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A SIMPLE PROOF OF THE FORMULA FOR THE SUM OF TRIBONACCI AND TETRANACCI NUMBERS SQUARED

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Abstract

We offer a very short derivation of the sum of the squares of the Tribonacci and the Tetranacci numbers with arbitrary initial conditions. Our method avoids generating functions and instead relies on elementary recurrence identities and telescoping sums.

1. Motivation and Preliminaries

In the literature, there are many articles related to the sum of squares of the Fibonacci numbers [1, 3], the Tribonacci numbers [2, 6], or the Tetranacci numbers [4, 5, 9]. Howard and Cooper [1] attempt to derive a sum of squares for arbitrary Fibonacci m-step numbers in their paper, but their result, presented in Theorem 3.3, leaves room for further improvement.

Schumacher [5] obtains the explicit formula for the squares of the Tetranacci numbers, but his (long) proof uses induction. To prove the identity, Schumacher had to guess the desired formula. Prodinger and Selkirk [4] use the generating function approach, but their paper relies on computer-assisted calculations. Furthermore, Shah [7] obtains a formula for the sum of squared Tribonacci numbers, but he also uses an induction. He also refers to Zeitlin's formula [10], which is more complicated. Soykan [9] finds the sum of squares and other related sums through a very long computation.

In this article, we are concerned with the sequences F_n, T_n and Q_n, which are the Fibonacci, the Tribonacci, and the Tetranacci numbers, respectively, defined for arbitrary initial conditions. Their respective recurrence relations are defined as usual; that is, for example,

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n, \qquad n \geq 0.$$

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These sequences can be extended to negative indices via the relation displayed for the Tribonacci case, which clearly translates to the remaining cases:

$$T_n = T_{n+3} - T_{n+2} - T_{n+1}, \qquad n \leq 0.$$

More information about these sequences can be found in [8] (entries A000045, A000073, and A000078).

The goal of this note is to present a very simple and direct derivation of the sums of squares of Tribonacci and Tetranacci numbers with arbitrary initial conditions. Our reasoning is a simplified version of the argument proposed by Jakubczyk [2] and also applies to sequences with arbitrary initial conditions. We also note that our proof is self-contained.

2. Main Results

2.1. Sum of Squares of the Tribonacci Numbers

Let us introduce the notation:

$$S(n) = \sum_{k=1}^{n} T_k^2, \qquad R(n) = \sum_{k=1}^{n} T_k T_{k+1}, \qquad P(n) = \sum_{k=1}^{n} T_k T_{k-2}.$$

Lemma 1. *The closed-form expression for the sum $P(n)$ is*

$$P(n) = \frac{(T_{n+1} - T_{n-1})^2 - (T_1 - T_{-1})^2}{4}. \tag{1}$$

Proof. Notice that

$$\begin{aligned}(T_{k+1} - T_{k-1})^2 &- (T_k - T_{k-2})^2 \\ &= (T_{k+1} - T_k - T_{k-1} + T_{k-2})(T_{k+1} + T_k - T_{k-1} - T_{k-2}) \\ &= 4T_k T_{k-2},\end{aligned}$$

so using the telescoping summation we immediately get Equation (1). □

Theorem 1. *The sum $S(n)$ admits the closed form*

$$S(n) = \frac{4T_n T_{n+1} - 4T_0 T_1 - (T_{n+1} - T_{n-1})^2 + (T_1 - T_{-1})^2}{4}. \tag{2}$$

Proof. Note that

$$R(n) = \sum_{k=1}^{n} (T_k^2 + T_k T_{k-1} + T_k T_{k-2}) = S(n) + (T_0 T_1 + R(n) - T_n T_{n+1}) + P(n).$$

This gives

$$S(n) = T_nT_{n+1} - T_0T_1 - P(n)$$

and the formula (2) follows. □

Let us note that the simpler version of the just-presented proof works for the Fibonacci numbers. Namely, we have

$$\bar{R}(n) := \sum_{k=1}^{n} F_kF_{k+1} = \sum_{k=1}^{n} F_k^2 + \sum_{k=1}^{n} F_kF_{k-1} = \sum_{k=1}^{n} F_k^2 + (F_0F_1 + \bar{R}(n) - F_nF_{n+1})$$

from which we obtain

$$\sum_{k=1}^{n} F_k^2 = F_nF_{n+1} - F_0F_1.$$

2.2. Sum of Squares of the Tetranacci Numbers

We extend the idea presented in the proof of the sum of squared Tribonacci numbers to the Tetranacci case. We begin with the following lemma.

Lemma 2. *The identity*

$$\begin{aligned} 3\sum_{k=1}^{n} Q_k(Q_{k-2} + Q_{k-3}) = (Q_{n+1} - Q_{n-1})^2 - (Q_1 - Q_{-1})^2 \\ - Q_{n-2}(Q_n + Q_{n-3}) + Q_{-2}(Q_0 - Q_{-3}) \end{aligned} \tag{3}$$

holds.

Proof. The identity again relies on telescoping summation. To find out how to apply it, we perform a few minor algebraic manipulations. This starts from a difference of squares with the same structure as in the Tribonacci case:

$$\begin{aligned} &(Q_{k+1} - Q_{k-1})^2 - (Q_k - Q_{k-2})^2 \\ &\quad = (Q_{k+1} - Q_k - Q_{k-1} + Q_{k-2})(Q_{k+1} + Q_k - Q_{k-1} - Q_{k-2}) \\ &\quad = (2Q_{k-2} + Q_{k-3})(2Q_k + Q_{k-3}) \\ &\quad = 4Q_kQ_{k-2} + 2Q_{k-2}Q_{k-3} + 2Q_kQ_{k-3} + Q_{k-3}^2 \\ &\quad = 3Q_k(Q_{k-2} + Q_{k-3}) + Q_{k-2}(Q_k + Q_{k-3}) - Q_{k-3}(Q_k - Q_{k-2} - Q_{k-3}) \\ &\quad = 3Q_k(Q_{k-2} + Q_{k-3}) + Q_{k-2}(Q_k + Q_{k-3}) - Q_{k-3}(Q_{k-1} + Q_{k-4}). \end{aligned}$$

Rearranging gives

$$\begin{aligned} 3Q_k(Q_{k-2} + Q_{k-3}) = (Q_{k+1} - Q_{k-1})^2 - (Q_k - Q_{k-2})^2 \\ - Q_{k-2}(Q_k + Q_{k-3}) + Q_{k-3}(Q_{k-1} + Q_{k-4}) \end{aligned}$$

from which the lemma is easily concluded. □

We are now ready to derive the sum of squares of the Tetranacci numbers. The motivation for considering Equation (3) becomes clear when we attempt to repeat the argument used for deriving $S(n)$.

Theorem 2. *It holds that*

$$\sum_{k=1}^{n} Q_k^2 = \frac{3Q_nQ_{n+1} - 3Q_0Q_1 - (Q_{n+1} - Q_{n-1})^2 + (Q_1 - Q_{-1})^2}{3} + \frac{Q_{n-2}(Q_n + Q_{n-3}) - Q_{-2}(Q_0 - Q_{-3})}{3}.$$

Proof. Note that

$$\begin{aligned}\sum_{k=1}^{n} Q_kQ_{k+1} &= \sum_{k=1}^{n}(Q_k^2 + Q_kQ_{k-1} + Q_k(Q_{k-2} + Q_{k-3})) \\ &= \sum_{k=1}^{n} Q_k^2 + \left(Q_0Q_1 + \sum_{k=1}^{n} Q_kQ_{k+1} - Q_nQ_{n+1}\right) + \sum_{k=1}^{n} Q_k(Q_{k-2} + Q_{k-3}).\end{aligned}$$

Rearranging gives

$$\sum_{k=1}^{n} Q_k^2 = Q_nQ_{n+1} - Q_0Q_1 - \sum_{k=1}^{n} Q_k(Q_{k-2} + Q_{k-2}),$$

and applying Lemma 2 we get the final result. □

References

[1] C. Cooper and F. T. Howard, Some identities for r-Fibonacci numbers, *Fibonacci Quart.* **49** (3) (2011), 231–242.

[2] Z. Jakubczyk, Sums of squares of Tribonacci numbers (Solution of Advanced Problem H-715), *Fibonacci Quart.* **51** (3) (2013), 285–286.

[3] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons, New York, 2001.

[4] H. Prodinger and S. J. Selkirk, Sums of squares of Tetranacci numbers: A generating function approach, *Fibonacci Quart.* **57** (4) (2019), 313–317.

[5] R. Schumacher, How to sum the squares of the Tetranacci numbers and the Fibonacci m-step numbers, *Fibonacci Quart.* **57** (2) (2019), 168–175.

[6] R. Schumacher, Explicit formulas for sums involving the squares of the first n Tribonacci numbers, *Fibonacci Quart.* **58** (3) (2020), 194–202.

[7] D. V. Shah, Some Tribonacci identities, *Math. Today* **27** (2011), 1–9.

[8] N. J. A. Sloane, *OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences,* `https://oeis.org`.

[9] Y. Soykan, A study on generalized Tetranacci numbers: Closed form formulas $\sum_{k=0}^{n} x^k W_k^2$ of sums of the squares of terms, *Asian Res. J. Math.* **16** (10) (2020), 109–136.

[10] D. Zeitlin, On summation formulas and identities for Fibonacci numbers, *Fibonacci Quart.* **5** (1) (1967), 1–43.