 2(2n+1)(n+1) \\
&\qquad + 2n(n+1)(2n-10)\Big] S_n \\
&\quad + \Big[n(2n+1)(n+1) + 4n(n-5)(2n-5) + 44n(n-1)\Big] S_{n+1} \\
&\quad + \Bigg[n(2n+1)(n+2) + 2n(5n+1)(n-5) + \frac{n(n-1)(2n+1)}{2} \\
&\qquad + \frac{11n(3n+1)}{2}\Bigg] S_{n+2} \\
&= \left(10n^3 + 6n^2 - n + 2\right) S_n + (10n^3 - 13n^2 + 57n) S_{n+1} \\
&\quad + (13n^3 - 27n^2 - 3n)\, S_{n+2} \\
&= (10n^3 + 6n^2 - n + 2) S_n + (23n^3 + 32n^2 + 18n) S_n \\
&= (33n^3 + 38n^2 + 17n + 2) S_n.
\end{aligned}$$

Therefore, by Lemma 1, the 2-Sylvester number is given as

$$\begin{aligned}
n_2(S_n, S_{n+1}, S_{n+2}) &= (33n^3 + 38n^2 + 17n + 2) - \frac{S_n - 1}{2} \\
&= 33n^3 + 35n^2 + 20n + 2.
\end{aligned}$$

□

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