#A11 INTEGERS 25 (2025)

A NOTE ON NUMERICAL SEMIGROUPS GENERATED BY k-TH POWERS

Fabián Arias
Departamento de Matemáticas, Universidad del Atlántico, Barranquilla, Colombia
fabianarias@mail.uniatlantico.edu.co

Jerson Borja
Departamento de Matemáticas y Estadística, Montería, Colombia
jersonborjas@correo.unicordoba.edu.co

Jaider Terán
Departamento de Matemáticas y Estadística, Montería, Colombia
jteranmelendez@correo.unicordoba.edu.co

Received: 4/18/24, Revised: 10/12/24, Accepted: 2/3/25, Published: 2/21/25

Abstract

Given a positive integer k, we associate a family of numerical semigroups $(S_n)_n$ with the sequence of k-th powers $x_n = n^k$, by setting $S_n = \langle\{x_{n+j} : j \in \mathbb{N}\}\rangle$. We prove that if $k = 2$, then the embedding dimension of S_n satisfies $\mathrm{e}(S_n) = O(n)$. For $k \geq 2$, we prove that the Frobenius number of S_n satisfies $\mathrm{F}(S_n) = O(n^{k+\epsilon})$ for all $\epsilon > 0$.

1. Introduction

We shall denote by $\mathbb{N}$ the set of natural numbers $0, 1, 2, \ldots$, and denote by $\mathbb{Z}^+$ the set of positive integers. Throughout this paper, all linear combinations are supposed to have natural coefficients.

A subset S of $\mathbb{N}$ is a *numerical semigroup* if S is closed under addition, contains 0 and $\mathbb{N} \setminus S$ is a finite set. If A is a subset of $\mathbb{N}$, we let $\langle A\rangle$ denote the set of all linear combinations of elements in A. This set $\langle A\rangle$ is a *submonoid* of $\mathbb{N}$. It is known that $\langle A\rangle$ is a numerical semigroup (that is, $\mathbb{N} \setminus \langle A\rangle$ is finite) if and only if $\gcd(A) = 1$.

If S is a numerical semigroup and A is a subset of S such that $S = \langle A\rangle$, we say that A is a *generating set* of S. It is well known that every numerical semigroup S has a subset $\beta(S)$ satisfying the following properties:

1. The set $\beta(S)$ is a generating set of S.

DOI: 10.5281/zenodo.14907270

2. If A is a generating set of S, then $\beta(S) \subseteq A$.

Because of these properties, the set $\beta(S)$ is called the *minimal generating set* of S. It can be proved that $\beta(S)$ is a finite set, and it consists of all nonzero elements in S that cannot be written as a sum of two nonzero elements of S. The cardinality of $\beta(S)$ is called the *embedding dimension* $\mathrm{e}(S)$ of S.

For a numerical semigroup S, the largest integer that does not belong to S is called the *Frobenius number of* S, which is denoted by $\mathrm{F}(S)$. When $S \neq \mathbb{N}$, we have $\mathrm{F}(S) = \max(\mathbb{N} \setminus S)$, which is a positive integer. A well known formula for the Frobenius number of a numerical semigroup generated by two relatively prime positive integers a and b is given by Sylvester's formula:

$$\mathrm{F}(\langle a, b\rangle) = ab - a - b. \tag{1}$$

More concepts and properties about numerical semigroups can be consulted in [11].

Now, we consider a strictly increasing sequence of positive integers $(x_n)_n$, indexed by the set of positive integers. For each $n \in \mathbb{Z}^+$, we define a submonoid S_n of $\mathbb{N}$ by

$$S_n = \big\langle \{x_{n+j} : j \in \mathbb{N}\} \big\rangle.$$

Clearly, S_n is a numerical semigroup if and only if $\gcd(x_n, x_{n+1}, \ldots) = 1$.

Recent works have dealt with the family $(S_n)_n$ of numerical semigroups associated with a sequence $(x_n)_n$ of a special form, and they have studied the minimal generating set, the embedding dimension, the Frobenius number, among other properties of the family $(S_n)_n$. Some of these works [1, 2, 4, 5, 8, 9, 10, 12, 13, 14] have considered sequences that satisfy a linear recurrence relation of the form $x_{n+1} = ax_n + b$, where a and b are fixed integer numbers. Some other works [3, 6, 7] have treated quadratic sequences.

In this work, we specialize to sequences of k-th powers, that is, $x_n = n^k$, where $k \geq 2$, and we study upper bounds for the embedding dimension and the Frobenius number of the numerical semigroups S_n, $n \in \mathbb{Z}^+$.

We prove some properties of the minimal generators of the family $(S_n)_n$ associated with a general sequence $(x_n)_n$. Then, we apply these results to the sequence of squares $x_n = n^2$ to obtain that

$$\mathrm{e}(S_n) = O(n).$$

Finally, by making slight modifications to the ideas of Ducth and Rickett [3], we prove that

$$\mathrm{F}(S_n) = O(n^{k+\epsilon}),$$

for all $\epsilon > 0$, where $(S_n)_n$ is the family of numerical semigroups associated with the sequence $x_n = n^k$, where $k \geq 2$.

n	$\beta(S_n)$
1	$\{1^2\}$
2	$\{2^2, 3^2\}$
3	$\{3^2, 4^2, 7^2\}$
4	$\{4^2, 5^2, 6^2, 7^2\}$
5	$\{5^2, 6^2, 7^2, 8^2, 9^2\}$
6	$\{6^2, 7^2, 8^2, 9^2, 13^2\}$
7	$\{7^2, 8^2, 9^2, 10^2, 11^2, 12^2, 13^2\}$
8	$\{8^2, 9^2, 10^2, 11^2, 12^2, 13^2, 14^2\}$
9	$\{9^2, 10^2, 11^2, 12^2, 13^2, 14^2, 16^2, 17^2, 19^2, 21^2, 23^2\}$
10	$\{10^2, 11^2, 12^2, 13^2, 14^2, 15^2, 16^2, 17^2, 18^2, 19^2, 21^2, 23^2\}$
11	$\{11^2, 12^2, 13^2, 14^2, 15^2, 16^2, 17^2, 18^2, 19^2, 21^2, 23^2\}$
12	$\{12^2, 13^2, 14^2, 15^2, 16^2, 17^2, 18^2, 19^2, 21^2, 23^2\}$
13	$\{13^2, 14^2, 15^2, 16^2, 17^2, 18^2, 19^2, 20^2, 21^2, 22^2, 23^2, 24^2, 27^2\}$
14	$\{14^2, 15^2, 16^2, 17^2, 18^2, 19^2, 20^2, 21^2, 22^2, 23^2, 24^2, 26^2, 27^2\}$

Table 1: Minimal generating sets associated with squares.

2. Upper Bounds for the Embedding Dimension of Numerical Semigroups Generated by Sequences of Positive Integers

Suppose $(x_n)_n$ is a strictly increasing sequence of positive integers. For each positive integer n, we define $S_n = \langle \{x_{n+j} : j \in \mathbb{N}\} \rangle$, and we assume $\gcd(x_n, x_{n+1}, \ldots) = 1$ for all $n \in \mathbb{Z}^+$, so S_n is a numerical semigroup. We are interested in studying the minimal generators and the embedding dimension of S_n.

Example 1. Consider the sequence of squares $x_n = n^2$. Because n^2 and $(n+1)^2$ are relatively prime, the submonoid S_n associated with $(x_n)_n$ is a numerical semigroup. In Table 1, we compute the minimal generating set of S_n, for $1 \leq n \leq 14$.

To have a better visualization of these minimal generating sets, in Figure 1 we have plotted lattice points (k, n) satisfying the condition $x_k = k^2 \in \beta(S_n)$. So, we can identify the elements of $\beta(S_n)$ on the horizontal line passing through the point $(0, n)$. For instance, if $n = 6$, we see that the elements of $\beta(S_6)$ corresponds to the following values of k: $6, 7, 8, 9, 13$.

Now, for k in the x-axis, we note that the points above k correspond to the integers n for which $x_k \in \beta(S_n)$. We observe that each x_k has a *first appearance* as a minimal generator element of some S_n. For instance, if $k = 13$, then $x_{13} \in \beta(S_n)$ if and only if $6 \leq n \leq 13$, so the first time $x_{13} = 13^2$ appears in some S_n occurs when $n = 6$.

Motivated by the above example, for a given strictly increasing sequence of positive integers $(x_n)_n$, we define, for every $k \in \mathbb{Z}^+$ an integer r_k to be the smallest n such that $x_k \in \beta(S_n)$.

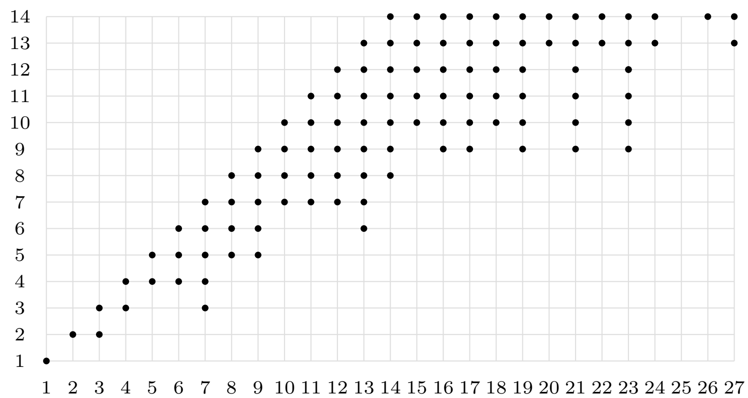

Figure 1: Diagram of minimal generating sets associated with $x_n = n^2$.

Lemma 1. *Let k and n be positive integers. Then, $x_k \in \beta(S_n)$ if and only if $r_k \le n \le k$. Thus,*

$$\mathrm{e}(S_n) = |\{k : r_k \le n \le k\}|\,.$$

Proof. It is clear that if $r_k \le n \le k$, then $x_k \in \beta(S_n)$. Now, by the definition of r_k, we have $x_k \notin \beta(S_n)$ whenever $n < r_k$. On the other hand, since $(x_n)_n$ is strictly increasing, it follows that if $n > k$, then $x_k \notin S_n$, so $x_k \notin \beta(S_n)$. This proves that if $x_k \in \beta(S_n)$, then $r_k \le n \le k$. □

We can determine information about $\beta(S_n)$ and $\mathrm{e}(S_n)$ with the aid of the numbers r_k, as we show in the next example and proposition.

Example 2. Let us suppose that $r_k = k - k_0$ for all $k > k_0$, where $k_0 \in \mathbb{Z}^+$. Then, $x_k \in \beta(S_n)$ if and only if $k - k_0 \le n \le k$, which is equivalent to $n \le k \le n + k_0$. Therefore, $\beta(S_n) \subseteq \{x_n, x_{n+1}, \ldots, x_{n+k_0}\}$ and $\mathrm{e}(S_n) \le k_0 + 1$.

In general, if we define r_n^* to be the greatest integer k such that $r_k \le n$, then we have $\beta(S_n) \subseteq \{x_k : n \le k \le r_n^*\}$ and $\mathrm{e}(S_n) \le r_n^* - n + 1$ for all $n \ge 1$.

Proposition 1. *Assume $r_k \ge \alpha k$ for all k, where α is a positive (real) constant that does not depend on k. Then*

$$\mathrm{e}(S_n) \le \left\lfloor \frac{1}{\alpha} n \right\rfloor - n + 1,$$

for all n.

Proof. The inequality $\alpha k \le n$ is equivalent to $k \le (1/\alpha)n$, so the greatest k for which $r_k \le n$ is at most $\lfloor (1/\alpha)n \rfloor$, that is, $r_n^* \le \lfloor (1/\alpha)n \rfloor$. □

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\mathrm{e}(S_n)$	1	2	3	4	5	5	7	7	11	12	11	10	13	13

Table 2: Embedding dimensions associated with $x_n = n^2$, $1 \leq n \leq 14$.

Now, we apply the previous results to the sequence $x_n = n^2$. In Table 2, we show the values of $\mathrm{e}(S_n)$, for $1 \leq n \leq 14$. This table suggests that $\mathrm{e}(S_n) = O(n)$. We will prove this assertion, for which we need the following result.

Lemma 2 ([7]). *There is a positive constant C such that for every positive integer n there are integers a_1, a_2, a_3 and a_4 such that*

$$n = a_1^2 + a_2^2 + a_3^2 + a_4^2,$$

and for each $i \in \{1, 2, 3, 4\}$, $a_i = 0$ or $a_i \geq \dfrac{\sqrt{n}}{C}$.

Remark 1. The proof of Lemma 2 in [7] relies on properties of the distribution of integral points in a 3-dimensional sphere of radius $\sqrt{n}$, as well as properties of the sum of divisors function $\sigma(n)$. It is also pointed out in [7] that the constant C is bounded below by 8.

Theorem 1. *For the family of numerical semigroups $(S_n)_n$ associated with the sequence of square numbers $x_n = n^2$, we have $\mathrm{e}(S_n) = O(n)$.*

Proof. We prove that if C is the constant of Lemma 2, then $r_k \geq \dfrac{k}{C}$ for all k. In fact, by Lemma 2, there is a representation $k^2 = a_1^2 + a_2^2 + a_3^2 + a_4^2$, where $a_j \geq \sqrt{k^2}/C = k/C$ whenever $a_j > 0$. If a_{j_0} is the least positive integer among the a_j's, then

$$r_k \geq a_{j_0} \geq \frac{k}{C},$$

so by Proposition 1, we get

$$\mathrm{e}(S_n) \leq \lfloor Cn \rfloor - n + 1 \leq (C-1)n + 1.$$

This proves that $\mathrm{e}(S_n) = O(n)$. □

3. The Frobenius Number of S_n Associated with $x_n = n^k$, $k \geq 2$

Dutch and Rickett [3] dealt with the sequence $x_n = n^2$ and proved that $\mathrm{F}(S_n) = O(n^{2+\epsilon})$ for all $\epsilon > 0$. Moscariello [7] proved that the ϵ can be dropped out, so $\mathrm{F}(S_n) = O(n^2)$ (his proof makes essential use of Lemma 2).

In this section, we make slight modifications to the ideas of Dutch and Rickett to prove that $F(S_n) = O(n^{k+\epsilon})$ for all $\epsilon > 0$, where $(S_n)_n$ is the family of numerical semigroups associated with the sequence $x_n = n^k$, where $k \geq 2$. Since n^k and $(n+1)^k$ are relatively prime, we see that the monoids S_n are numerical semigroups.

For an arbitrary numerical semigroup S, let $\tau(S)$ be the *conductor* of S, namely, the smallest integer m such that all integers $n \geq m$ are in S. Clearly, $\tau(S) = \mathrm{F}(S) + 1$. From this equality and Sylvester's formula (Equation (1)), we see that if a and b are relatively prime positive integers, then

$$\tau(\langle a, b\rangle) = (a-1)(b-1). \tag{2}$$

For simplicity, if A is a subset of $\mathbb{N}$ such that $\gcd(A) = 1$, we write $\tau(A)$ instead of $\tau(\langle A\rangle)$.

Lemma 3 ([3]). *Let A and B be nonempty subsets of $\mathbb{N}$ such that $\gcd(A \cup B) = 1$, and let t be a divisor of all elements of A. If $C = \{a/t : a \in A\}$, then*

$$\tau(A \cup B) \leq t \cdot \tau(C \cup B) + \tau(B \cup \{t\}).$$

Now, we consider the sequence $x_n = n^k$, where $k \geq 2$ is fixed, and the associated family of numerical semigroups $(S_n)_n$.

Proposition 2 ([3]). *For positive integers n and k we have*

$$\tau(S_n) \leq 2^k \tau\left(S_{\lceil \frac{n}{2} \rceil}\right) + (2^k - 1)(2n)^k. \tag{3}$$

Proof. In Lemma 3, let $A = \{m^k : m \geq n, m \text{ is even}\}$, $B = \{m^k : m \geq n, m \text{ is odd}\}$ and $t = 2^k$. If $C = \{m^k/2^k : m \geq n\ m \text{ is even}\}$, we observe that $B \subseteq C$. It is easily seen that $C = \left\{m^k : m \geq \lceil \frac{n}{2} \rceil\right\}$, so

$$\tau(B \cup C) = \tau(C) = \tau\left(S_{\lceil \frac{n}{2} \rceil}\right).$$

On the other hand, the minimum element of B, say β, is either equal to n^k or $(n+1)^k$. By using Equation (2), we have

$$\tau\left(B \cup \{2^k\}\right) \leq \tau(\{2^k, \beta\}) = (2^k - 1)(\beta - 1) \leq (2^k - 1)(n+1)^k \leq (2^k - 1)(2n)^k.$$

The result follows by applying Lemma 3. □

Theorem 2. *For the family of numerical semigroups $(S_n)_n$ associated with the sequence of k-th powers $x_n = n^k$, where $k \geq 2$, we have $\mathrm{F}(S_n) = O\left(n^{k+\epsilon}\right)$ for all $\epsilon > 0$.*

Proof. Let $\epsilon > 0$. We are going to prove that there is a positive constant K such that $\tau(S_n) \leq K n^\lambda$ for all n, where $\lambda = k + \epsilon$. First of all, choose a positive integer N such that

$$N > \frac{1}{2^{(1-k/\lambda)} - 1}.$$

The reason for this choice will appear later. For all real $x \geq N$, we have $\frac{1}{x} < 2^{(1-k/\lambda)} - 1$, which is equivalent to

$$x^\lambda - 2^k \left[(x+1)/2\right]^\lambda > 0.$$

By setting

$$K_1 = \max\left(\tau(S_1), \frac{\tau(S_2)}{2^\lambda}, \ldots, \frac{\tau(S_N)}{N^\lambda}\right),$$

we immediately see that $\tau(S_n) \leq K_1 n^\lambda$ for all $n \in \{1, \ldots, N\}$. Next, as cited above, we shall prove the existence of K that satisfies $\tau(S_n) \leq K n^\lambda$ for all n. To do this, we shall introduce the real function

$$f(x) = \frac{\left(2^k - 1\right)(2x)^k}{x^\lambda - 2^k \left[(x+1)/2\right]^\lambda},$$

defined on $[N, +\infty)$. Clearly, f is continuous and tends to 0 as x tends to $+\infty$. In particular, there exists $M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in [N, +\infty)$. Let

$$K = \max\left(K_1, M\right).$$

We shall prove that K satisfies the inequality above. This is clear for $n \in \{1, \ldots, N\}$, and we shall prove by induction that it is also true for $n > N$. Let $n > N$ and assume that $\tau(S_m) \leq K m^\lambda$ for all $m < n$. In particular, for $m = \lceil \frac{n}{2} \rceil$, we have

$$\tau\left(S_{\lceil \frac{n}{2} \rceil}\right) \leq K\left(\left\lceil \frac{n}{2} \right\rceil\right)^\lambda \leq K\left(\frac{n+1}{2}\right)^\lambda.$$

Then, by Proposition 2, we obtain

$$\begin{aligned}
\tau(S_n) &\leq 2^k \tau\left(S_{\lceil \frac{n}{2} \rceil}\right) + (2^k - 1)(2n)^k \\
&\leq 2^k K \left(\frac{n+1}{2}\right)^\lambda + (2^k - 1)(2n)^k \\
&= K n^\lambda \left[2^k \left(\frac{n+1}{2n}\right)^\lambda + \frac{(2^k - 1)(2n)^k}{K n^\lambda}\right] \\
&\leq K n^\lambda \left[2^k \left(\frac{n+1}{2n}\right)^\lambda + \frac{(2^k - 1)(2n)^k}{f(n) n^\lambda}\right]
\end{aligned}$$

$$
\begin{aligned}
&= Kn^{\lambda}\left[2^k\left(\frac{n+1}{2n}\right)^{\lambda}+\frac{1}{n^{\lambda}}\left(n^{\lambda}-2^k\left(\frac{n+1}{2}\right)^{\lambda}\right)\right]\\
&= Kn^{\lambda}.
\end{aligned}
$$

This ends the proof. □

4. Final Remarks

We have proved the following two results related to the family of numerical semigroups $(S_n)_n$ associated with a sequence of the form $x_n = n^k$, where $k \geq 2$:

1. For the sequence $x_n = n^2$, we have $\mathrm{e}(S_n) = O(n)$;
2. For the sequence $x_n = n^k$, where $k \geq 2$, we have $\mathrm{F}(S_n) = O(n^{k+\epsilon})$ for all $\epsilon > 0$.

We have the following two questions, concerning the case $k \geq 3$:

1. Is it true that $\mathrm{e}(S_n) = O(n)$?
2. Is it true that $\mathrm{F}(S_n) = O(n^k)$?

Moscariello's proof of the relation $\mathrm{F}(S_n) = O(n^2)$, for $x_n = n^2$, strongly relies on Lemma 2. As we mentioned above, the proof of this lemma is based on properties of the distribution of integral points in a 3-dimensional sphere of radius $\sqrt{n}$ and the function $\sigma(n)$. Our proof of Theorem 2 also makes use of this lemma. We do not see a straightforward generalization of the proof of Lemma 2, for k-th powers, where $k \geq 3$.

We know, by the Hilbert-Waring theorem on the Waring problem, that for every $k \geq 2$, there exists a positive integer $g(k)$ such that every $n \in \mathbb{Z}^+$ can be expressed as a sum of $g(k)$ non-negative k-th powers. We ask the following question: Is there a positive constant C_k such that for every $n \in \mathbb{Z}^+$, there is a representation

$$
n = \sum_{j=1}^{g(k)} a_i^k,
$$

where $a_j = 0$ or $a_j \geq \frac{\sqrt[k]{n}}{C_k}$ for all $j \in \{1, \ldots, g(k)\}$? A positive answer to this question would settle our questions about $\mathrm{e}(S_n)$ and $\mathrm{F}(S_n)$, for $k \geq 3$.

Acknowledgement. The authors would like to thank the anonymous referee for the comments and suggestions which improved the quality and presentation of this paper.

References

[1] F. Arias, J. Borja, and C. Rhenals, The Frobenius problem for numerical semigroups generated by sequences of the form $ca^n - d$, *Semigroup Forum* **107** (2023), 581-608.

[2] M. B. Branco, I. Colaço, and I. Ojeda, The Frobenius problem for generalized repunit numerical semigroups, *Mediterr. J. Math.* **20** (16) (2023), #A16.

[3] K. Dutch and C. Rickett, Conductors for sets of large integer squares, *Notes on Number Theory and Discrete Mathematics* **18** (1) (2012), 16-21.

[4] Z. Gu, On the numerical semigroup generated by $\left\{b^{n+i+1} + \frac{b^{n+i}-1}{b-1}\right\}$, *Discrete Math. Appl.* **30** (4) (2020), 257-264.

[5] Z. Gu and X. Tang, The Frobenius problem for a class of numerical semigroups, *Int. J. Number Theory* **13** (2017), 1-13.

[6] M. Hashuga, M. Herbine, and A. Jensen, Numerical semigroups generated by quadratic sequences, *Semigroup Forum* **104** (2022), 330-357.

[7] A. Moscariello, On integers which are representable as sums of large squares, *Int. J. Number Theory*, **11** (8) (2015), 2505-2511.

[8] J. C. Rosales, M. B. Branco, and D. Torrão, The Frobenius problem for Mersenne numerical semigroups, *Math. Z.* **286** (2017), 1-9.

[9] J. C. Rosales, M. B. Branco, and D. Torrão, The Frobenius problem for repunit numerical semigroups, *Ramanujan J.* **40** (2016), 323-334.

[10] J. C. Rosales, M. B. Branco, and D. Torrão, The Frobenius problem for Thabit numerical semigroups, *J. Number Theory* **155** (2015), 85-99.

[11] J. C. Rosales and P. A. García-Sánchez, *Numerical Semigroups*, Springer-Verlag, New York, 2009.

[12] K. Song, The Frobenius problem for extended Thabit numerical semigroups, *Integers* **21** (2021), #A17.

[13] K. Song, The Frobenius problem for numerical semigroups generated by the Thabit numbers of the first, second kind base b and the Cunningham numbers, *Bull. Korean Math. Soc.* **57** (3) (2020), 623-647.

[14] P. Srivastava and D. Thakkar, The Frobenius problem for the Proth numbers, in *Algorithms and Discrete Applied Mathematics: 10th Internationa Conference, CALDAM 2024*, Bhilai, India, 2024.