DOUBLE PERFECT PARTITIONS OF HIGHER ORDER

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Abstract

A partition of a positive integer n is called double-perfect if the summands contain two partitions of every integer between 2 and $n-2$. In this paper we give new derivations of known results on double-perfect partitions. Then we consider generalized double-perfect partitions of n in which the summands contain two partitions of every integer between r and $n-r$, where $2 \leq r < n/2$. Our results include explicit characterizations of double-perfect partitions of all orders and a seemingly new class of pseudo-perfect partitions that produce double-perfect partitions. We also state an inclusive enumeration formula in terms of ordered factorization functions.

1. Introduction

A *partition* of a positive integer n is any nondecreasing sequence of positive integers whose sum is n. The summands are called *parts*, and n is the *weight*, of the partition. Thus, a partition λ of n (also expressed as $\lambda \vdash n$) into k parts will be denoted by

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k),\ 0 < \lambda_1 \leq \cdots \leq \lambda_k$$

or

$$\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_r^{m_r}),\ 0 < \lambda_1 < \cdots < \lambda_r,\ 1 \leq r \leq k,$$

where m_i denotes the multiplicity of λ_i for all i.

The definition of a perfect partition first appeared in the works of P. A. MacMahon [5, 6]. Subsequently other mathematicians studied and found several properties and generalizations of perfect partitions (see for example, [1, 2, 4, 7, 8]).

Definition 1. A *perfect partition* of n is a partition in which the parts contain exactly one partition of every positive integer less than or equal to n.

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For example, $(1^3, 4) \vdash 7$ is a perfect partition since it contains the partitions (1), (1^2), (1^3), $(4), (1, 4), (1^2, 4), (1^3, 4)$ with weights $1, 2, \ldots, 7$, respectively.

There is a known bijection between the set of perfect partitions of n and the set of ordered factorizations of $N = n + 1$, that is, representations of N as ordered products of positive integers without unit factors [3, 6, 9]. For example, $N = 12$ has eight ordered factorizations, namely, $12, 2\cdot 6, 6\cdot 2, 3\cdot 4, 4\cdot 3, 2\cdot 2\cdot 3, 2\cdot 3\cdot 2, 3\cdot 2\cdot 2$. Let $n+1 = a_1 a_2 \cdots a_r$, $a_i > 1$, be an ordered factorization of $n + 1$. Then the bijection is given by

$$a_1 a_2 \cdots a_r \longrightarrow \left(1^{a_1-1}, a_1^{a_2-1}, (a_1 a_2)^{a_3-1}, \ldots, (a_1 a_2 \cdots a_{r-1})^{a_r-1}\right). \tag{1}$$

Let $f(n, k)$ be the number of ordered factorizations of n into k factors, and let the prime-power factorization of n be $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$. The formula for $f(n, k)$ is given by (see [6] or [3, p. 59])

$$f(n,k) = \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} \prod_{j=1}^{r} \binom{\alpha_j + k - i - 1}{\alpha_j}.$$

We define $f(n) := \sum_k f(n, k)$, where $f(0) = 0$ and $f(1) = 1$. So the formula for the number $\operatorname{per}(n)$ of perfect partitions of n is given by

$$\operatorname{per}(n) = f(n + 1).$$

Example 1. Table 1 shows the ordered factorizations of 6 which correspond to the perfect partitions of 5.

Ordered Factorization of 6	6	$2\cdot 3$	$3\cdot 2$
Perfect Partition of 5	(1^5)	$(1, 2^2)$	$(1^2, 3)$

Table 1: Factorizations of 6 and perfect partitions of 5

Park [8] generalized perfect partitions to "complete partitions" by removing the uniqueness condition from subpartitions, that is, contained partitions.

Definition 2 (Park)**.** A complete partition of n is a weakly increasing partition λ with $\lambda_1 = 1$, such that each integer m, $1 \le m \le n$, can be expressed as a sum of parts of λ, that is, each m can be expressed as $\sum_{j=1}^{k} \alpha_j \lambda_j$, where $\alpha_j \in \{0, 1\}$.

For example, of the 7 partitions of $n = 5$, four are complete partitions, namely, (1^5), $(1^3, 2)$, $(1^2, 3)$, $(1, 2^2)$.

Another extension of perfect partitions was introduced by Lee [4] based on the following observation.

Lemma 1 (Lee)**.** *Let $H(n, v)$ be the set of partitions of n that contain exactly v partitions of m, $v \leq m \leq n - v$, and exactly one partition of every other positive integer less than n. Then $H(n, v) \neq \emptyset$ if and only if $v = 1$ or $v = 2$.*

The case $v = 1$ gives perfect partitions. Naturally, Lee decided to study the seemingly overlooked case of $v = 2$.

Definition 3. A *double-perfect* partition is a partition $(\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ such that each integer m, $2 \leq m \leq n-2$, can be represented exactly twice as $m = \sum_{i=1}^{k} \alpha_i \lambda_i$, where $\alpha_i \in \{0, 1\}$.

For example, $(1^5, 2)$ is a double-perfect partition of 7 because it contains two partitions of $2, 3, 4, 5$ and one partition of $1, 6, 7$:

$$(1), (1^2), (2), (1^3), (1, 2), (1^4), (1^2, 2), (1^5), (1^3, 2), (1^4, 2), (1^5, 2). \tag{2}$$

Proposition 1 (Lee [4])**.** *A double-perfect partition has the form*

$$(1^{q_1}, 2^{q_2}, (q_1{+}2q_2{-}1)^{q_3}, \{(q_1{+}2q_2{-}1)(q_3{+}1)\}^{q_4}, \{(q_1{+}2q_2{-}1)(q_3{+}1)(q_4{+}1)\}^{q_5}, \ldots), \tag{3}$$

where $q_1 \geq 2$ and $q_2, q_3, \ldots$ are positive integers such that $q_1 \neq 3$ implies $q_2 = 1$.

Theorem 1 (Lee [4])**.** *Let $d(n)$ be the number of double-perfect partitions of a positive integer n. We have*

$$d(n) = \begin{cases} f(n-1) & \text{if } n \not\equiv 1 \pmod 4, \\ f(n-1) - f(\frac{n-1}{4}) & \text{if } n \equiv 1 \pmod 4. \end{cases} \tag{4}$$

In the course of proving Theorem 1, Lee separated (3) into two forms of double-perfect partitions λ by setting $q_2 > 1$ (with $q_1 = 3$) and $q_2 = 1$, as follows:

$$\lambda = (1^3, 2^{q_2}, (2(q_2+1))^{q_3}, (2(q_2+1)(q_3+1))^{q_4}, \ldots, (2(q_2+1)\cdots(q_{k-1}+1))^{q_k}); \tag{5}$$

$$\lambda = (1^{q_1}, 2, (q_1+1)^{q_3}, ((q_1+1)(q_3+1))^{q_4}, \ldots, ((q_1+1)(q_3+1)\cdots(q_{k-1}+1))^{q_k}). \tag{6}$$

These forms were then shown to correspond to the following ordered factorizations:

$$n - 1 = 2(q_2+1)(q_3+1)\cdots(q_{k-1}+1)(q_k+1), \;\; q_2 > 1; \tag{7}$$

$$n - 1 = (q_1+1)(q_3+1)(q_4+1)\cdots(q_{k-1}+1)(q_k+1), \;\; q_1 > 1. \tag{8}$$

Notice that (7) and (8) exclude factorizations of the type $n - 1 = 2 \cdot 2 \cdot (q_3 + 1) \cdots$. The number of such factorizations, $f(\frac{n-1}{2})$, is therefore subtracted from the total count in (4).

The aim of this paper is to study generalized double-perfect partitions of n that contain two partitions of every integer between $r+1$ and $n-r-1$, where $1 \leq r \leq \lfloor \frac{n-2}{2} \rfloor$. These will be called *double-perfect partitions of order* r.

We will first give new proofs of Theorem 1 and Proposition 1 in Section 2. Then in Section 3 we adapt the new approach to the study of double-perfect partitions of order r and characterize the first sub-class of the partitions followed by an enumeration result (Theorems 2 and 3). In Section 4 we discuss alternative methods of generating the partitions (Theorem 4). Section 5 deals with a special ordered factorization which leads to the second sub-class of double-perfect partitions of order r. Finally, we state an inclusive enumeration formula (Theorem 7).

2. New Proofs of Lee's Results

Let $G(\lambda)$ be the set of nonempty subpartitions of $\lambda \vdash n$. Thus, if λ is complete, then $G(\lambda)$ contains at least one partition of every positive integer less than or equal to n.

We will show that double-perfect partitions of N arise from perfect partitions of $N-2$, and hence from ordered factorizations of $N-1$ by (1). Let $\mathrm{Per}(n)$ denote the set of perfect partitions of n, and let $D(N)$ be the set of double-perfect partitions of N.

Proposition 2. *A double-perfect partition* $\lambda \vdash N > 3$ *may be obtained from a partition* $\beta \in \mathrm{Per}(N-2)$ *in two ways:*

- *I. If the multiplicity of 1 in* β *is 1, then insert* 1^2 *into* β*. Denote the resulting set by* $E(1^2)$*.*
- *II. If* β *does not contain 2 as a part, insert* 2 *into* β*. Denote the resulting set by* $W(2)$*.*

Then

$$D(N) = E(1^2) \cup W(2).$$

Proof. Let $h(m)$ be the partition of m contained in $G(\beta)$ and write $\lambda \cup \gamma$ for the partition obtained by combining the parts of two partitions λ and γ. Assume that $\lambda \vdash N$ is obtained from $\beta \in \mathrm{Per}(N-2)$ by insertion of 1^2 or 2 according to I or II respectively.

Then from $G(\beta)$ to $G(\lambda)$ we find one additional partition of each $j \in \{2, 3, \ldots, N-2\}$, namely $(1^2) \cup h(j-2)$ or $(2) \cup h(j-2)$. Then one new partition of each of $N-1$ and N appears, that is, $(1^2) \cup h(N-3)$ or $(2) \cup h(N-3)$ and $(1^2) \cup h(N-2)$ or $(2) \cup h(N-2)$.

So the resulting partition λ is double-perfect by definition. For example, let $\beta = (1^2, 3)$. Then from II, $\lambda = (2) \cup \beta = (1^2, 2, 3)$, and our construction is shown in Table 2.

j	$h(j) \in G(\beta)$	$\gamma \in G(\lambda) \setminus G(\beta)$
1	(1)	–
2	(1^2)	$((2), h(0)) = (2)$
3	(3)	$((2), h(1)) = (1, 2)$
4	$(1, 3)$	$((2), h(2)) = (1^2, 2)$
5	$(1^2, 3)$	$((2), h(3)) = (2, 3)$
6	–	$((2), h(4)) = (1, 2, 3)$
7	–	$((2), h(5)) = (1^2, 2, 3)$.

Table 2: The construction in the proof of Proposition 2 for $\beta = (1^2, 3)$

□

Example 2. We illustrate Proposition 2 further by extending Table 1 to the corresponding double-perfect partitions (see Table 3).

Ordered Factorization of 6	6	$2 \cdot 3$	$3 \cdot 2$
Per(5)	(1^5)	$(1, 2^2)$	$(1^2, 3)$
Insert Parts	2	1^2	2
Double-Perfect Partition of 7	$(1^5, 2)$	$(1^3, 2^2)$	$(1^2, 2, 3)$

Table 3: Factorizations of 6 and double-perfect partitions of 7

The following corollary is equivalent to Proposition 1.

Corollary 1. *A double-perfect partition has one of the forms in* (5) *and* (6)*.*

Proof. Consider an ordered factorization of the form $N - 1 = 2a_2a_3 \cdots a_k$, $a_2 > 2$. From (1) the corresponding perfect partition is

$$(1, 2^{a_2-1}, (2a_2)^{a_3-1}, (2a_2a_3)^{a_4-1}, \ldots, (2a_2a_3 \cdots a_{k-1})^{a_k-1}). \tag{9}$$

Secondly, an ordered factorization of the form $N - 1 = a_1a_2a_3 \cdots a_k$, $a_1 > 2$, corresponds to the perfect partition

$$(1^{a_1-1}, a_1^{a_2-1}, (a_1a_2)^{a_3-1}, \ldots, (a_1a_2a_3 \cdots a_{k-1})^{a_k-1}). \tag{10}$$

Observe that the partitions in (9) and (10) fulfill the asserted properties of β in parts I and II of Proposition 2. Lastly, insertion of 1^2 and 2 into these partitions restores (5) and (6) respectively (on setting $a_i = q_i + 1$ for all i). □

Proof of Theorem 1. Proposition 2 implies that $d(N) = f(N-1)$ with the exception of certain duplicated partitions. Note that the factorizations $N-1 = 2 \cdot 2 \cdot m$ and $N-1 = 4 \cdot m$ produce the same double-perfect partition:

$$N-1 = 2 \cdot 2 \cdot m \implies \beta = (1, 2, 4^{m-1}) \longmapsto (1^3, 2, 4^{m-1}) \in E(1^2)$$

and

$$N-1 = 4 \cdot m \implies \beta = (1^3, 4^{m-1}) \longmapsto (1^3, 2, 4^{m-1}) \in W(2).$$

The number of factorizations of the form $N-1 = 2 \cdot 2 \cdot m$ is given by $f(\frac{N-1}{4})$. Thus, when $N-1 \equiv 0 \pmod 4$, we have $d(N) = f(N-1) - f(\frac{N-1}{4})$. This completes the proof. □

3. Double Perfect Partitions of Higher Order

We propose the following extension of double-perfect partitions.

Definition 4. A *double-perfect partition of order* r is a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash N$ such that each integer m with $r+1 \leq m \leq N-r-1$ can be represented exactly twice as $m = \sum_{i=1}^{k} \alpha_i \lambda_i$, $\alpha_i \in \{0, 1\}$, and other integers less than or equal to N can be uniquely represented.

In particular, double-perfect partitions of order $r = 1$ are the original double-perfect partitions discussed above.

The representation scheme of a double-perfect partition of N of order r is

$$\underbrace{1, 2, \ldots, r}_{1 \text{ time}}, \underbrace{r+1, r+2, \ldots, N-r-1}_{2 \text{ times}}, \underbrace{N-r, N-r+1, \ldots, N-1, N}_{1 \text{ time}}. \tag{11}$$

Let $U_r(N)$ be the set of double-perfect partitions of N of order r with $u_r(N) = |U_r(N)|$. Then (11) implies that

$$U_r(N) \neq \emptyset \iff N \geq 2(r+1). \tag{12}$$

Let $D_r(N)$ be the subset of $U_r(N)$ containing partitions which may be found by insertions of (1^{r+1}) and $(r+1)$ into perfect partitions (thus extending the construction in Section 2 that corresponds to $r = 1$). Then define $E_r(N) := U_r(N) \setminus D_r(N)$.

Partitions in $D_r(N)$ and $E_r(N)$ will also be referred to as *Type-A* and *Type-B* respectively. The rest of this section is devoted to the characterization and enumeration of $D_r(N)$. Properties of $E_r(N)$ will be explored in detail in Sections 4 and 5.

Theorem 2. *A Type-A double-perfect partition $\lambda \vdash N$ of order $r > 0$ may be obtained from a partition $\beta \in \mathrm{Per}(N-r-1)$ in two ways:*
I. If the multiplicity of 1 in β is r, then insert 1^{r+1} into β, and denote the resulting set by $A(1^{r+1})$.
II. If the multiplicity of 1 in β is different from r, then insert $r+1$ into β, and denote the resulting set by $B(r+1)$.
Then

$$D_r(N) = A(1^{r+1}) \cup B(r+1). \tag{13}$$

Proof. Let $h(m) \in G(\beta)$, $1 \leq m \leq N-r-1$. We show that any $\lambda \vdash N$ obtained from I or II is double-perfect of order r by accounting for new partitions arising between $G(\beta)$ and $G(\lambda)$.

The single partitions of $1, 2, \ldots, r$ are not affected by insertion of additional parts into β, but one new partition of each $m \in \{r+1, r+2, \ldots, N-r-1\}$ appears from $A(1^{r+1})$ or $B(r+1)$, namely, (1^{r+1}) and $(1^{r+1}) \cup h(m-r-1)$ or $(r+1)$ and $(r+1) \cup h(m-r-1))$, respectively.

Finally we obtain one new partition of each $m \in \{N-r, \ldots, N\}$ by symmetry (since $G(\lambda)$ already contains partitions of $j = 0, 1, \ldots, r$). This shows that weights of partitions in $G(\lambda)$ are distributed as in (11), as desired. □

Remark 1. In the proof of Theorem 2 consider the effect of inserting $\gamma \in P(r+1) \setminus \{(1^{r+1}), (r+1)\}$ into β, where $r > 1$. So γ has the form $\gamma = (\gamma_1, \ldots, \gamma_k)$, $k > 1$ and $\gamma_i > 1$ for some i.

We claim that $\lambda = \gamma \cup \beta$ is not a Type-A double-perfect partition of order r.

Assume that the multiplicity of 1 in β is $x \geq r$. Then λ would contain at least two partitions of γ_i instead of one, namely (1^{γ_i}) and (γ_i).

The case when the multiplicity of 1 in β is $x < r$ affects only type II. Observe that already $x+1 \in \beta$ since β is perfect. Thus, if $1 \in \gamma$, then λ would contain at least two partitions of $x+1$: (1^{x+1}) and $(x+1)$.

However, if $1 \notin \gamma$ when $x < r$, it is possible for λ to be double perfect of order r, but not of Type-A. For example, consider $N = 23$, $r = 5$ with $\beta = (1^2, 3, 6^2)$ and $\gamma = (3^2)$. Then it may be verified that $\lambda = (1^2, 3^3, 6^2) \in E_5(23)$.

A systematic method of obtaining all members of $E_r(N)$ is discussed in Section 5.

Corollary 2. *A Type-A double-perfect partition $\lambda \in D_r(N)$ has either of the following forms:*

$$\lambda = (1^{2r+1}, (r+1)^{a_2-1}, ((r+1)a_2)^{a_3-1}, \ldots, ((r+1)a_2a_3\cdots a_{k-1})^{a_k-1}), \tag{14}$$

$$\lambda = (1^{a_1-1}, (r+1), a_1^{a_2-1}, (a_1a_2)^{a_3-1}, (a_1a_2a_3)^{a_4-1}, \ldots, (a_1a_2\cdots a_{k-1})^{a_k-1}), \tag{15}$$

where $a_1 \neq r+1$ and the location of $r+1$ in (15) *depends on its relative size.*

Proof. The two forms are consequences of converting the following ordered factorizations to perfect partitions by means of the bijection (1), and then inserting 1^{r+1} and $r+1$ respectively.

$$N - r = (r+1)a_2a_3\cdots a_k; \tag{16}$$

$$N - r = a_1a_2a_3\cdots a_k, \quad a_1 \neq r+1. \tag{17}$$

□

Note that the two sets on the right-hand-side of (13) are not always disjoint as certain partitions may be obtained twice using the two methods. The following result gives the exact cardinality of $D_r(N)$ after excluding duplicates.

Theorem 3. *The number $d_r(N)$ of Type-A double-perfect partitions of N of order r, $1 \le r \le \lfloor \frac{n-2}{2} \rfloor$, is given by*

$$d_r(N) = \begin{cases} f(N-r) - f(\frac{N-r}{2(r+1)}) - (f(r+1)-1)f(\frac{N-r}{r+1}) & \text{if } N \equiv r \pmod{2(r+1)}, \\ f(N-r) - (f(r+1)-1)f(\frac{N-r}{r+1}) & \text{if } N \equiv -1 \pmod{2(r+1)}, \\ f(N-r) & \text{otherwise.} \end{cases} \tag{18}$$

In particular when $1 \le r \le 3$, we obtain $d_1(N) = d(N)$ (same as (4));

$$d_2(N) = \begin{cases} f(N-2) - f(\frac{N-2}{6}) & \text{if } N \equiv 2 \pmod{6}, \\ f(N-2) & \text{otherwise;} \end{cases} \tag{19}$$

$$d_3(N) = \begin{cases} f(N-3) - f(\frac{N-3}{8}) - f(\frac{N-3}{4}) & \text{if } N \equiv 3 \pmod{8}, \\ f(N-3) - f(\frac{N-3}{4}) & \text{if } N \equiv 7 \pmod{8}, \\ f(N-3) & \text{otherwise.} \end{cases} \tag{20}$$

Proof. From Theorem 2, $\lambda \in D_r(N)$ is obtained by inserting 1^{r+1} or $r+1$ into suitable perfect partitions of $n = N - r - 1$. The latter may be constructed from the ordered factorizations of $N - r$; see Corollary 2.

Thus, $d_r(N) = f(N-r)$ subject to the following exceptions.
(i) If $N - r \equiv 0 \pmod{2(r+1)}$, the factorizations $N - r = (r+1)\cdot 2 \cdot m$ and $N - r = (2(r+1)) \cdot m$ produce the same λ:

$$(r+1)\cdot 2\cdot m \implies (1^r, r+1, (2(r+1))^{m-1}) \mapsto (1^{2r+1}, r+1, (2(r+1))^{m-1}) \in A(1^{r+1}),$$

and

$$(2(r+1))\cdot m \implies (1^{2r+1}, (2(r+1))^{m-1}) \mapsto (1^{2r+1}, r+1, (2(r+1))^{m-1}) \in B(r+1).$$

We remove the first type of such factorizations which is counted by $f(\frac{N-r}{2(r+1)})$.

(ia) Furthermore, since $N-r \equiv 0 \pmod{2(r+1)}$ implies $N-r \equiv 0 \pmod{r+1}$ we isolate a set of perfect partitions that do not contribute to discovering additional λ, namely, factorizations of the form $a_1a_2\cdots a_tm$, $t>1$, where $a_1a_2\cdots a_t = r+1$. Note that $a_1a_2\cdots a_tm$ translates into the perfect partition $\beta = (1^{a_1-1}, a_1^{a_2-1}, (a_1a_2)^{a_3-1}, \ldots, (a_1a_2\cdots a_t)^{m-1})$. However, β contains 1^{a_1-1} but $a_1-1 \neq r$, so Method I does not apply. Also, β already contains the part $r+1 = a_1a_2\cdots a_t$, so Method II does not apply. The number of such non-contributing perfect partitions of $N-r-1$ is equal to the number of ordered factorizations of $r+1$ into two or more factors times the number of ordered factorizations of $(N-r)/(r+1)$, that is, $(f(r+1)-1)f(\frac{N-r}{r+1})$.

Parts (i) and (ia) together give the first line of the stated formula.

(ii) When $N-r \equiv r+1 \pmod{2(r+1)}$, we obtain part (ia) independently. Hence the second line of the formula follows.

There are no other exceptions. The remaining factorizations all yield valid partitions λ. Hence their number is $f(N-r)$. □

Example 3. Let $N = 17$ with $1 \leq r \leq 7$. The members of $D_r(N)$ are shown in Table 4. The derivation of members of $D_2(17)$ is shown in Table 5. It may be verified that the distribution of weights of members of $G(\lambda)$, for every $\lambda \in D_2(17)$, corresponds to the scheme (cf. (11))

$$\underbrace{1,2}_{\text{1 time}}, \underbrace{3,4,\ldots,14}_{\text{2 times}}, \underbrace{15,16,17}_{\text{1 time}}.$$

r	$D_r(17)$	$d_r(17)$
1	$(1^3,2^7),(1^3,2,4^3),(1^3,2^3,8),(1^3,2,4,8),(1^{15},2),(1^7,2,8)$	6
2	$(1^{14},3),(1^5,3^4),(1^4,3,5)$	3
3	$(1^{13},4),(1,2^6,4),(1^6,4,7)$	3
4	$(1^{12},5)$	1
5	$(1^{11},6),(1,2^5,6),(1^2,2^3,6),(1^3,4^2,6),(1,2,4^2,6)$	5
6	$(1^{10},7)$	1
7	$(1^9,8),(1,2^4,8),(1^4,5,8)$	3

Table 4: Type-A double-perfect partitions of 17 of all orders

Remark 2. Note that $D_r(17) = U_r(17)$ when $r \neq 2$, that is, $E_r(17) = \emptyset$; but $E_2(17) = \{(1,2^2,3^4)\}$. Hence $d_2(17) = 3$ but $u_2(17) = 4$ (see Example 4).

Ordered Factorization of 15	15	$3 \cdot 5$	$5 \cdot 3$
Per(14)	(1^{14})	$(1^2, 3^4)$	$(1^4, 5^2)$
Insert Parts	3	1^3	3
$D_2(17)$	$(1^{14}, 3)$	$(1^5, 3^4)$	$(1^4, 3, 5^2)$

Table 5: Derivation of $D_2(17)$

4. Further Properties of Double-Perfect Partitions

We record the following assertion which corresponds to many members of $D_r(N)$.

Proposition 3. *Let $\lambda \in D_r(n)$, $r > 0$. Then*

$$\lambda \cup ((n-r)^m) \in D_r(n + m(n-r)),\ m \geq 0.$$

Proof. Refer to the construction in Theorem 2. The perfect partition $\beta \vdash n - r - 1$ is classified according to the multiplicity of 1. The latter does not change if $(n-r)^m$ is inserted to give $\gamma = \beta \cup ((n-r)^m)$, $m \geq 1$. However, γ is still a perfect partition since $n - r$ exceeds the weight of β by 1. The weight of γ is $n - r - 1 + m(n - r)$. This shows that $(1^{r+1}) \cup \gamma$ or $(r + 1) \cup \gamma$ belongs to $D_r(n + m(n - r))$. □

We remark that Proposition 3 cannot be nested, that is, if $\lambda \in D_r(n)$ and $n^* = n + m(n - r)$, then $\gamma = \lambda \cup ((n - r)^m) \cup ((n^* - r)^s) \notin D_r(n + m(n - r) + s(n^* - r))$ for any $s > 0$. However, γ is still double-perfect of order r, but belongs to $E_r(n + m(n - r) + s(n^* - r))$. The following assertion relates to all members of $U_r(N)$ and admits nesting when a partition is mapped to $E_r(N)$.

Theorem 4. *Let $\lambda \in U_r(n)$, $r > 0$. Then*

$$\lambda \cup ((n-r)^m) \in U_r(n + m(n-r)),\ m \geq 0.$$

Proof. The representation scheme of weights of members of $G(\lambda)$ is

$$\underbrace{1, 2, \ldots, r}_{\text{1 time}}, \underbrace{r+1, r+2, \ldots, n-r-1}_{\text{2 times}}, \underbrace{n-r, n-r+1, \ldots, n-1, n}_{\text{1 time}}. \tag{21}$$

The sequence of multiplicities has the form $1, \ldots, 1, 2, \ldots, 2, 1, \ldots, 1$ or $1^r, 2^{n-2r-1}, 1^{r+1}$. We denote the empty partition of 0 by $\emptyset$. Let S_{10}, S_{20}, S_{30} be the sets of partitions whose weights are represented by the three segments in (21) from left to right, and assume that $\emptyset \in S_{10}$ so that $|S_{10}| = r + 1 = |S_{30}|$ and $|S_{20}| = n - 2r - 1$. Let $\alpha \in S_{10}$, $\gamma_1, \gamma_2 \in S_{20}$ and $\rho \in S_{30}$, where γ_1 and γ_2 represent a pair of different partitions of the same integer. Furthermore define $S_{ij}\pi := \{\theta \cup \pi \mid \theta \in S_{ij}\}$.

With $N_0 = n$, we claim that $|S_{1\ell}| = r + 1 = |S_{3\ell}|$ and $|S_{2\ell}| = N_\ell - 2r - 1$ for $0 \leq \ell \leq m$.

We will prove the claim by induction on ℓ. The result is obviously true for $\ell = 0$. Assume that $\ell = 1$. The effect of inserting $N_0 - r$ into λ is that members of S_{10} and S_{20} remain unchanged. Then S_{20} is extended to include partitions of N_0 with $r+1$ new pairs of partitions $\rho \in S_{30}, \alpha \cup (N_0 - r) \in S_{10}(N_0 - r)$. This is followed by a further extension with $N_0 - 2r - 1$ new pairs of partitions $\gamma_1 \cup (N_0 - r), \gamma_2 \cup (N_0 - r) \in S_{20}(N_0 - r)$. Lastly, we find $r+1$ new partitions $\rho \cup (N_0 - r) \in S_{30}(N_0 - r)$.

Note that the scheme (21) remains unchanged relative to $\lambda \cup (N_0 - r)$ whose weight is $N_1 = 2N_0 - r$. Indeed we have $S_{11} = S_{10}$, $S_{31} = S_{30}(N_0 - r)$ and

$$S_{21} = S_{20} \cup (S_{30} \dot{\cup} S_{10}(N_0 - r)) \cup S_{20}(N_0 - r), \tag{22}$$

where $\dot{\cup}$ denotes a combination (by pairwise equal weights) of two sets of partitions having exactly the same multiset of weights, and

$$|S_{21}| = (N_0 - 2r - 1) + (r + 1) + (N_0 - 2r - 1) = 2N_0 - 3r - 1 = N_1 - 2r - 1.$$

Hence $\lambda \cup (N_0 - r)$ is double-perfect.

If $\ell = 2$, we insert $N_0 - r$ into the relevant partition of N_1 corresponding to (22) and obtain the following results:

$$\begin{aligned} S_{12} &= S_{11} \ (= S_{10}); \\ S_{22} &= S_{21} \cup S_{30}(N_0 - r) \dot{\cup} S_{10}((N_0 - r)^2) \cup S_{20}((N_0 - r)^2); \\ S_{32} &= S_{30}((N_0 - r)^2). \end{aligned}$$

We have

$$\begin{aligned} |S_{22}| &= |S_{21}| + |S_{30}((N_0 - r)) \dot{\cup} S_{10}((N_0 - r)^2)| + |S_{20}((N_0 - r)^2)| \\ &= (N_1 - 2r - 1) + (r + 1) + (N_0 - 2r - 1) \\ &= N_1 + N_0 - 3r - 1 \\ &\equiv N_2 - 2r - 1. \end{aligned}$$

So the basic scheme (21) is preserved. The result follows by a straightforward application of the principle of mathematical induction. □

Remark 3. Let $\lambda \in U_r(n)$, and set $\lambda = \lambda^1$, $n = n_1$. Then according to Theorem 4, $\lambda^k \in U_r(n_k)$, $k \geq 1$, where

$$\lambda^{j+1} = \lambda^j \cup ((n_j - r)^{e_j}), \ j \geq 1, \, e_j \geq 0 \qquad \text{and} \qquad n_{j+1} = n_j + e_j(n_j - r).$$

That is,

$$\begin{aligned} \lambda^2 &= \lambda^1 \cup ((n_1 - r)^{e_1}) \\ \lambda^3 &= \lambda^1 \cup ((n_1 - r)^{e_1}) \cup ((n_2 - r)^{e_2}) \\ \lambda^4 &= \lambda^1 \cup ((n_1 - r)^{e_1}) \cup ((n_2 - r)^{e_2}) \cup ((n_3 - r)^{e_3}) \\ &\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \end{aligned}$$

Thus, given $\lambda^{j+1} = \lambda^j \cup ((n_j - r)^{e_j})$, one may insert additional copies of the existing largest part to obtain $\gamma = \lambda^j \cup ((n_j - r)^{e_j+s})$, $s \geq 1$, or create a new largest part by inserting copies of $n_{j+1} - r$ to obtain $\gamma = \lambda^j \cup ((n_j - r)^{e_j}) \cup (n_{j+1} - r)^{e_{j+1}}$, $e_{j+1} \geq 1$. In either case γ is a double-perfect partition of order r.

The emerging partitions originate from $\lambda = \lambda^1$. For example, $(1^3, 4^2, 6) \in D_5(17)$ and $(1^3, 4^2, 6^2) \in E_5(23)$, and subsequently $(1^3, 4^2, 6^x) \in E_5(11 + 6x)$, $x \geq 3$. In addition, a new set may start with $(1^3, 4^2, 6) \cup (12^v)$ which gives $(1^3, 4^2, 6, 12^v) \in E_5(17 + 12v)$, $v \geq 0$. Since $(1^3, 4^2, 6^3) \in E_5(29)$, a further set may start with $(1^3, 4^2, 6^3) \cup (24^y)$ which gives $(1^3, 4^2, 6^3, 24^y) \in E_5(29 + 24y)$, $y \geq 0$. As a final example, note that $(1^3, 4^2, 6^3, 24) \cup (48)^z = (1^3, 4^2, 6^3, 24, 48^z) \in E_5(53 + 48z)$.

Observe that each of these E_r-partitions is "halved perfect" in the sense that if we double every part greater than or equal to $(r+1)$, we obtain a perfect partition. For example, $(1^3, 4^2, 6^3, 24^y)$ is double-perfect of order 5 but $(1^3, 4^2, (2 \cdot 6)^3, (2 \cdot 24)^y) = (1^3, 4^2, 12^3, 48^y)$ is a perfect partition. Such partitions will be discussed in the next section.

Corollary 3. *Let $\lambda \in U_r(n)$ with largest part $\ell(\lambda) \neq r + 1$. Then*

$$\lambda \cup (\ell(\lambda)) \in U_r(n + \ell(\lambda)).$$

The condition $\ell(\lambda) \neq r + 1$ is necessary in general. For example, $(1^2, 3^4, 6) \in D_5(20)$ but $(1^2, 3^4, 6^2) \notin U_5(26)$. Exceptions occur with $\ell(\lambda) = r + 1$ only when λ contains a perfect partition $\beta \vdash m$ such that $r + 1 = m - r$. So from Proposition 3, $\beta \cup (r + 1) = \beta \cup (m - r) \in D_r(2m - r)$. For example, since $\beta = (1, 2^3)$ implies $\lambda = (1, 2^3, 4) \in D_3(11)$, it follows that $(1, 2^3, 4^2) \in E_3(15)$, $(1, 2^3, 4^3) \in E_3(19), \ldots$.

5. Halved Perfect Partitions

If $N \equiv r \pmod{(r + 1)}$, we consider the following restricted ordered factorizations:

$$2(N - r) = a_1 a_2 \cdots a_k,\ k > 2, \tag{23}$$

$$a_1 a_2 \cdots a_j = 2(r + 1),\ 1 < j < k, \tag{24}$$

$$a_j > 2,\ a_{j+1} > 2. \tag{25}$$

Theorem 5. *The following partition is double-perfect of order r and belongs to $E_r(N)$:*

$$\lambda = (1^{a_1-1}, a_1^{a_2-1}, \ldots, (a_1 \cdots a_{j-1})^{a_j-1}, (\frac{a_1 \cdots a_j}{2})^{a_{j+1}-1}, \ldots, (\frac{a_1 \cdots a_{k-1}}{2})^{a_k-1}). \tag{26}$$

Note that λ cannot be obtained from Theorem 2 since $a_1 - 1 < r$ and $r + 1$ is already a part.

The restrictions imposed on the factorizations (23), namely (24) and (25), ensure the determination of $\lambda \in E_r(N)$ with $E_r(n) \cup D_r(N) = \emptyset$. The number of factors is $k > 2$, otherwise $(r+1)$ would not be a part of λ. Also $a_j \neq 2$, otherwise λ would contain a repeated base: $(r+1)^1, (r+1)^{a_{j+1}-1}$ which is not reversible. If $a_j = 2$ when $k = j+1$, then λ reduces to an ordinary perfect partition. Lastly, we must have $a_{j+1} \neq 2$, otherwise λ would contain a single copy of $r+1 = \frac{1}{2}a_1 \cdots a_j$ which would duplicate a member of $B(r+1)$.

Proof of Theorem 5. We first show that the weight $\sum \lambda$ of λ is N. We have

$$\begin{aligned}\sum\lambda &= a_1 - 1 + a_1(a_2-1) + a_1a_2(a_3-1) + \cdots + a_1\cdots a_{j-1}(a_j-1)\\ &\quad + \frac{a_1\cdots a_j}{2}(a_{j+1}-1) + \cdots + \frac{a_1\cdots a_{k-1}}{2}(a_k-1)\\ &= a_1a_2\cdots a_j - 1 + \frac{a_1\cdots a_j}{2}(a_{j+1}-1) + \cdots + \frac{a_1\cdots a_{k-1}}{2}(a_k-1)\\ &= \frac{a_1a_2\cdots a_j}{2} + \frac{a_1a_2\cdots a_j}{2}a_{j+1}a_{j+2}\cdots a_k - 1\\ &= \frac{2(r+1)}{2} + \frac{2(r+1)}{2}\cdot\frac{2(N-r)}{2(r+1)} - 1\\ &= N.\end{aligned}$$

Next, we show that λ is double-perfect of order r. Define, for every t, $j < t \le k$,

$$\lambda^t = (1^{a_1-1}, a_1^{a_2-1}, \ldots, (a_1\cdots a_{j-1})^{a_j-1}, (\frac{a_1\cdots a_j}{2})^{a_{j+1}-1}, \ldots, (\frac{a_1\cdots a_{t-1}}{2})^{a_t-1}),$$

that is,

$$\begin{aligned}\lambda^t = (&1^{a_1-1}, a_1^{a_2-1}, \ldots, (a_1\cdots a_{j-1})^{a_j-1}, (r+1)^{a_{j+1}-1}, ((r+1)a_{j+1})^{a_{j+2}-1},\\ &\ldots, ((r+1)a_{j+1}\cdots a_{t-1})^{a_t-1}).\end{aligned}$$

Let the weight of λ^t be N_t. Then from the calculation of $\sum \lambda$ we deduce that

$$N_t = r + (r+1)a_{j+1}\cdots a_t.$$

It is clear that $\beta = (1^{a_1-1}, a_1^{a_2-1}, (a_1a_2)^{a_3-1}, \ldots, (a_1\cdots a_{j-1})^{a_j-1})$ is a perfect partition with weight $a_1a_2\cdots a_j - 1 = 2(r+1) - 1$. Since $a_1 \neq r+1$ Theorem 2 implies $\beta \cup (r+1) \in D_r(3r+2)$. Thus, from the remark immediately following Corollary 3 the partition $\beta \cup ((r+1)^x)$ is double-perfect of the same order for all $x > 0$. The case $x = a_{j+1} - 1 > 1$ is

$$\lambda^{j+1} = (1^{a_1-1}, a_1^{a_2-1}, (a_1a_2)^{a_3-1}, \ldots, (a_1\cdots a_{j-1})^{a_j-1}, (r+1)^{a_{j+1}-1}) \vdash N_{j+1}.$$

The fact that the full partition $\lambda = \lambda^k$ is double-perfect of order r now follows from Theorem 4 (or Remark 3). □

Example 4. If $N = 17$, $r = 2$, the only ordered factorization of $2(17-2) = 30$ satisfying (23) to (25) is $2\cdot 3\cdot 5$. So by Theorem 5, $(1, 2^2, (6/2)^4) = (1, 2^2, 3^4) \in E_2(17)$.

5.1. Enumeration of $E_r(N)$

If $n-r$ is inserted into $\lambda \in D_r(n)$ to give $\gamma \in E_r(N)$, then from (12) and Theorem 4 we have

$$D_r(n) \neq \emptyset \implies n = \frac{N+r}{2} \geq 2(r+1) \implies r \leq \left\lfloor \frac{N-4}{3} \right\rfloor .$$

So from the factorizations (23) and Theorem 5 we deduce that if $N \equiv r \pmod{r+1}$, then $E_r(N) \neq \emptyset$ provided that $r \leq \lfloor \frac{N-4}{3} \rfloor$. We claim that $e_r(N) = |E_r(N)|$ is given by the following formula.

Theorem 6. *The number $e_r(N)$ of Type-B double-perfect partitions of N of order r, $1 \leq r \leq \lfloor \frac{n-4}{3} \rfloor$, is given by*

$$e_r(N) =$$

$$\begin{cases} (f(2(r+1)) - 1 - f(r+1)) \left(f(\frac{N-r}{r+1}) - f(\frac{N-r}{2(r+1)}) \right) & \text{if } N \equiv r \pmod{2(r+1)}, \\ (f(2(r+1)) - 1 - f(r+1))\, f(\frac{N-r}{r+1}) & \text{if } N \equiv -1 \pmod{2(r+1)}, \\ \qquad 0 & \text{otherwise.} \end{cases} \tag{27}$$

Proof. It will suffice to count the factorizations (23) subject to the restrictions (24) and (25).

The number of ordered factorizations of $2(r+1)$ into two or more factors is $f(2(r+1)) - 1$. Note that the condition $a_j \neq 2$ forbids factorizations that end in 2. The number of such factorizations is $f(r+1) \cdot 1$. So the number of factorizations of $2(r+1)$ into two or more factors that do not end in 2 is $f(2(r+1)) - 1 - f(r+1)$.

If $N \equiv r \pmod{2(r+1)}$, the condition $a_{j+1} \neq 2$ can be violated. The number of factorizations of $\frac{a_1 a_2 \cdots a_k}{2(r+1)} = \frac{2(N-r)}{2(r+1)}$ that do not start with 2 is given by $f(\frac{2(N-r)}{2(r+1)}) - f(\frac{2(N-r)}{4(r+1)})$. Thus, together with the previous paragraph, the first line of (27) is proved.

Finally, if $N - r \equiv r + 1 \pmod{2(r+1)}$, then $a_{j+1} \neq 2$ is automatically satisfied and we simply multiply the first result by $f(\frac{2(N-r)}{2(r+1)})$. □

Remark 4. Equation (27) gives $e_1(N) = 0$ for all $N > 0$. So if $2 \leq r \leq \lfloor \frac{N-4}{3} \rfloor$, then $e_r(N) > 0$ whenever $N \equiv r \pmod{r+1}$. The sequence of such weights N is given by [10, A254671] and described as "Numbers that can be represented as $xy + x + y$, where $x \geq y \geq 1$".

Lastly, we state the full formula for $u_r(N) = d_r(N) + e_r(N)$, after simplification.

Theorem 7. *The number $u_r(N)$ of all double-perfect partitions of N of order r, $1 \leq r \leq \lfloor \frac{n-2}{2} \rfloor$, is given by*

$u_r(N) =$

$$\begin{cases} f(N-r) + (f(2(r+1)) - 2f(r+1))f(\frac{N-r}{r+1}) \\ \qquad\qquad -(f(2(r+1)) - f(r+1))f(\frac{N-r}{2(r+1)}) & \textit{if } N \equiv r \pmod{2(r+1)}, \\ f(N-r) + (f(2(r+1)) - 2f(r+1))f(\frac{N-r}{r+1}) & \textit{if } N \equiv -1 \pmod{2(r+1)}, \\ f(N-r) & \textit{otherwise.} \end{cases} \tag{28}$$

Example 5. It may be verified that $E_r(22) = \emptyset$ for all r; so $U_r(22) = D_r(22)$. But $E_r(23) \neq \emptyset$ when $r = 2, 3, 5$; so $U_r(23) = D_r(23)$ for $r \neq 2, 3, 5$. The classification of all members of $U_r(23), 1 \leq r \leq 10$, is shown in Table 6. Note that when $r = 2, 3, 5$, the rth row is split into two parts showing $U_r(23)$ on top and $E_r(23)$ at the bottom of each row.

r	$U_r(23) = D_r(23) \cup E_r(23)$		$u_r(23)$
1	$(1^3, 2^{10}), (1^{21}, 2), (1^{10}, 2, 11)$	3	3
2	$(1^5, 3^6), (1^{20}, 3), (1^6, 3, 7^2)$	3	
	$(1, 2^2, 3^6)$	1	4
3	$(1^7, 4^4), (1^{19}, 4), (1, 2^9, 4), (1^4, 4, 5^3), (1^9, 4, 10), (1, 2^4, 4, 10), (1^4, 4, 5, 10)$	7	
	$(1, 2^3, 4^4)$	1	8
4	$(1^{18}, 5)$	1	1
5	$(1^{11}, 6^2), (1^{17}, 6), (1, 2^8, 6), (1^2, 3^5, 6), (1^8, 6, 9), (1^2, 3^2, 6, 9)$	6	
	$(1, 2, 4^2, 6^2), (1^3, 4^2, 6^2), (1^2, 3^3, 6^2), (1, 2^5, 6^2)$	4	10
6	$(1^{16}, 7)$	1	1
7	$(1^{15}, 8), (1, 2^7, 8), (1^3, 4^3, 8), (1, 2, 4^3, 8)$	4	4
8	$(1^{14}, 9), (1^2, 3^4, 9), (1^4, 5^2, 9)$	3	3
9	$(1^{13}, 10), (1, 2^6, 10), (1^6, 7, 10)$	3	3
10	$(1^{12}, 11)$	1	1

Table 6: Type-A and Type-B double-perfect partitions of 23 of all orders

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