ON THE TWO-COLOR DISJUNCTIVE RADO NUMBER FOR THE EQUATIONS $\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c_j$, $j = 1, 2$

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Abstract

Given a system of linear equations $\mathscr{S}$, the disjunctive Rado number for the system $\mathscr{S}$ is the least positive integer $R = \mathscr{R}(\mathscr{S})$, if it exists, such that every 2-coloring of the integers in $[1, R]$ admits a monochromatic solution to at least one equation in $\mathscr{S}$. We determine $\mathscr{R}(\mathscr{S})$ when $\mathscr{S}$ is the pair of equations $\big\{ \sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c_1, \sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c_2 \big\}$ for some range of values of c_1 and c_2.

1. Introduction

By an *r-coloring* of $\{1, \ldots, N\}$ we mean a mapping $\chi : \{1, \ldots, N\} \to \{1, \ldots, r\}$. In 1916, Schur showed that for every positive integer r, there exists a least positive integer $s = s(r)$ such that for every r-coloring of the integers in the interval $[1, s]$, there exists $x, y, x + y \in [1, s]$ such that $\chi(x) = \chi(y) = \chi(x + y)$. Schur's theorem was generalized in a series of results in the 1930's by Rado leading to a complete resolution to the following problem: characterize systems of linear homogeneous equations with integral coefficients $\mathscr{S}$ such that for a given positive integer r, there exists a least positive integer $n = \mathscr{R}(\mathscr{S}; r)$ such that every r-coloring of the integers in the interval $[1, n]$ yields a monochromatic solution to the system $\mathscr{S}$. There has been a growing interest in the determination of the Rado numbers $\mathscr{R}(\mathscr{S}; r)$, particularly when $\mathscr{S}$ is a single equation and $r = 2$; for instance, see [1, 6, 7, 8, 9, 10, 12]. When $r = 2$, we denote this number simply by $\mathscr{R}(\mathscr{S})$.

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The problem of disjunctive Rado numbers was introduced by Johnson and Schaal in [11]. The 2-color *disjunctive Rado number* for the set of equations $\mathcal{E}_1, \ldots, \mathcal{E}_k$ is the least positive integer N such that any 2-coloring of $\{1, \ldots, N\}$ admits a monochromatic solution to at least one of the equations $\mathcal{E}_1, \ldots, \mathcal{E}_k$; we denote this by $\mathscr{R}(\mathcal{E}_1, \ldots, \mathcal{E}_k)$. Johnson and Schaal gave necessary and sufficient conditions for the existence of the 2-color disjunctive Rado number for the additive equations $x_1 - x_2 = a$ and $x_1 - x_2 = b$ for all pairs of distinct positive integers a, b, and also determined exact values when it exists. They also determined exact values for the pair of multiplicative equations $ax_1 = x_2$ and $bx_1 = x_2$ whenever a, b are distinct positive integers; for alternate proofs, see [2]. Dileep, Moondra and Tripathi [3] extended the results of Johnson and Schaal to the set of equations $x_1 - x_2 = a_i$, $1 \le i \le k$, giving conditions for the existence of the 2-color disjunctive Rado number, exact values in some cases, and upper and lower bounds in all cases. They also investigated and obtained parallel results for the set of multiplicative equations $y = a_i x$, $1 \le i \le k$. Further, they gave a general search-based algorithm with a run time of $O(ka_k \log a_k)$ for the case of additive equations, which is exponentially better than the brute-force algorithm for the problem. Lane-Harvard and Schaal [14] determined exact values of 2-color disjunctive Rado number for the pair of equations $ax_1 + x_2 = x_3$ and $bx_1 + x_2 = x_3$ for all distinct positive integers a, b. Sabo, Schaal and Tokaz [15] determined exact values of 2-color disjunctive Rado number for $x_1 + x_2 - x_3 = c_1$, and $x_1 + x_2 - x_3 = c_2$, whenever c_1, c_2 are distinct positive integers. Kosek and Schaal [13] determined the exact value of 2-color disjunctive Rado number for the equations $x_1 + \cdots + x_{m-1} = x_m$, and $x_1 + \cdots + x_{n-1} = x_n$, for all pairs of distinct positive integers m, n.

Schaal and Zinter [16] studied the 2-color Rado number for the equation $x_1 + 3x_2 + c = x_3$ for $c \ge 3$, giving a lower bound in all cases and upper bounds in some. Dwivedi and Tripathi [4] generalized this to investigate the 2-color Rado number for the equation $x_1 + ax_2 - x_3 = c$ for positive integers a, giving conditions for existence, upper and lower bounds in all cases, and exact results in a few. The same authors [5] further generalized this to investigate the 2-color Rado number for the equation $\sum_{i=1}^{m-2} x_i + ax_{m-1} + x_m = c$ when $4 \le m \le a$. They give a necessary and sufficient condition for the Rado number to exist, give upper and lower bounds in all cases, and exact values in many cases. This paper investigates the disjunctive Rado number problem for the pair of equations $\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c_1$ and $\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c_2$. We reproduce some pertinent results from [5] for ready reference.

Theorem 1 ([5, Theorem 1]). Let $a, c, m \in \mathbb{Z}$, and $4 \le m \le a$. If $a + m$, and c are both odd, then

$$\mathscr{R}\left(\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c\right)$$

does not exist.

Proposition 2 ([5, Proposition 1]). For $a \in \mathbb{N}$, and $4 \leq m \leq a$,

$$\mathscr{R}\left(\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = a + m - 3\right) = 1.$$

Theorem 3 ([4, Theorem 3], [5, Theorem 5]).
Let a, m be integers of the same parity, with $a \geq 3$, and $m \geq 3$. Let $a' = a + m - 3$. If either of

(i) $m = 3$, and $c \leq -\frac{a(a-3)}{2}$;
(ii) $m \geq 4$, and $c < -(a' + 3)(a - 2)$

is true, then

$$\mathscr{R}\left(\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c\right) = (a' + 3)(a' - c) + 1.$$

2. Results for $\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c_j$, $j = 1, 2$

We study the disjunctive Rado number for the pair of equations

$$\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c_1, \tag{1a}$$

$$\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c_2, \tag{1b}$$

where $a \geq 3$, $m \geq 3$, and c_1, c_2 are any integers. Throughout this paper, we denote this 2-color Rado number by $\texttt{Rad}_2\big(\underbrace{1, \ldots, 1}_{m-2 \text{ times}}, a, -1; c_1, c_2\big)$, or more briefly by $\mathscr{R}(c_1, c_2)$.

By assigning the color of x_i in the solution of Equation (1a) and Equation (1b) to $x_i - 1$, we note that this is equivalent to determining the smallest positive integer R for which every 2-coloring of $[0, R-1]$ contains a monochromatic solution to

$$\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c_1', \tag{2a}$$

or

$$\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c_2', \tag{2b}$$

where $c_j' = c_j - a'$, $j \in \{1, 2\}$, and $a' = a + m - 3$.

Proposition 4. *Let $a, \lambda, n \in \mathbb{N}$ such that $a \geq 3$, $n \geq 1$, and $\lambda \geq a-1$. Then for each $N \in \{0, \dots, \lambda(a+n)\}$ the equation*

$$\sum_{i=1}^{n} x_i + a x_{n+1} = N \tag{3}$$

admits a solution with each $x_i \in \{0, \dots, \lambda\}$.

Proof. If $N = \lambda(a+n)$, then $x_i = \lambda$ for $i \in \{1, \dots, n+1\}$ is a solution to Equation (3). If $0 \leq N < \lambda(a+n)$, we can write $N = q(a+n) + \epsilon a + r$, where $0 \leq q < \lambda$, $0 \leq r \leq n$, and $\epsilon \in \{0, 1\}$. Then, $x_i = q+1$ for $1 \leq i \leq r$, $x_i = q$ for $r+1 \leq i \leq n$, and $x_{n+1} = q + \epsilon$ is a solution to Equation (3). □

Theorem 5. *Let $4 \leq m \leq a$, and $c_j = k_j(a+m-3)$ with $1 < k_j \leq a+m-2$ for $j \in \{1, 2\}$. Then,*

$$\mathscr{R}(c_1, c_2) = \min\{k_1, k_2\}.$$

Proof. Let $k = \min\{k_1, k_2\}$. The coloring $\Delta : [1, k-1] \to \{0, 1\}$ defined by $\Delta(x) = 0$ is a valid coloring, since

$$\begin{aligned} \sum_{i=1}^{m-2} x_i + a x_{m-1} - x_m &\leq (a+m-2)(k-1) - 1 \\ &= k(a+m-3) + (k-2) - (a+m-3) \\ &< k(a+m-3). \end{aligned}$$

Hence, $\mathscr{R}(c_1, c_2) \geq k$.

On the other hand, since $x_1 = \cdots = x_m = k$ satisfies Equations (1a) or (1b) for $c_j = k(a+m-3)$, $j = 1, 2$, every coloring $\chi : [1, k] \to \{0, 1\}$ admits a monochromatic solution to Equations (1a) or (1b). Hence, $\mathscr{R}(c_1, c_2) \leq k$. □

Theorem 6. *Let a, m be integers of the same parity, with $a \geq 3$, and $m \geq 4$. Let $a' = a+m-3$, and $c'_j = c_j - a'$ for $j \in \{1, 2\}$. Then, for $c_1 < -(a'+3)(a-2)$,*

$$\mathscr{R}(c_1, c_2) =$$

$$\begin{cases} (a'+3)(a'-c_1)+1 & \text{if } c_1 - a' \leq c_2 \leq c_1, & \text{(4a)} \\ (a'+2)(a'-c_1)+1 & \text{if } (a'+2)c_1 - a'(a'+1) \leq c_2 < c_1 - a', & \text{(4b)} \\ (a'-c_2)+1 & \text{if } (a'+3)c_1 - a'(a'+2) < c_2 < (a'+2)c_1 - a'(a'+1), & \text{(4c)} \\ (a'+3)(a'-c_1)+1 & \text{if } c_2 \leq (a'+3)c_1 - a'(a'+2). & \text{(4d)} \end{cases}$$

Proof. We note that $a' = a+m-3$, and that

$$\mathscr{R}(c_1, c_2) \leq \min\{\mathscr{R}(c_1), \mathscr{R}(c_2)\} = (a'+3)(a'-c_1)+1 = -(a'+3)c'_1 + 1$$

by Theorem 1. We consider two cases: (I) given by Equation (4a) and Equation (4d), and (II) given by Equation (4b) and Equation (4c). Thus, in Case I, it suffices to prove that $\mathscr{R}(c_1, c_2) \geq (a'+3)(a'-c_1)+1$ to complete the proof of Case I.

`Case I.` We exhibit a valid coloring of $[1,(a+m)(a+m-c_1-3)]$ with respect to Equation (1a) and (1b). Let $\Delta : [1,(a+m)(a+m-c_1-3)] \to \{0,1\}$ be defined by

$$\Delta(x) = \begin{cases} 0 & \text{if } x \in [1, a+m-c_1-3] \\ & \quad \bigcup\, [(a+m-1)(a+m-c_1-3)+1, (a+m)(a+m-c_1-3)], \\ 1 & \text{if } x \in [a+m-c_1-2, (a+m-1)(a+m-c_1-3)]. \end{cases}$$

Let $A = [1, a+m-c_1-3]$, $B = [a+m-c_1-2, (a+m-1)(a+m-c_1-3)]$, and $C = [(a+m-1)(a+m-c_1-3)+1, (a+m-1)(a+m-c_1-3)]$.

Suppose $x_1, \ldots, x_m$ is a solution to Equation (1a), with $\Delta(x_1) = \cdots = \Delta(x_m)$. Suppose $\Delta(x_i) = 0$ for $i \in \{1, \ldots, m\}$. If $x_1, \ldots, x_{m-1}$ all belong to A, then

$$\begin{aligned} a+m-c_1-2 \leq x_m &= \sum_{i=1}^{m-2} x_i + a x_{m-1} - c_1 \\ &\leq (a+m-2)(a+m-c_1-3) - c_1 \\ &\leq (a+m-1)(a+m-c_1-3). \end{aligned}$$

Hence, $x_m \in B$, and so $\chi(x_m) = 1$.

If at least one of $x_1, \ldots, x_{m-1}$ belongs to C and $\min C$ denotes the least element in C, then,

$$x_m = \left(\sum_{i=1}^{m-2} x_i\right) + a x_{m-1} - c_1 \geq (a+m-c_1-3) + \min C = (a+m)(a+m-c_1-3)+1.$$

Hence, x_m is outside the domain of Δ. Therefore, $\Delta(x_i) = 1$ for $i \in \{1, \ldots, m\}$, and so

$$x_m = \left(\sum_{i=1}^{m-2} x_i\right) + a x_{m-1} - c_1 \geq (a+m-2)\cdot \min B - c_1 \geq (a+m-1)(a+m-c_1-3)+1,$$

where $\min B$ denotes the least element in B. Hence, $x_m \in$ C. This proves that Δ is a valid coloring of $[1,(a+m)(a+m-c_1-3)]$ with respect to Equation (1a). For Equation (4a), the same argument applies with respect to Equation (1b). For Equation (4d)

$$x_m = \sum_{i=1}^{m-2} x_i + a x_{m-1} - c_2 > (a+m)(a+m-c_1-3).$$

Hence, x_m is outside the domain of Δ. Thus, Δ is a valid coloring of $[1,(a+m)(a+m-c_1-3)]$ with respect to Equation (1b). This concludes the proof of Case I.

`Case II.` The coloring Δ, with suitable modifications, also provides a valid coloring for Equation (4b) and (4c). For Equation (4b), we consider the function Δ, restricted to $[1,(a+m-1)(a+m-c_1-3)] = A \cup B$. A sub-argument used in Equation (4a) shows that this is a valid coloring for Equation (1a) and (1b). For Equation (4c), we consider the function Δ, restricted to $[1, a+m-c_2-3] = A \cup B \cup C'$, where $C' = [(a+m-1)(a+m-c_1-3)+1, a+m-c_2-3]$. An argument similar to the one for Equation (4a) shows that this is a valid coloring for Equation (1a) and (1b). Since we have provided valid colorings for Case II, it only remains to prove the upper bounds in this case.

By assigning the color of x_i in the solution of Equation (1a) and (1b) to $x_i - 1$, we equivalently consider monochromatic solutions to Equation (2a) and (2b), respectively, under colorings that start with $x = 0$. Let $\chi : [0, -(a'+3)c_1'] \to \{0,1\}$ be any 2-coloring of $[0, -(a'+3)c_1']$. Without loss of generality, let $\chi(0) = 0$. Each step in the following sequence forces a color on some number in the given range in order to avoid a mononchromatic solution to Equation (2a) or (2b):

- $x_i = 0$ for $1 \le i \le m-1$ implies $\chi(-c_1') = 1$ and $\chi(-c_2') = 1$;
- $x_i = -c_1'$ for $1 \le i \le m-1$ implies $\chi\big(-(a'+2)c_1'\big) = 0$;
- $x_i = 0$ for $2 \le i \le m-1$, and $x_m = -(a'+2)c_1'$ implies $\chi\big(-(a'+1)c_1'\big) = 1$.

We capture this information in Table 1.

0	**1**
0	$-c_j'$
$-(a'+2)c_j'$	$-(a'+1)c_j'$

Table 1

To complete the proof in Case II, we must show that:

- every 2-coloring of $\chi : [0, -(a'+2)c_1'] \to \{0,1\}$ must yield a monochromatic solution to one of Equation (2a), (2b) for $-(c_1' - a') < -c_2' \le -(a'+2)c_1'$, and
- every 2-coloring of $\chi : [0, -c_2'] \to \{0,1\}$ must yield a monochromatic solution to one of Equation (2a), (2b) for $-(a'+2)c_1' < -c_2' < -(a'+3)c_1'$.

We have assumed, without loss of generality, that $\chi(0) = 0$. There are two possibilities for $\chi(1)$, of which the case $\chi(1) = 1$ is common to Equation (4b) and (4c). We first assume $\chi(1) = 1$. We claim that

$$\chi\big(-tc_1' - a'\big) = \begin{cases} 0 & \text{if } t \text{ is odd;} \\ 1 & \text{if } t \text{ is even} \end{cases}$$

for $t \in \{1, \ldots, a'\}$. With each $x_i = 1$, $2 \le i \le m-1$, and $x_m = -(a'+1)c_1'$ in Equation (2a), we have $x_1 = -a'(c_1'+1)$, forcing $\chi\big(-a'c_1' - a'\big) = 0$ in order to avoid a monochromatic coloring. This proves the claim for $t = a'$.

Suppose $t \in \{3, \ldots, a'\}$, t is odd, and that $\chi\big(-tc_1' - a'\big) = 0$. We begin the inductive step at $t = a'$. To complete the claim, we show that if $\chi\big(-tc_1' - a'\big) = 0$, then $\chi\big(-(t-1)c_1' - a'\big) = 1$ and $\chi\big(-(t-2)c_1' - a'\big) = 0$ for $t \in \{3, \ldots, a'\}$. Each step in the following sequence forces a color on some number in the given range in order to avoid a mononchromatic solution to Equation (2a):

- $x_i = 0$ for $2 \le i \le m-1$, and $x_m = -tc_1' - a'$ implies $\chi\big(-(t-1)c_1' - a'\big) = 1$;
- $x_1 = -(t-1)c_1' - a'$, and $x_i = 1$ for $2 \le i \le m-1$ implies $\chi\big(-tc_1'\big) = 0$;
- $x_i = 0$ for $2 \le i \le m-1$, and $x_m = -tc_1'$ implies $\chi\big(-(t-1)c_1'\big) = 1$;
- $x_i = 1$ for $2 \le i \le m-1$, and $x_m = -(t-1)c_1'$ implies $\chi\big(-(t-2)c_1' - a'\big) = 0$.

In particular, from the above claim, $\chi\big(-c_1' - a'\big) = 0$. We note that $\chi\big((a+m-1)c_1'\big) = 0$ from Table 1. Each step in the following sequence forces a color on some number in the given range in order to avoid a mononchromatic solution to Equation (2a) or (2b):

- $x_1 = -c_1' - a'$, $x_i = -c_1' + 1$ for $2 \le i \le m-1$, and $x_m = -(a+m-1)c_1'$ implies $\chi\big(-c_1' + 1\big) = 1$;
- $x_1 = -c_1' + 1$, and $x_i = 1$ for $2 \le i \le m-1$ implies $\chi\big(-2c_1' + (a'+1)\big) = 0$;
- $x_i = 0$ for $2 \le i \le m-1$, and $x_m = -2c_1' + (a'+1)$ implies $\chi\big(-c_1' + (a'+1)\big) = 1$.

Now $x_i = 1$ for $1 \le i \le m-1$, and $x_m = -c_1' + (a'+1)$ forms a monochromatic solution to Equation (2a). This completes the proof when $\chi(1) = 1$. For the remainder of the proof, we consider the case when $\chi(1) = 0$. We claim that

$$\chi(n) = 0 \text{ for } 0 \le n \le \left\lfloor \frac{-2c_1'}{a'} \right\rfloor = K. \tag{5}$$

By way of contradiction, assume $\chi(n) = 1$ for some $n \le K$. We claim this implies

$$\chi\big(-tc_1'\big) = \begin{cases} 0 & \text{if } t \text{ is even;} \\ 1 & \text{if } t \text{ is odd} \end{cases}$$

for $t \in \{1, \ldots, a'\}$. From Table 1, we have $\chi(-c_1') = 1$. Let $t \in \{1, \ldots, a'-2\}$. Assuming $\chi(-tc_1') = 1$ when t is odd, we show that $\chi\big(-(t+1)c_1'\big) = 0$ and $\chi\big(-(t+2)c_1'\big) = 1$. Each step in the following sequence forces a color on some number in the given range in order to avoid a mononchromatic solution to Equation (2a):

- $x_1 = -tc_1'$, and $x_i = n$ for $2 \le i \le m-1$ implies $\chi\big(-(t+1)c_1' + na'\big) = 0$;

- $x_1 = -(t+1)c_1'+na'$, and $x_i = 0$ for $2 \le i \le m-1$ implies $\chi\big(-(t+2)c_1'+na'\big) = 1$;
- $x_i = n$ for $2 \le i \le m-1$, and $x_m = -(t+2)c_1'+na'$ implies $\chi\big(-(t+1)c_1'\big) = 0$;
- $x_1 = -(t+1)c_1'$, and $x_i = 0$ for $2 \le i \le m-1$ implies $\chi\big(-(t+2)c_1'\big) = 1$.

In particular, we have $\chi\big(-a'c_1'\big) = 1$. We note that the maximum allowable value of numbers used is $-a'c_1' + Ka'$ from the second step, and this lies in the domain of χ. In order that the numbers lie in the domain of χ, we must have $-a'c_1' + na' \le -(a'+2)c_1'$, in particular. This implies $n \le -\frac{2c_1'}{a'}$. To complete the claim that $\chi(n) = 0$ for $0 \le n \le K$, we show that $\chi\big(-a'c_1'\big) = 0$ by using Table 1. Each step in the following sequence forces a color on some number in the given range in order to avoid a mononchromatic solution to Equation (2a) or (2b):

- $x_i = n$ for $2 \le i \le m-1$, and $x_m = -(a'+1)c_1'$ implies $\chi\big(-a'c_1' - na'\big) = 0$;
- $x_i = 0$ for $2 \le i \le m-1$, and $x_m = -a'c_1'-na'$ implies $\chi\big(-(a'-1)c_1'-na'\big) = 1$;
- $x_1 = -(a'-1)c_1' - na'$, and $x_i = n$ for $2 \le i \le m-1$ implies $\chi\big(-a'c_1'\big) = 0$.

This contradiction completes the proof of the claim that $\chi(n) = 0$ for $0 \le n \le K$. It can be shown that $a \le K$, and so we have $\chi(n) = 0$ for $0 \le n \le a$, in particular. For the rest of this proof, we consider Equation (4b) and (4c) separately.

We first consider Equation (4b). By assigning the color of x_i in the solution of Equation (1a) and (1b) to $x_i - 1$, we note that the ranges of c_1 and c_2 translate to

$$-c_1' + a' + 1 \le -c_2' \le -(a'+2)c_1'.$$

We use $\chi(n) = 0$ for $0 \le n \le a$, and prove that $\chi(n) = 0$ for $a+1 \le n \le -c_1'-1$. Let $t+1 = \min\{n : \chi(n) = 1\}$; we have shown that $t+1 > a$. By way of contradiction, we may assume $t+1 \le -c_1'-1$. By Proposition 4, the expression $\sum_{i=1}^{m-2} x_i + ax_{m-1}$ assumes every value in the interval $[0, (a'+1)t]$ as each x_i runs over the set $\{0, \dots, t\}$. Under the same range for the x_i's, the expression

$$\sum_{i=1}^{m-2} x_i + ax_{m-1} - c_2' = x_m$$

assumes every value in the interval $\mathcal{I} = [-c_1' + a' + 1, (a'+1)t - (a'+2)c_1']$. So in order to avoid a monochromatic solution to Equation (2b), we must have $\chi(n) = 1$ for each $n \in \mathcal{I}$. Now choosing $x_i = t+1$, $1 \le i \le m-1$ in Equation (2a) forces $\chi\big((a'+1)(t+1) - c_1'\big) = 0$ in order to avoid a mononchromatic solution. But $(a'+1)(t+1) - c_1'$ lies within $[-c_1' + a' + 1, (a'+1)t - (a'+2)c_1']$, and this is a contradiction to the conclusion from the previous paragraph. Therefore, we have the claim that $\chi(n) = 0$ for $0 \le n \le -c_1' - 1$.

From the above argument for $t = -c_1' - 1$, the expression

$$\sum_{i=1}^{m-2} x_i + ax_{m-1} - c_2' = x_m$$

assumes every value in the interval $\mathcal{J} = [-c_1' + a' + 1, -(a'+1)(c_1'+1) - (a'+2)c_1']$. Since $-(a'+2)c_1' \in \mathcal{J}$, there exist $x_1, \ldots, x_{m-1}$, with each $x_i \in \{0, \ldots, -c_1'-1\}$, such that $\sum_{i=1}^{m-2} x_i + ax_{m-1} - c_2' = -(a'+2)c_1'$. This gives a monochromatic solution to Equation (2b), since $\chi\big(-(a'+2)c_1'\big) = 0$ by Table 1. This completes the argument for Equation (4b).

We now consider Equation (4c). By assigning the color of x_i in the solution of Equation (1a) and (1b) to $x_i - 1$, we note that the ranges of c_1 and c_2 translate to

$$-(a'+2)c_1' < -c_2' \leq -(a'+3)c_1' - 1.$$

Recall that $\chi(n) = 0$ for $0 \leq n \leq \left\lfloor \frac{-2c_1'}{a'} \right\rfloor$. Each step in the following sequence forces a color on some number in the given range in order to avoid a mononchromatic solution to Equation (2a):

- $x_i = -c_1'$ for $2 \leq i \leq m-1$, and $x_m = -c_2'$ implies $\chi\big((a'+1)c_1' - c_2'\big) = 0$;
- $x_1 = (a'+1)c_1' - c_2'$, and $x_i = 0$ for $2 \leq i \leq m-1$ implies $\chi\big(a'c_1' - c_2'\big) = 1$.

We capture this information in Table 2.

0	**1**
0	$-c_1'$
$-(a'+2)c_1'$	$-c_2'$
$(a'+1)c_1' - c_2'$	$-(a'+1)c_1'$

Table 2

Arguing as in Equation (4b), with $t = K$, the expression

$$\sum_{i=1}^{m-2} x_i + ax_{m-1} - c_2' = x_m$$

assumes every value in the interval $\mathcal{K} = [-c_2', (a'+1)K - c_2']$. Since $(a'+1)c_1' - c_2' \in \mathcal{K}$, there exist $x_1, \ldots, x_{m-1}$, with each $x_i \in \{0, \ldots, K\}$, such that $\sum_{i=1}^{m-2} x_i + ax_{m-1} - c_2' = (a'+1)c_1' - c_2'$. This gives a monochromatic solution to Equation (2b), since $\chi\big((a'+1)c_1' - c_2'\big) = 0$ by Table 2. This completes the argument in Equation (4c), thereby completing the proof of Theorem 6. □

3. Open Problems

We close our paper by suggesting a few directions for further investigation to the interested reader. To keep the list short, we do not discuss variants of the classical Rado numbers. Although we discuss these problems in the case of two colors, each can be asked in the general case. However, given the difficulty for a complete resolution even in the case of two colors, the general case seems quite challenging at present.

1. The problem of determination of the Rado number for the equation $x_1+ax_2-x_3=c$, and more generally, for the equation $\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c$, was considered in [4] and [5]. However, exact results could not be determined in some cases. It would be desirable to determine the exact value of the Rado number in all cases.

2. The results in this paper concern the Rado number of the pair of equations $\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c_j$, $j = 1, 2$. Theorem 6 gives exact values for the Rado number, but only for all values of c_1 and c_2 in a limited range. It would be desirable to determine the exact value of the Rado number for a larger range of values of c_1 and c_2, perhaps even for all such pairs.

3. The choice of the pair of equations in this paper is based on the existence of the Rado number for the equation $\sum_{i=1}^{m-2} x_i+ax_{m-1}-x_m = c$ for a and c in certain range; see [5]. For a fixed value of a, we have considered various c in this paper. We may consider instead a fixed value of c, and various a, or even vary both a and c, i.e., consider the pair of equations $\sum_{i=1}^{m-2} x_i + a_j x_{m-1} - x_m = c_j$, $j = 1, 2$.

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