BALANCING AND LUCAS-BALANCING NUMBERS AS THE DIFFERENCE OF TWO REPDIGITS

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Abstract

Positive integers with all digits equal are called repdigits. In this paper, we find all balancing and Lucas-balancing numbers which can be expressed as the difference of two repdigits. The method of proof involves the application of Baker's theory for linear forms in logarithms of algebraic numbers and the Baker–Davenport reduction procedure.

1. Introduction

The balancing number sequence $(B_n)_{n\geq 0}$ and the Lucas-balancing sequence $(C_n)_{n\geq 0}$ are defined by the binary recurrences

$$B_{n+1} = 6B_n - B_{n-1}, \quad B_0 = 0, \quad B_1 = 1$$

and

$$C_{n+1} = 6C_n - C_{n-1}, \quad C_0 = 1, \quad C_1 = 3.$$

The Binet formulas for the sequences are given by

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$$B_n = \frac{\alpha^n - \beta^n}{4\sqrt{2}} \text{ and } C_n = \frac{\alpha^n + \beta^n}{2} \text{ for } n \geq 1,$$

where $(\alpha, \beta) = (3+2\sqrt{2}, 3-2\sqrt{2})$ is the pair of roots of the characteristic equation $x^2 - 6x - 1 = 0$. This easily implies that the inequalities

$$\alpha^{n-1} \leq B_n < \alpha^n \text{ and } \alpha^n < 2C_n < \alpha^{n+1},$$

hold for all $n \geq 1$.

A repdigit is a positive integer whose digits are all equal. Investigations of repdigits in second-order linear recurrence sequences have been of interest to mathematicians. All balancing and Lucas-balancing numbers which are repdigits were found in [12]. Rayaguru and Panda [14] identified all balancing and Lucas-balancing numbers which can be expressed as the sums of two repdigits. Mohapatra et al. [10] investigated the existence of repdigits as the difference of two balancing or Lucas-balancing numbers. Erduvan et al. [8] found all Fibonacci and Lucas numbers which are the difference of two repdigits. Edjeou and Faye [7] explored all Pell and Pell-Lucas numbers which can be written as the difference of two repdigits. Recently, Duman [6] obtained all Padovan numbers which are the difference of two repdigits. In this paper, we explore all balancing and Lucas-balancing numbers which can be written as the difference of two repdigits. For this purpose, we consider the following two equations

$$B_k = d_1\left(\frac{10^n - 1}{9}\right) - d_2\left(\frac{10^m - 1}{9}\right) \tag{1}$$

and

$$C_k = d_1\left(\frac{10^n - 1}{9}\right) - d_2\left(\frac{10^m - 1}{9}\right), \tag{2}$$

where k, m, n are positive integers with $n \geq 2$ and $d_1, d_2 \in \{1, 2, \ldots, 9\}$ such that B_k and C_k are positive integers.

2. Auxiliary Results

To solve the Diophantine equations involving repdigits and terms of binary recurrence sequences, many authors have used Baker's theory to reduce lower bounds concerning linear forms in logarithms of algebraic numbers. These lower bounds play an important role in solving such Diophantine equations. We begin by recalling some basic definitions and results from algebraic number theory.

Let λ be an algebraic number with minimal primitive polynomial

$$f(X) = a_0(X - \lambda^{(1)}) \cdots (X - \lambda^{(k)}) \in \mathbb{Z}[X],$$

where $a_0 > 0$ and $\lambda^{(i)}$'s are conjugates of λ. Then

$$h(\lambda)=\frac{1}{k}\left(\log a_0+\sum_{j=1}^{k}\max\{0,\log|\lambda^{(j)}|\}\right)$$

is called the logarithmic height of λ. If $\lambda=a/b$ is a rational number with $\gcd(a,b)=1$ and $b>1$, then $h(\lambda)=\log(\max\{|a|,b\})$.

We give some properties of the logarithmic height whose proofs can be found in [2, Theorem B5]. Let γ and η be two algebraic numbers, then

(i) $h(\gamma\pm\eta)\le h(\gamma)+h(\eta)+\log 2$,

(ii) $h(\gamma\eta^{\pm 1})\le h(\gamma)+h(\eta)$, and

(iii) $h(\gamma^k)=|k|h(\gamma)$.

Now we give a theorem which is deduced from Corollary 2.3 of Matveev [9] and provides a large upper bound for the subscript k in Equations (1) and (2) (see also Theorem 9.4 in [3]).

Theorem 1 ([9])**.** *Let $\gamma_1,\ldots,\gamma_l$ be positive real algebraic numbers in an algebraic number field $\mathbb{L}$ of degree $d_{\mathbb{L}}$ and $b_1,\ldots,b_l$ be nonzero integers. If*

$$\Gamma=\prod_{i=1}^{l}\gamma_i^{b_i}-1$$

is not zero, then

$$\log|\Gamma|>-1.4\cdot 30^{l+3}\cdot l^{4.5}\cdot d_{\mathbb{L}}^2(1+\log d_{\mathbb{L}})(1+\log D)A_1A_2\cdots A_l,$$

where $D\ge\max\{|b_1|,\ldots,|b_l|\}$ and $A_1,\ldots,A_l$ are positive integers such that

$$A_j\ge h'(\gamma_j)=\max\{d_{\mathbb{L}}h(\gamma_j),|\log\gamma_j|,0.16\},$$

for $j=1,\ldots,l$.

Next, we introduce a lemma proved in [1], which is a modification of the lemma given by Dujella and Pethő in [5]. It reduces the upper bound of the variable k in Equations (1) and (2).

Lemma 1 ([1])**.** *Let M be a positive integer and p/q be a convergent of the continued fraction of the irrational number τ such that $q>6M$. Let A,B,μ be some real numbers with $A>0$ and $B>1$. Let $\epsilon:=\|\mu q\|-M\|\tau q\|$, where $\|.\|$ denotes the distance from the nearest integer. If $\epsilon>0$, then there exists no solution to the inequality*

$$0<|u\tau-v+\mu|<AB^{-w},$$

in positive integers u,v, and w with

$$u\le M \text{ and } w\ge\frac{\log(Aq/\epsilon)}{\log B}.$$

We also need the following lemmas in order to achieve the objectives of this paper.

Lemma 2 ([4])**.** *Let* $a, x \in \mathbb{R}$*. If* $0 < a < 1$ *and* $|x| < a$*, then*

$$|\log(1+x)| < \frac{-|\log(1-a)|}{a}|x|$$

and

$$|x| < \frac{a}{1-e^{-a}}|e^x - 1|.$$

Lemma 3 ([13])**.** *The only balancing number which is the concatenation of two repdigits is 35.*

Lemma 4 ([11])**.** *The only balancing numbers which are concatenation of three repdigits are 204 and 1189.*

Lemma 5 ([15])**.** *The only Lucas-balancing numbers which are the concatenation of two repdigits are 17 and 577.*

Lemma 6 ([11])**.** *The only Lucas-balancing number which is concatenation of three repdigits is 3363.*

3. Balancing Numbers as the Difference of Two Repdigits

In this section, we will show that 6 and 35 are the only balancing numbers which can be expressed as the difference of two repdigits. Specifically, we will prove the following theorem.

Theorem 2. *If* B_k *is expressible as the difference of two repdigits, then* $B_k \in \{6, 35\}$*. In particular,* $B_2 = 6 = 11 - 5$ *and* $B_3 = 35 = 44 - 9$*.*

Proof. Assume that Equation (1) holds. Let $1 \leq k \leq 25$ and $n \geq 2$. Using $Mathematica$, one can obtain the solutions listed in Theorem 2. From now on, assume that $k > 25$.

If $n = m$, it follows that $d_1 > d_2$, which implies that B_k is a repdigit. However, the largest possible repdigit in B_k is 6 (see [12]), leading to a contradiction since $k > 25$. Next, consider the case where $n - m = 1$. If $d_1 \geq d_2$, we encounter balancing numbers which are the concatenation of two repdigits. This is impossible by Lemma 3. If $d_1 < d_2$, we obtain balancing numbers which are concatenation of three repdigits, contradicting Lemma 4. Therefore, we conclude that $n - m \geq 2$.

The inequality

$$\frac{\alpha^n}{20} < \frac{10^{n-1}}{2} < 10^{n-1} - 10^{m-1} < \frac{d_1(10^n - 1)}{9} - \frac{d_2(10^m - 1)}{9} = B_k < \alpha^k$$

implies that $n < k+2$. On the other hand, Equation (1) can be rewritten as

$$\frac{\alpha^k - \beta^k}{4\sqrt{2}} = \frac{d_1(10^n - 1)}{9} - \frac{d_2(10^m - 1)}{9}$$

to obtain

$$\frac{9\alpha^k}{4\sqrt{2}} - d_1 10^n = \frac{9\beta^k}{4\sqrt{2}} - d_2 10^m - (d_1 - d_2). \tag{3}$$

By taking the absolute value of both sides of Equation (3), we get

$$\left|\frac{9\alpha^k}{4\sqrt{2}} - d_1 10^n\right| \le \frac{9|\beta|^k}{4\sqrt{2}} + d_2 10^m + |d_1 - d_2|. \tag{4}$$

Division of both sides of Equation (4) by $d_1 10^n$ results in

$$\begin{aligned}\left|\frac{9 \cdot 10^{-n} \cdot \alpha^k}{d_1 \cdot 4\sqrt{2}} - 1\right| &\le \frac{9|\beta|^k}{d_1 10^n \cdot 4\sqrt{2}} + \frac{d_2 10^m}{d_1 10^n} + \frac{|d_1 - d_2|}{d_1 10^n} \\ &\le \frac{9|\beta|^k}{10^{n-m+1} \cdot 4\sqrt{2}} + \frac{9}{10^{n-m}} + \frac{8}{10^{n-m+1}},\end{aligned}$$

which implies

$$\left|\frac{9 \cdot 10^{-n} \cdot \alpha^k}{d_1 \cdot 4\sqrt{2}} - 1\right| < \frac{9.81}{10^{n-m}}. \tag{5}$$

Put $(\gamma_1, \gamma_2, \gamma_3) = (\alpha, 10, 9/(d_1 \cdot 4\sqrt{2}))$ and $(b_1, b_2, b_3) = (k, -n, 1)$. Notably, γ_1, γ_2, and γ_3 are positive real numbers and elements of the field $\mathbb{K} = \mathbb{Q}(\sqrt{2})$. Consequently, the degree of the field $\mathbb{K}$ is $d_{\mathbb{L}} = 2$. Define

$$\Gamma_1 = \frac{9 \cdot 10^{-n} \cdot \alpha^k}{d_1 \cdot 4\sqrt{2}} - 1.$$

If $\Gamma_1 = 0$, then $\alpha^k = 10^n d_1 \cdot 4\sqrt{2}/9$. By conjugating in $\mathbb{Q}(\sqrt{2})$, we obtain $\beta^k = -10^n d_1 \cdot 4\sqrt{2}/9$, which implies that $C_k = (\alpha^k + \beta^k)/2 = 0$. This leads to a contradiction. Therefore, $\Gamma_1 \neq 0$.

By the properties of the absolute logarithmic height, the logarithmic heights of γ_1, γ_2, and γ_3 are calculated as $h(\gamma_1) = (\log\alpha)/2$, $h(\gamma_2) = \log 10$, and

$$h(\gamma_3) \le h(4d_1\sqrt{2}) + h(9) < 6.2.$$

Accordingly, we set $A_1 = \log\alpha$, $A_2 = 2\log 10$, and $A_3 = 12.4$. Since $n < k+2$, we take $D = k+2 \ge \max\{n, k, 1\}$. Thus, using (5) and Theorem 1, we obtain

$$\log\left(\frac{9.81}{10^{n-m}}\right) > \log|\Gamma_1| > C \cdot (1+\log 2)(1+\log(k+2))(\log\alpha)(2\log 10)(12.4),$$

where $C = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2$. Through a simple calculation, the above inequality leads to

$$\begin{aligned}(m-n)\log(\alpha) &< \log(9.81) + 9.8 \cdot 10^{13}(1+\log(k+2)) \\ &< 9.9 \cdot 10^{13}(1+\log(k+2)).\end{aligned} \tag{6}$$

Rearranging Equation (1) as

$$\frac{\alpha^k}{4\sqrt{2}} - \frac{d_1 10^n - d_2 10^m}{9} = \frac{\beta^k}{4\sqrt{2}} - \frac{(d_1 - d_2)}{9} \tag{7}$$

and taking the absolute value of both sides of Equation (7), we get

$$\left|\frac{\alpha^k}{4\sqrt{2}} - \frac{d_1 10^n - d_2 10^m}{9}\right| \leq \frac{|\beta^k|}{4\sqrt{2}} + \frac{|d_1 - d_2|}{9}. \tag{8}$$

Dividing both sides of Equation (8) by $\alpha^k/4\sqrt{2}$, we obtain

$$\left|1 - \frac{(d_1 - d_2 10^{m-n}) \cdot 10^n \cdot \alpha^{-k} \cdot 4\sqrt{2}}{9}\right| \leq \frac{1}{\alpha^{2k}} + \frac{8 \cdot 4\sqrt{2}}{9\alpha^k} < \frac{6}{\alpha^k}. \tag{9}$$

Now, take

$$(\gamma_1, \gamma_2, \gamma_3) = \left(10, \alpha, \frac{(d_1 - d_2 10^{m-n}) \cdot 4\sqrt{2}}{9}\right) \text{ and } (b_1, b_2, b_3) = (n, -k, 1).$$

Note that γ_1, γ_2, and γ_3 are positive real numbers and elements of the field $\mathbb{K} = \mathbb{Q}(\sqrt{2})$. Thus, the degree of the field $\mathbb{K}$ is $d_{\mathbb{L}} = 2$. Let

$$\Gamma_2 = 1 - \frac{(d_1 - d_2 10^{m-n}) \cdot 10^n \cdot \alpha^{-k} \cdot 4\sqrt{2}}{9}.$$

If $\Gamma_2 = 0$, then $\alpha^{2k} \in \mathbb{Q}$, which is false for $k > 0$. Therefore, $\Gamma_2 \neq 0$.

Using the properties of the absolute logarithmic height, we get

$$h(\gamma_1) = \log 10, \quad h(\gamma_2) = h(\alpha) = \frac{\log \alpha}{2},$$

and

$$\begin{aligned}h(\gamma_3) &= h\left(\frac{(d_1 - d_2 10^{m-n}) \cdot 4\sqrt{2}}{9}\right) \\ &\leq h(9) + h(4\sqrt{2}) + h(d_1) + h(d_2) + (n-m)\log 10 + \log 2 \\ &< 9.02 + (n-m)\log 10.\end{aligned}$$

Thus, we can set $A_1 = 2\log 10$, $A_2 = \log \alpha$, and $A_3 = 18.04 + 2(n-m)\log 10$. Since $n < k+2$, we take $D = k+2 \geq \max\{n, k, 1\}$. Taking into account Equation (9) and Theorem 1, we arrive at

$$6 \cdot \alpha^{-k} > |\Gamma_2| > e^{(C \cdot (1+\log 2)(1+\log(k+2))(\log \alpha)(2\log 10)(18.04+2(n-m)\log 10))},$$

where $C = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2$. A simple computation shows that

$$k \log \alpha - \log 6 < 7.9 \cdot 10^{12} \cdot (1 + \log(k+2))(18.04 + 2(n-m)\log 10). \tag{10}$$

Using Equations (6) and (10), and a computer search with *Mathematica*, we find $k < 1.7 \cdot 10^{28}$.

Now we reduce the upper bound on k by using the Baker–Davenport algorithm given in Lemma 1. Define

$$\Lambda_1 = k \log \alpha - n \log 10 + \log\left(\frac{9}{d_1 \cdot 4\sqrt{2}}\right).$$

In view of Equation (5), we have

$$|x| = |e^{\Lambda_1} - 1| < \frac{9.81}{10^{n-m}} < \frac{1}{10},$$

for $n - m \geq 2$. Choosing $a = 0.1$ and using Lemma 2, we get the inequality

$$|\Lambda_1| = |\log(x+1)| < \frac{\log(10/9)}{1/10} \cdot \frac{9.81}{10^{n-m}} < (10.34) \cdot 10^{m-n}.$$

Consequently,

$$0 < \left| k \log \alpha - n \log 10 + \log\left(\frac{9}{d_1 \cdot 4\sqrt{2}}\right) \right| < (10.34) \cdot 10^{m-n}.$$

Dividing this inequality by $\log 10$, we obtain

$$0 < \left| k\left(\frac{\log \alpha}{\log 10}\right) - n + \frac{\log\left(\frac{9}{d_1 \cdot 4\sqrt{2}}\right)}{\log 10} \right| < (4.5) \cdot 10^{m-n}. \tag{11}$$

We can take $\tau = \log \alpha / \log 10 \notin \mathbb{Q}$ and $M = 1.7 \cdot 10^{28}$. Then, we find that

$$q_{59} = 808643106803003389273254071835,$$

which is the denominator of the 59^{th} convergent of τ, and is greater than $6M$. Now, let

$$\mu = \frac{\log(9/(d_1 \cdot 4\sqrt{2}))}{\log 10}.$$

Considering the fact that $1 \leq d_1 \leq 9$, a quick computation with *Mathematica* gives

$$\epsilon(\mu) := \|\mu q_{59}\| - M\,\|\tau q_{59}\| = 0.03855.$$

Let $A = 4.5$, $B = 10$, and $\omega = n - m$ in Lemma 1. Using *Mathematica*, we conclude that Equation (11) has no solution if

$$\frac{\log(A q_{59}/\epsilon(\mu))}{\log B} < 31.97 < n - m.$$

So, $n-m \leq 31$. Substituting this upper bound for $n-m$ in Equation (10), we obtain $k < 3.1 \cdot 10^{15}$. Next, define

$$\Lambda_2 = n\log 10 - k\log\alpha + \log\left(\frac{(d_1 - d_2 10^{m-n})\cdot 4\sqrt{2}}{9}\right).$$

In view of Equation (9), we have

$$|x| = |e^{\Lambda_2} - 1| < \frac{6}{\alpha^k} < \frac{1}{10},$$

for $k > 25$. By choosing $a = 0.1$ and using Lemma 2, we obtain

$$|\Lambda_2| = |\log(x+1)| < \frac{\log(10/9)}{1/10}\cdot\frac{6}{\alpha^k} < (6.33)\cdot\alpha^{-k},$$

from which, it follows that

$$0 < \left| n\log 10 - k\log\alpha + \log\left(\frac{(d_1 - d_2 10^{m-n})\cdot 4\sqrt{2}}{9}\right)\right| < (6.33)\cdot\alpha^{-k}. \tag{12}$$

Dividing both sides of Equation (12) by $\log\alpha$, we obtain

$$0 < \left| n\left(\frac{\log 10}{\log\alpha}\right) - k + \frac{\log\left(\frac{(d_1 - d_2 10^{m-n})\cdot 4\sqrt{2}}{9}\right)}{\log\alpha}\right| < (3.6)\cdot\alpha^{-k}. \tag{13}$$

Putting $\tau = \log 10/\log\alpha \notin \mathbb{Q}$ and taking $M = 3.1\cdot 10^{15}$, we find that

$$q_{36} = 73257846218558279,$$

the denominator of the 36^{th} convergent of τ, exceeds $6M$. Now, let

$$\mu = \frac{\log\left(\frac{(d_1 - d_2 10^{m-n})\cdot 4\sqrt{2}}{9}\right)}{\log\alpha}.$$

In this case, considering the fact that $1 \leq d_1,\, d_2 \leq 9$ and $2 \leq n-m \leq 31$, a quick computation gives

$$\epsilon(\mu) := \|\mu q_{36}\| - M\,\|\tau q_{36}\| = 0.327562.$$

Let $A = 3.6$, $B = \alpha$, and $\omega = k$ in Lemma 1. Using *Mathematica*, we conclude that Equation (13) has no solution if

$$\frac{\log(Aq_{36}/\epsilon(\mu))}{\log B} < 23.38 < k.$$

This implies $k \leq 23$, which contradicts the assumption that $k > 25$. This completes the proof. □

4. Lucas-balancing Numbers as the Difference of Two Repdigits

In this section, we will show that 3 and 17 are the only Lucas-balancing numbers which can be expressed as the difference of two repdigits. Specifically, we will prove the following theorem.

Theorem 3. *If C_k is expressible as the difference of two repdigits, then $C_k \in \{3, 17\}$. In particular, $C_2 = 3 = 11 - 8$ and $C_3 = 17 = 22 - 5$.*

Proof. The proof bears similarities to that of Theorem 2. At times, we may leave out some details.

Assume that Equation (2) holds. Let $1 \leq k \leq 25$ and $n \geq 2$. Using $Mathematica$, one can verify the solutions provided in Theorem 3. So, from now on, assume that $k > 25$.

If $n = m$, it follows that $d_1 > d_2$, implying that C_k is a repdigit. However, the largest repdigit in C_k is 99 [12], which contradicts the assumption that $k > 25$. If $n - m = 1$ and $d_1 \geq d_2$, we encounter Lucas-balancing numbers which are the concatenation of two repdigits. This is impossible by Lemma 5. If $n - m = 1$ and $d_1 < d_2$, then we have Lucas-balancing numbers which are concatenation of three repdigits, which is not feasible according to Lemma 6. Consequently, we can assume that $n - m \geq 2$.

The inequality

$$\frac{\alpha^n}{20} < \frac{10^{n-1}}{2} < 10^{n-1} - 10^{m-1} < \frac{d_1(10^n - 1)}{9} - \frac{d_2(10^m - 1)}{9} = C_k < \alpha^{k+1}$$

implies that $n < k+3$. By applying the Binet formula for Lucas-balancing numbers, we can rearrange Equation (2) into two distinct equations as

$$\frac{9\alpha^k}{2} - d_1 10^n = -\frac{9\beta^k}{2} - d_2 10^m - (d_1 - d_2)$$

and

$$\frac{\alpha^k}{2} - \frac{d_1 10^n - d_2 10^m}{9} = -\frac{\beta^k}{2} - \frac{(d_1 - d_2)}{9}.$$

Using the same steps as in the proof of Theorem 2, we deduce that $k < 2.4 \cdot 10^{28}$. This upper bound on k can be further reduced using the Baker–Davenport algorithm given in Lemma 1, whose procedure is comparable to that of Theorem 2.

Continuing in this manner, we arrive at $k \leq 23$, which contradicts our initial assumption that $k > 25$. This ends the proof. □

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