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ON A RAMANUJAN'S CONTINUED FRACTION OF ORDER SIXTEEN AND NEW EISENSTEIN SERIES IDENTITIES

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Abstract

We prove many new identities associated with Ramanujan's continued fraction of order sixteen and the Ramanujan-Göllnitz-Gordon continued fraction. We further establish several new Eisenstein series identities associated with Ramanujan's continued fraction of order sixteen.

1. Introduction

Throughout this paper, we assume that $|q| < 1$ and use the standard product notation

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j), \quad \text{and} \quad (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

For convenience, we sometimes use the multiple q-shifted factorial notation, which is defined as

$$(a_1, a_2, \ldots, a_m; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_m;q)_\infty.$$

The famous Rogers-Ramanujan continued fraction $R(q)$ was studied by Ramanu-

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jan and Rogers [24]:

$$R(q) := q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = \frac{q^{\frac{1}{5}}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \cdots.$$

Ramanujan made some significant contributions to the theory of the Rogers-Ramanujan continued fraction expansion in his notebooks [23, Vol. II, Chapter 16, Section 15] and "Lost Notebook" [24]. Motivated by Ramanujan's work, Liu [18] and Chan, Chan, and Liu [11] established many new identities associated with the Rogers-Ramanujan continued fraction $R(q)$. Further, they proved new Eisenstein series identities which involve $R(q)$. Recently, Cao et al. [7] established a Rogers-Ramanujan-Slater type theta function identity.

The beautiful Ramanujan's cubic continued fraction $X_1(q)$ was first introduced by Ramanujan in his second letter to Hardy [23, p. xxvii], and is defined by

$$X_1(q) := q^{1/3}\frac{(q,q^5;q^6)_\infty}{(q^3,q^3;q^6)_\infty} = \frac{q^{1/3}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \frac{q^3+q^6}{1} + \cdots.$$

Ramanujan's cubic continued fraction has several properties that are analogous to those of the Rogers-Ramanujan continued fraction. Adiga et al. [2], Bhargava et al. [6], Chan [9], and Cooper [14] established numerous elegant theorems for $X_1(q)$, many of which are analogues of well known properties satisfied by the Rogers-Ramanujan continued fraction.

In [29], Vasuki et al. studied the following continued fraction of order six:

$$X_2(q) := q^{1/4}\frac{(q,q^5;q^6)_\infty}{(q^2,q^4;q^6)_\infty} = \frac{q^{-1/4}(1-q^2)}{(1-q^{3/2})} + \frac{(1-q^{1/2})(1-q^{7/2})}{q^{1/2}(1-q^{3/2})(1+q^3)} + \cdots.$$

The continued fraction $X_2(q)$ is a special case of an interesting continued fraction identity recorded by Ramanujan in his second notebook [24], [1, p. 24]. Furthermore, they established modular relations between the continued fractions $X_2(q)$ and $X_2(q^n)$ for $n = 2, 3, 5, 7,$ and 11. Motivated by these, Adiga et al. [5] established two new identities associated with $X_1(q)$ and $X_2(q)$ using the quintuple product identity. Also, they derived Eisenstein series identities associated with $X_1(q)$ and $X_2(q)$.

The celebrated Ramanujan-Göllnitz-Gordon continued fraction [24] is defined by

$$G(q) := q^{\frac{1}{2}}\frac{(q,q^7;q^8)_\infty}{(q^3,q^5;q^8)_\infty} = \frac{q^{1/2}}{1+q} + \frac{q^2}{1+q^3} + \frac{q^4}{1+q^5} + \cdots.$$

The theory of the Ramanujan-Göllnitz-Gordon continued fraction has been further developed by various mathematicians including Chan and Huang [10], Cooper [14], and Vasuki and Srivatsa Kumar [30]. Chamaraju [8], in his thesis, established new identities associated with $G(q)$ and derived a new Eisenstein series identity involving $G(q)$. In [12], Chaudhary and Choi established certain identities associated with Eisenstein series, the Ramanujan-Göllnitz-Gordon continued fraction, and combinatorial partition identities. Recently, Chaudhary [13] established some Rogers-Ramanujan type identities.

In [19], Naika et al. established the following continued fraction of order 12:

$$U(q) := \frac{q\ (q, q^{11}; q^{12})_\infty}{(q^5, q^7; q^{12})_\infty} = \frac{q(1-q)}{(1-q^3)} + \frac{q^3(1-q^2)(1-q^4)}{(1-q^3)(1+q^6)} + \cdots.$$

The continued fraction $U(q)$ is a special case of a fascinating continued fraction identity recorded by Ramanujan in his second notebook [24]. The above continued fraction was studied by Naika et al. [19], Vasuki et al. [31], Dharmendra et al. [15], and Adiga et al. [4]. Recently, Adiga et al. [3] established two new identities associated with $U(q)$ of order 12, using two elementary trigonometric sums and the Jacobi theta function θ_1. They also derived several Eisenstein series identities involving $U(q)$. In [25], Shpot et al. established the integrals of products of Hurwitz zeta functions and the Casimir effect in φ^4 field theories.

Surekha [27] and Vanitha [28] studied two continued fractions $I_1(q)$ and $I_2(q)$ of order sixteen, which are defined as follows:

$$I_1(q) := \frac{q^{1/2}(q^3, q^{13}; q^{16})_\infty}{(q^5, q^{11}; q^{16})_\infty} = \frac{q^{1/2}(1-q^3)}{(1-q^4)} + \frac{q^4(1-q)(1-q^7)}{(1-q^4)(1+q^8)} + \cdots, \quad (1)$$

and

$$I_2(q) := \frac{q^{3/2}(q, q^{15}; q^{16})_\infty}{(q^7, q^9; q^{16})_\infty} = \frac{q^{3/2}(1-q)}{(1-q^4)} + \frac{q^4(1-q^3)(1-q^5)}{(1-q^4)(1+q^8)} + \cdots. \quad (2)$$

The continued fractions $I_1(q)$ and $I_2(q)$, are a special case of a fascinating continued fraction identity recorded by Ramanujan in his second notebook [24].

Surekha [27] derived modular relations for $I_1(q)$ and $I_2(q)$ and also proved the 2-, 4-, 8-, and 16-dissections for the continued fraction $I_1(q)$ of order sixteen and its reciprocal. Vanitha [28] established the 2-, 4-, 8-, and 16-dissections of a continued fraction $I_2(q)$ of order sixteen and its reciprocal. Also, Vanitha gave combinatorial interpretations for the coefficients in the power series expansion of a continued fraction $I_2(q)$ and its reciprocal. Park [21] studied the continued fractions $I_1(q)$ and $I_2(q)$ by using the theory of modular functions. He proved the modularities of $I_1(q)$ and $I_2(q)$. Further, he proved that the values of $2(I_1(q)^2 + 1/I_1(q)^2)$ and $2(I_2(q)^2+1/I_2(q)^2)$ are algebraic integers for a certain imaginary quadratic quantity q. Recently, Rajkhowa and Saikia [22] established theta function identities, explicit values, partition-theoretic results and some matching coefficients of the continued fractions $I_1(q)$ and $I_2(q)$.

In this paper, we study the following. In Section 3, we derive several new identities associated with Ramanujan's continued fractions $I_1(q)$ and $I_2(q)$ of order sixteen, and the Ramanujan-Göllnitz-Gordon continued fraction $G(q)$, using the Jacobi theta function θ_1. In Section 4, we establish two new identities associated with $I_1(q)$ and $I_2(q)$ by using Ramanujan's ${}_1\Psi_1$ summation formula. Finally, in Section 5, we establish several Eisenstein series identities associated with Ramanujan's continued fractions $I_1(q)$ and $I_2(q)$ of order sixteen.

2. Definitions and Preliminary Results

In this section, we present some basic definitions and preliminary results on Ramanujan's theta functions. Ramanujan's general theta function is

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}\ b^{n(n-1)/2}, \quad |ab| < 1. \tag{3}$$

Then it is easy to verify that

$$f(a,b) = f(b,a),\ f(1,a) = 2f(a,a^3),\ f(-1,a) = 0.$$

The Jacobi triple product identity states, for $z \neq 0$, that

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(n-1)}{2}} z^n = (q;q)_\infty\ (z;q)_\infty\ (q/z;q)_\infty.$$

In Ramanujan's notation, the Jacobi triple product identity takes the shape

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

In [26], Srivastava et al. derived some theta function identities related to the Jacobi triple product identity. If n is an integer,

$$f(a,b) = a^{n(n+1)/2}\ b^{n(n-1)/2}\ f(a(ab)^n, b(ab)^{-n}). \tag{4}$$

The most interesting special cases of $f(a,b)$ are [1, Entry 22]

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{5}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{6}$$

Note that $\eta(\tau) = q^{1/24} f(-q)$ where $q = e^{2\pi i\tau}$, $Im\tau > 0$, and $\eta(\tau)$ is the Dedekind-eta function. Also, Ramanujan, define

$$\chi(q) := (-q; q^2)_\infty. \tag{7}$$

For convenience, we define, for a positive integer n,

$$f_n := f(-q^n) = (q^n; q^n)_\infty.$$

The following lemma is a consequence of the product representations of Identities (5)-(7).

Lemma 1. *We have*

$$\psi(q) = \frac{f_2^2}{f_1}, \qquad f(q) = \frac{f_2^3}{f_1 f_4}, \qquad \chi(q) = \frac{f_2^2}{f_1 f_4}, \qquad \textit{and} \qquad \chi(-q) = \frac{f_1}{f_2}.$$

Lemma 2 ([1, Entry 30 (i -iii)]). *We have*

$$f(a, ab^2) f(b, a^2 b) = f(a, b)\psi(ab), \tag{8}$$

$$f(a, b) + f(-a, -b) = 2f(a^3 b, ab^3), \tag{9}$$

and

$$f(a, b) - f(-a, -b) = 2af\left(\frac{b}{a}, a^5 b^3\right). \tag{10}$$

3. Main Results

The Jacobi theta function, θ_1, is defined as

$$\begin{aligned}\theta_1(z|\tau) &= 2\sum_{n=0}^{\infty} (-1)^n q^{\frac{(2n+1)^2}{8}} \sin(2n+1)z \\ &= 2q^{\frac{1}{8}} \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} \sin(2n+1)z. \end{aligned} \tag{11}$$

In [11], Chan et al. showed that

$$\begin{aligned} 2\sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} \sin(2n+1)z &= \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} \sin(2n+1)z \\ &= 2(\sin z)(q; q)_\infty (qe^{2iz}; q)_\infty (qe^{-2iz}; q)_\infty. \end{aligned} \tag{12}$$

Combining Identities (11) and (12) together, we find the infinite product representation of θ_1:

$$\begin{aligned}\theta_1(z|\tau) &= 2q^{\frac{1}{8}}(\sin z)(q;q)_\infty(qe^{2iz};q)_\infty(qe^{-2iz};q)_\infty \\ &= iq^{\frac{1}{8}}e^{-iz}(q;q)_\infty(e^{2iz};q)_\infty(qe^{-2iz};q)_\infty. \end{aligned} \tag{13}$$

Putting $z=\frac{\pi}{16}$, $z=\frac{2\pi}{16}$, $z=\frac{3\pi}{16}$, $z=\frac{4\pi}{16}$, $z=\frac{5\pi}{16}$, $z=\frac{6\pi}{16}$, and $z=\frac{7\pi}{16}$, respectively, in Identity (13), we obtain

$$\theta_1\left(\frac{\pi}{16}|\tau\right) = 2q^{\frac{1}{8}}\left(\sin\frac{\pi}{16}\right)(q;q)_\infty(qe^{\frac{2\pi i}{16}};q)_\infty(qe^{\frac{-2\pi i}{16}};q)_\infty, \tag{14}$$

$$\theta_1\left(\frac{2\pi}{16}|\tau\right) = 2q^{\frac{1}{8}}\left(\sin\frac{2\pi}{16}\right)(q;q)_\infty(qe^{\frac{4\pi i}{16}};q)_\infty(qe^{\frac{-4\pi i}{16}};q)_\infty, \tag{15}$$

$$\theta_1\left(\frac{3\pi}{16}|\tau\right) = 2q^{\frac{1}{8}}\left(\sin\frac{3\pi}{16}\right)(q;q)_\infty(qe^{\frac{6\pi i}{16}};q)_\infty(qe^{\frac{-6\pi i}{16}};q)_\infty, \tag{16}$$

$$\theta_1\left(\frac{4\pi}{16}|\tau\right) = 2q^{\frac{1}{8}}\left(\sin\frac{4\pi}{16}\right)(q;q)_\infty(qe^{\frac{8\pi i}{16}};q)_\infty(qe^{\frac{-8\pi i}{16}};q)_\infty, \tag{17}$$

$$\theta_1\left(\frac{5\pi}{16}|\tau\right) = 2q^{\frac{1}{8}}\left(\sin\frac{5\pi}{16}\right)(q;q)_\infty(qe^{\frac{10\pi i}{16}};q)_\infty(qe^{\frac{-10\pi i}{16}};q)_\infty, \tag{18}$$

$$\theta_1\left(\frac{6\pi}{16}|\tau\right) = 2q^{\frac{1}{8}}\left(\sin\frac{6\pi}{16}\right)(q;q)_\infty(qe^{\frac{12\pi i}{16}};q)_\infty(qe^{\frac{-12\pi i}{16}};q)_\infty, \tag{19}$$

and

$$\theta_1\left(\frac{7\pi}{16}|\tau\right) = 2q^{\frac{1}{8}}\left(\sin\frac{7\pi}{16}\right)(q;q)_\infty(qe^{\frac{14\pi i}{16}};q)_\infty(qe^{\frac{-14\pi i}{16}};q)_\infty. \tag{20}$$

Multiplying Identities (14)-(20) together, and using the identities

$$\sin\frac{\pi}{16}\sin\frac{2\pi}{16}\sin\frac{3\pi}{16}\sin\frac{4\pi}{16}\sin\frac{5\pi}{16}\sin\frac{6\pi}{16}\sin\frac{7\pi}{16} = \frac{\sqrt{2}}{64}, \tag{21}$$

and

$$\begin{aligned}&(1-x)(1-xe^{\frac{2\pi i}{16}})(1-xe^{\frac{-2\pi i}{16}})(1-xe^{\frac{4\pi i}{16}})(1-xe^{\frac{-4\pi i}{16}})(1-xe^{\frac{6\pi i}{16}})(1-xe^{\frac{-6\pi i}{16}}) \\ &\times(1-xe^{\frac{8\pi i}{16}})(1-xe^{\frac{-8\pi i}{16}})(1-xe^{\frac{10\pi i}{16}})(1-xe^{\frac{-10\pi i}{16}})(1-xe^{\frac{12\pi i}{16}})(1-xe^{\frac{-12\pi i}{16}}) \\ &\times(1-xe^{\frac{14\pi i}{16}})(1-xe^{\frac{-14\pi i}{16}})(1-xe^{\frac{16\pi i}{16}}) = (1-x^{16}),\end{aligned}$$

in the resulting equation, then after some simplifications, we obtain the following identity:

$$\theta_1\left(\frac{\pi}{16}|\tau\right)\theta_1\left(\frac{2\pi}{16}|\tau\right)\theta_1\left(\frac{3\pi}{16}|\tau\right)\theta_1\left(\frac{4\pi}{16}|\tau\right)\theta_1\left(\frac{5\pi}{16}|\tau\right)\theta_1\left(\frac{6\pi}{16}|\tau\right)\theta_1\left(\frac{7\pi}{16}|\tau\right)$$
$$=\frac{2\sqrt{2}\ \ \eta^7(\tau)\eta(16\tau)}{\eta(2\tau)}. \tag{22}$$

Taking

$$\alpha=-2\cos\frac{2\pi}{16}=-\sqrt{2+\sqrt{2}}, \qquad \beta=-2\cos\frac{4\pi}{16}=-\sqrt{2},$$
$$\gamma=-2\cos\frac{6\pi}{16}=-\sqrt{2-\sqrt{2}}, \qquad \delta=-2\cos\frac{8\pi}{16}=0,$$
$$\varepsilon=-2\cos\frac{10\pi}{16}=\sqrt{2-\sqrt{2}}, \qquad \lambda=-2\cos\frac{12\pi}{16}=\sqrt{2},$$
$$\mu=-2\cos\frac{14\pi}{16}=\sqrt{2+\sqrt{2}},$$

and using $\eta(\tau)=q^{1/24}f(-q)$, we may rewrite Identities (14)-(20) as follows:

$$P_1(q):=\prod_{n=1}^{\infty}(1+\alpha q^n+q^{2n})=q^{\frac{-1}{12}}\frac{\theta_1\left(\frac{\pi}{16}|\tau\right)}{\eta(\tau)2\left(\sin\frac{\pi}{16}\right)}, \tag{23}$$

$$P_2(q):=\prod_{n=1}^{\infty}(1+\beta q^n+q^{2n})=q^{\frac{-1}{12}}\frac{\theta_1\left(\frac{2\pi}{16}|\tau\right)}{\eta(\tau)2\left(\sin\frac{2\pi}{16}\right)}, \tag{24}$$

$$P_3(q):=\prod_{n=1}^{\infty}(1+\gamma q^n+q^{2n})=q^{\frac{-1}{12}}\frac{\theta_1\left(\frac{3\pi}{16}|\tau\right)}{\eta(\tau)2\left(\sin\frac{3\pi}{16}\right)}, \tag{25}$$

$$P_4(q):=\prod_{n=1}^{\infty}(1+\delta q^n+q^{2n})=q^{\frac{-1}{12}}\frac{\theta_1\left(\frac{4\pi}{16}|\tau\right)}{\eta(\tau)2\left(\sin\frac{4\pi}{16}\right)}, \tag{26}$$

$$P_5(q):=\prod_{n=1}^{\infty}(1+\varepsilon q^n+q^{2n})=q^{\frac{-1}{12}}\frac{\theta_1\left(\frac{5\pi}{16}|\tau\right)}{\eta(\tau)2\left(\sin\frac{5\pi}{16}\right)}, \tag{27}$$

$$P_6(q):=\prod_{n=1}^{\infty}(1+\lambda q^n+q^{2n})=q^{\frac{-1}{12}}\frac{\theta_1\left(\frac{6\pi}{16}|\tau\right)}{\eta(\tau)2\left(\sin\frac{6\pi}{16}\right)}, \tag{28}$$

and

$$P_7(q) := \prod_{n=1}^{\infty}(1+\mu q^n+q^{2n}) = q^{\frac{-1}{12}}\frac{\theta_1\left(\frac{7\pi}{16}|\tau\right)}{\eta(\tau)2\left(\sin\frac{7\pi}{16}\right)}. \tag{29}$$

Multiplying Identities (23)-(29) together, and then using Identity (21) and Identity (22) in the resulting identity, we find that

$$P_1(q)P_2(q)P_3(q)P_4(q)P_5(q)P_6(q)P_7(q) = q^{\frac{-7}{12}}\frac{\eta(16\tau)}{\eta(2\tau)}. \tag{30}$$

We are now ready to prove the main results.

Theorem 1. *Let $P_i(q)$, where $1 \le i \le 7$, be defined as in (23)-(29) where $\alpha = -\sqrt{2+\sqrt{2}}$, $\beta = -\sqrt{2}$, $\gamma = -\sqrt{2-\sqrt{2}}$, $\varepsilon = \sqrt{2-\sqrt{2}}$, $\lambda = \sqrt{2}$, $\mu = \sqrt{2+\sqrt{2}}$, and $G_*(q) = \sqrt{G(q)}$. Then, we have*

$$\begin{aligned} &P_1(q^{\frac14})\prod_{i=1,i\neq 4}^{7}P_i(q^{\frac14}) - P_7(q^{\frac14})\prod_{i=1,i\neq 4}^{7}P_i(q^{\frac14}) \\ &\quad = \frac{2\sqrt{2}\ \eta_*(\tau)}{\sqrt{2-\sqrt{2}}}\left[\sqrt{\frac{I_1(q)}{G_*(q)}} - \frac{1}{\sqrt{I_1(q)G_*(q)}} - \sqrt{2\ I_2(q)G_*(q)}\right], \end{aligned} \tag{31}$$

$$\begin{aligned} &P_3(q^{\frac14})\prod_{i=1,i\neq 4}^{7}P_i(q^{\frac14}) - P_5(q^{\frac14})\prod_{i=1,i\neq 4}^{7}P_i(q^{\frac14}) \\ &\quad = \frac{2\sqrt{2}\ \eta_*(\tau)}{\sqrt{2+\sqrt{2}}}\left[\sqrt{\frac{I_1(q)}{G_*(q)}} - \frac{1}{\sqrt{I_1(q)G_*(q)}} + \sqrt{2\ I_2(q)G_*(q)}\right], \end{aligned} \tag{32}$$

$$\begin{aligned} &(1+\mu)\ P_7(q^{\frac14})\prod_{i=1,i\neq 4}^{7}P_i(q^{\frac14}) + (1+\alpha)\ P_1(q^{\frac14})\prod_{i=1,i\neq 4}^{7}P_i(q^{\frac14}) \\ &\quad = 2\eta_*(\tau)\left[\left(\sqrt{\frac{G_*(q)}{I_2(q)}} - \sqrt{I_1(q)G_*(q)}\right)\right. \\ &\qquad \left. +\ (1+\sqrt{2})\left(\sqrt{\frac{G_*(q)}{I_1(q)}} + \sqrt{I_2(q)G_*(q)}\right)\right], \end{aligned} \tag{33}$$

$$\begin{aligned}
&\mu\ P_7(q^{\frac{1}{4}}) \prod_{i=1, i\neq 4}^{7} P_i(q^{\frac{1}{4}}) - \alpha\ P_1(q^{\frac{1}{4}}) \prod_{i=1, i\neq 4}^{7} P_i(q^{\frac{1}{4}}) \\
&\quad = \frac{2\sqrt{2}\ \eta_*(\tau)}{\sqrt{2-\sqrt{2}}} \left[\left(\sqrt{\frac{G_*(q)}{I_2(q)}} - \frac{1}{\sqrt{I_1(q)G_*(q)}} \right) \right. \\
&\qquad \left. + (1+\sqrt{2}) \left(\sqrt{\frac{I_1(q)}{G_*(q)}} - \sqrt{I_2(q)G_*(q)} \right) \right], \qquad (34)
\end{aligned}$$

and

$$\begin{aligned}
&\varepsilon\ P_3(q^{\frac{1}{4}}) \prod_{i=1, i\neq 4}^{7} P_i(q^{\frac{1}{4}}) - \gamma\ P_5(q^{\frac{1}{4}}) \prod_{i=1, i\neq 4}^{7} P_i(q^{\frac{1}{4}}) \\
&\quad = \frac{2\sqrt{2}\ \eta_*(\tau)}{\sqrt{2+\sqrt{2}}} \left[\left(\sqrt{\frac{G_*(q)}{I_2(q)}} - \frac{1}{\sqrt{I_1(q)G_*(q)}} \right) \right. \\
&\qquad \left. + (1-\sqrt{2}) \left(\sqrt{\frac{I_1(q)}{G_*(q)}} - \sqrt{I_2(q)G_*(q)} \right) \right], \qquad (35)
\end{aligned}$$

where

$$\eta_*(\tau) = q^{\frac{-7}{48}} \frac{\eta(4\tau)\eta(16\tau)}{\eta(\tau/4)\eta(\tau)} \sqrt[4]{\frac{\eta(\tau)}{\eta(2\tau)}}.$$

Proof. Subtracting Identity (29) from Identity (23), we obtain

$$\begin{aligned}
P_1(q) - P_7(q) &:= \prod_{n=1}^{\infty} (1 + \alpha q^n + q^{2n}) - \prod_{n=1}^{\infty} (1 + \mu q^n + q^{2n}) \\
&= \frac{q^{\frac{-1}{12}}}{\eta(\tau)} \left(\frac{\theta_1\left(\frac{\pi}{16}|\tau\right)}{2\sin\frac{\pi}{16}} - \frac{\theta_1\left(\frac{7\pi}{16}|\tau\right)}{2\sin\frac{7\pi}{16}} \right). \qquad (36)
\end{aligned}$$

Using Identity (11), the right-hand side of Identity (36) can be written as

$$\frac{q^{\frac{-1}{12}}}{\eta(\tau)} \sum_{n=0}^{\infty} (-1)^n A(n) q^{\frac{(2n+1)^2}{8}}, \qquad (37)$$

where

$$A(n) = \frac{\sin(2n+1)\frac{\pi}{16}}{\sin\frac{\pi}{16}} - \frac{\sin(2n+1)\frac{7\pi}{16}}{\sin\frac{7\pi}{16}}.$$

Now, by using maple computations, we find that

$$A(16m+0)=0, \qquad A(16m+8)=0,$$
$$A(16m+1)=\frac{2\sqrt{2}}{\sqrt{2-\sqrt{2}}}, \qquad A(16m+9)=-\frac{2\sqrt{2}}{\sqrt{2-\sqrt{2}}},$$
$$A(16m+2)=\frac{2\sqrt{2}}{\sqrt{2-\sqrt{2}}}, \qquad A(16m+10)=-\frac{2\sqrt{2}}{\sqrt{2-\sqrt{2}}},$$
$$A(16m+3)=\frac{4}{\sqrt{2-\sqrt{2}}}, \qquad A(16m+11)=-\frac{4}{\sqrt{2-\sqrt{2}}},$$
$$A(16m+4)=\frac{4}{\sqrt{2-\sqrt{2}}}, \qquad A(16m+12)=-\frac{4}{\sqrt{2-\sqrt{2}}},$$
$$A(16m+5)=\frac{2\sqrt{2}}{\sqrt{2-\sqrt{2}}}, \qquad A(16m+13)=-\frac{2\sqrt{2}}{\sqrt{2-\sqrt{2}}},$$

$$A(16m+6)=\frac{2\sqrt{2}}{\sqrt{2-\sqrt{2}}}, \quad A(16m+14)=-\frac{2\sqrt{2}}{\sqrt{2-\sqrt{2}}}, \quad A(16m+7)=0,$$
$$A(16m+15)=0.$$

Therefore,

$$\sum_{n=0}^{\infty}(-1)^n A(n) q^{\frac{(2n+1)^2}{8}} = \frac{2\sqrt{2}}{\sqrt{2-\sqrt{2}}}\Bigg\{-\sum_{m=0}^{\infty} q^{\frac{(32m+3)^2}{8}} + \sum_{m=0}^{\infty} q^{\frac{(32m+5)^2}{8}}$$
$$-\sqrt{2}\sum_{m=0}^{\infty} q^{\frac{(32m+7)^2}{8}} + \sqrt{2}\sum_{m=0}^{\infty} q^{\frac{(32m+9)^2}{8}} - \sum_{m=0}^{\infty} q^{\frac{(32m+11)^2}{8}}$$
$$+\sum_{m=0}^{\infty} q^{\frac{(32m+13)^2}{8}} + \sum_{m=0}^{\infty} q^{\frac{(32m+19)^2}{8}} - \sum_{m=0}^{\infty} q^{\frac{(32m+21)^2}{8}} + \sqrt{2}\sum_{m=0}^{\infty} q^{\frac{(32m+23)^2}{8}}$$
$$-\sqrt{2}\sum_{m=0}^{\infty} q^{\frac{(32m+25)^2}{8}} + \sum_{m=0}^{\infty} q^{\frac{(32m+27)^2}{8}} - \sum_{m=0}^{\infty} q^{\frac{(32m+29)^2}{8}}\Bigg\}.$$

In the right-hand side of the above equation, changing m to $m-1$ in the first six summations and also changing m to $-m$ in the last six summations, we obtain

$$\sum_{n=0}^{\infty}(-1)^n A(n) q^{\frac{(2n+1)^2}{8}} = \frac{2\sqrt{2}}{\sqrt{2-\sqrt{2}}}\Bigg\{\sum_{m=-\infty}^{\infty} q^{\frac{(32m-19)^2}{8}} - \sum_{m=-\infty}^{\infty} q^{\frac{(32m-21)^2}{8}}$$
$$+\sqrt{2}\sum_{m=-\infty}^{\infty} q^{\frac{(32m-23)^2}{8}} - \sqrt{2}\sum_{m=-\infty}^{\infty} q^{\frac{(32m-25)^2}{8}} + \sum_{m=-\infty}^{\infty} q^{\frac{(32m-27)^2}{8}} - \sum_{m=-\infty}^{\infty} q^{\frac{(32m-29)^2}{8}}\Bigg\}.$$

Now, using Identities (3) and (4) in the right-hand side of the above equation, we find that

$$\sum_{n=0}^{\infty}(-1)^n A(n) q^{\frac{(2n+1)^2}{8}} = \frac{2\sqrt{2}\ q^{\frac{9}{8}}}{\sqrt{2-\sqrt{2}}}\left\{-[f(q^{104},q^{152})-q^{20}f(q^{24},q^{232})]\right.$$
$$\left.+q^2[f(q^{88},q^{168})-q^{12}f(q^{40},q^{216})]-\sqrt{2}q^5[f(q^{72},q^{184})-q^4f(q^{56},q^{200})]\right\}. \quad (38)$$

Subtracting Identity (10) from Identity (9), we deduce

$$f(-a,-b)=f(a^3b,ab^3)-af\left(\frac{b}{a},a^5b^3\right). \quad (39)$$

Putting $\{a=q^{20},b=q^{44}\}$, $\{a=q^{12},b=q^{52}\}$, and $\{a=q^4,b=q^{60}\}$ in the above equation, we obtain

$$f(-q^{20},-q^{44})=f(q^{104},q^{152})-q^{20}f(q^{24},q^{232}), \quad (40)$$
$$f(-q^{12},-q^{52})=f(q^{88},q^{168})-q^{12}f(q^{40},q^{216}), \quad (41)$$
$$f(-q^{4},-q^{60})=f(q^{72},q^{184})-q^{4}f(q^{56},q^{200}). \quad (42)$$

Employing Identities (40)-(42) in Identity (38), we obtain

$$\sum_{n=0}^{\infty}(-1)^n A(n) q^{\frac{(2n+1)^2}{8}} = \frac{2\sqrt{2}\ q^{\frac{9}{8}}}{\sqrt{2-\sqrt{2}}}\left\{-f(-q^{20},-q^{44})+q^2f(-q^{12},-q^{52})\right.$$
$$\left.-\sqrt{2}q^5f(-q^4,-q^{60})\right\}. \quad (43)$$

Combining Identities (36), (37), and (43), we find that

$$P_1(q)-P_7(q)=\frac{2\sqrt{2}\ q^{\frac{25}{24}}}{(\sqrt{2-\sqrt{2}})\eta(\tau)}$$
$$\times\left\{-f(-q^{20},-q^{44})+q^2f(-q^{12},-q^{52})-\sqrt{2}q^5f(-q^4,-q^{60})\right\}. \quad (44)$$

Multiplying both sides of Identity (44) by Identity (30), we deduce

$$P_1^2(q)P_2(q)P_3(q)P_4(q)P_5(q)P_6(q)P_7(q)-P_1(q)P_2(q)P_3(q)P_4(q)P_5(q)P_6(q)P_7^2(q)$$
$$=\frac{2\sqrt{2}\ q^{\frac{25}{24}}\eta(16\tau)}{\sqrt{2-\sqrt{2}}\ \eta(\tau)\eta(2\tau)}\left\{-f(-q^{20},-q^{44})+q^2f(-q^{12},-q^{52})\right.$$
$$\left.-\sqrt{2}q^5f(-q^4,-q^{60})\right\}.$$

Now, using $\delta=0$ in the left-hand side of the above equation and using the fact that $(-q^2;q^2)_\infty = q^{-\frac{2}{24}}\dfrac{\eta(4\tau)}{\eta(2\tau)}$, then, after some simplification and changing q to

$q^{1/4}$ throughout, we get Identity (31). The proof of Identity (32) is identical to the proof of Identity (31), so we omit it.

From Identities (23) and (29), we obtain

$$\begin{aligned}
&(1+\mu)P_7(q)+(1+\alpha)P_1(q)\\
&\quad=(1+\mu)\prod_{n=1}^{\infty}(1+\mu q^n+q^{2n})+(1+\alpha)\prod_{n=1}^{\infty}(1+\alpha q^n+q^{2n})\\
&\quad=\frac{q^{\frac{-1}{12}}}{\eta(\tau)}\left[\frac{(1+\mu)\ \theta_1\left(\frac{7\pi}{16}|\tau\right)}{2\sin\frac{7\pi}{16}}+\frac{(1+\alpha)\ \theta_1\left(\frac{\pi}{16}|\tau\right)}{2\sin\frac{\pi}{16}}\right]. \qquad (45)
\end{aligned}$$

Using Identity (11), the right-hand side of Identity (45), can be written as

$$\frac{q^{\frac{-1}{12}}}{\eta(\tau)}\sum_{n=0}^{\infty}(-1)^nB(n)q^{\frac{(2n+1)^2}{8}}, \qquad (46)$$

where

$$B(n)=\frac{(1+\mu)\sin(2n+1)\frac{7\pi}{16}}{\sin\frac{7\pi}{16}}+\frac{(1+\alpha)\sin(2n+1)\frac{\pi}{16}}{\sin\frac{\pi}{16}}.$$

Now, by using Maple computations, we find that

$$\begin{aligned}
B(16m+0)&=2, & B(16m+8)&=-2,\\
B(16m+1)&=-\frac{2}{\sqrt{2}-1}, & B(16m+9)&=\frac{2}{\sqrt{2}-1},\\
B(16m+2)&=-2, & B(16m+10)&=2,\\
B(16m+3)&=-\frac{2}{\sqrt{2}-1}, & B(16m+11)&=\frac{2}{\sqrt{2}-1},\\
B(16m+4)&=-\frac{2}{\sqrt{2}-1}, & B(16m+12)&=\frac{2}{\sqrt{2}-1},\\
B(16m+5)&=-2, & B(16m+13)&=2,\\
B(16m+6)&=-\frac{2}{\sqrt{2}-1}, & B(16m+14)&=\frac{2}{\sqrt{2}-1},\\
B(16m+7)&=2, & B(16m+15)&=-2.
\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty}(-1)^n B(n) q^{\frac{(2n+1)^2}{8}} = 2\Bigg\{\sum_{m=0}^{\infty} q^{\frac{(32m+1)^2}{8}} + \frac{1}{\sqrt{2}-1}\sum_{m=0}^{\infty} q^{\frac{(32m+3)^2}{8}}$$
$$-\sum_{m=0}^{\infty} q^{\frac{(32m+5)^2}{8}} + \frac{1}{\sqrt{2}-1}\sum_{m=0}^{\infty} q^{\frac{(32m+7)^2}{8}} - \frac{1}{\sqrt{2}-1}\sum_{m=0}^{\infty} q^{\frac{(32m+9)^2}{8}}$$
$$+\sum_{m=0}^{\infty} q^{\frac{(32m+11)^2}{8}} - \frac{1}{\sqrt{2}-1}\sum_{m=0}^{\infty} q^{\frac{(32m+13)^2}{8}} - \sum_{m=0}^{\infty} q^{\frac{(32m+15)^2}{8}}$$
$$-\sum_{m=0}^{\infty} q^{\frac{(32m+17)^2}{8}} - \frac{1}{\sqrt{2}-1}\sum_{m=0}^{\infty} q^{\frac{(32m+19)^2}{8}} + \sum_{m=0}^{\infty} q^{\frac{(32m+21)^2}{8}}$$
$$-\frac{1}{\sqrt{2}-1}\sum_{m=0}^{\infty} q^{\frac{(32m+23)^2}{8}} + \frac{1}{\sqrt{2}-1}\sum_{m=0}^{\infty} q^{\frac{(32m+25)^2}{8}} - \sum_{m=0}^{\infty} q^{\frac{(32m+27)^2}{8}}$$
$$+\frac{1}{\sqrt{2}-1}\sum_{m=0}^{\infty} q^{\frac{(32m+29)^2}{8}} + \sum_{m=0}^{\infty} q^{\frac{(32m+31)^2}{8}}\Bigg\}.$$

In the right-hand side of the above, changing m to $m-1$ in the first eight summations and also changing m to $-m$ in last eight summations, we obtain

$$\sum_{n=0}^{\infty}(-1)^n B(n) q^{\frac{(2n+1)^2}{8}} = 2\Bigg\{-\sum_{m=-\infty}^{\infty} q^{\frac{(32m-17)^2}{8}} - \frac{1}{\sqrt{2}-1}\sum_{m=-\infty}^{\infty} q^{\frac{(32m-19)^2}{8}}$$
$$+\sum_{m=-\infty}^{\infty} q^{\frac{(32m-21)^2}{8}} - \frac{1}{\sqrt{2}-1}\sum_{m=-\infty}^{\infty} q^{\frac{(32m-23)^2}{8}} + \frac{1}{\sqrt{2}-1}\sum_{m=-\infty}^{\infty} q^{\frac{(32m-25)^2}{8}}$$
$$-\sum_{m=-\infty}^{\infty} q^{\frac{(32m-27)^2}{8}} + \frac{1}{\sqrt{2}-1}\sum_{m=-\infty}^{\infty} q^{\frac{(32m-29)^2}{8}} + \sum_{m=-\infty}^{\infty} q^{\frac{(32m-31)^2}{8}}\Bigg\}.$$

Using Identities (3) and (4), in the right-hand side of the above equation, we find that

$$\sum_{n=0}^{\infty}(-1)^n B(n) q^{\frac{(2n+1)^2}{8}} = 2q^{\frac{1}{8}}[f(q^{120}, q^{136}) - q^{28} f(q^8, q^{248})]$$
$$+\frac{2}{\sqrt{2}-1} q^{\frac{9}{8}}[f(q^{104}, q^{152}) - q^{20} f(q^{24}, q^{232})] - 2q^{\frac{25}{8}}[f(q^{88}, q^{168}) - q^{12} f(q^{40}, q^{216})]$$
$$+\frac{2}{\sqrt{2}-1} q^{\frac{49}{8}}[f(q^{72}, q^{184}) - q^4 f(q^{56}, q^{200})]. \tag{47}$$

Using Identity (39), with setting $\{a = q^{28}, b = q^{36}\}$, $\{a = q^{20}, b = q^{44}\}$, $\{a = q^{12}, b = q^{52}\}$, and $\{a = q^{4}, b = q^{60}\}$, respectively, we obtain

$$f(-q^{28}, -q^{36}) = f(q^{120}, q^{136}) - q^{28} f(q^{8}, q^{248}), \tag{48}$$
$$f(-q^{20}, -q^{44}) = f(q^{104}, q^{152}) - q^{20} f(q^{24}, q^{232}), \tag{49}$$
$$f(-q^{12}, -q^{52}) = f(q^{88}, q^{168}) - q^{12} f(q^{40}, q^{216}), \tag{50}$$
$$f(-q^{4}, -q^{60}) = f(q^{72}, q^{184}) - q^{4} f(q^{56}, q^{200}). \tag{51}$$

Employing Identities (48)-(51) in Identity (47), we obtain

$$\sum_{n=0}^{\infty} (-1)^n B(n) q^{\frac{(2n+1)^2}{8}} = 2q^{\frac{1}{8}} \left\{ f(-q^{28}, -q^{36}) - q^3 f(-q^{12}, -q^{52}) \right.$$
$$\left. + q(\sqrt{2}+1)[f(-q^{20}, -q^{44}) + q^5 f(-q^4, -q^{60})] \right\}. \tag{52}$$

Combining Identities (45), (46), and (52), we find that

$$(1+\mu)\ P_7(q) + (1+\alpha)\ P_1(q) = 2\frac{q^{\frac{1}{24}}}{\eta(\tau)} \left\{ f(-q^{28}, -q^{36}) - q^3 f(-q^{12}, -q^{52}) \right.$$
$$\left. + (\sqrt{2}+1)[q f(-q^{20}, -q^{44}) + q^6 f(-q^4, -q^{60})] \right\}. \tag{53}$$

Multiplying both sides of Identity (53) by Identity (30), we obtain

$$\begin{aligned}
&\{(1+\mu) P_1(q) P_2(q) P_3(q) P_4(q) P_5(q) P_6(q) P_7^2(q)\} \\
&\quad + \{(1+\alpha) P_1^2(q) P_2(q) P_3(q) P_4(q) P_5(q) P_6(q) P_7(q)\} \\
&= 2\frac{q^{\frac{1}{24}}}{\eta(\tau)} \left\{ f(-q^{28}, -q^{36}) - q^3 f(-q^{12}, -q^{52}) \right. \\
&\quad \left. + (\sqrt{2}+1)[q f(-q^{20}, -q^{44}) + q^6 f(-q^4, -q^{60})] \right\}.
\end{aligned}$$

Now, using $\delta = 0$ in the left-hand side of the above equation and using the fact that $(-q^2; q^2)_\infty = q^{\frac{-2}{24}} \dfrac{\eta(4\tau)}{\eta(2\tau)}$, changing q to $q^{1/4}$ throughout, we obtain Identity (33). Proofs of Identities (34) and (35) are identical to the proof of Identity (33), so we omit them here. □

4. New Identities Associated with $I_1(q)$ and $I_2(q)$

In this section, we establish the following two new identities associated with $I_1(q)$ and $I_2(q)$ by using Ramanujan's ${}_1\Psi_1$ summation formula.

Theorem 2. *Let $|q| < 1$. Then, we have the identities*

$$\sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \frac{q^{3n}+q^{5n}}{1-q^{16n}} - \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \frac{q^{11n}+q^{13n}}{1-q^{16n}} = \frac{\eta^4(32\tau)}{\eta^2(16\tau)} \left(I_1(q^2) + \frac{1}{I_1(q^2)} \right), \tag{54}$$

and

$$\sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \frac{q^{n}+q^{7n}}{1-q^{16n}} - \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \frac{q^{9n}+q^{15n}}{1-q^{16n}} = \frac{\eta^4(32\tau)}{\eta^2(16\tau)} \left(I_2(q^2) + \frac{1}{I_2(q^2)} \right). \tag{55}$$

Proof. Changing n to $-n$ in the second summation in the left-hand side of Identity (54), we obtain

$$\begin{aligned} &\sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \frac{q^{3n}+q^{5n}}{1-q^{16n}} - \sum_{\substack{n=-\infty \\ n\equiv 1 \pmod 2}}^{-1} \frac{q^{-11n}+q^{-13n}}{1-q^{-16n}} \\ &= \sum_{n=-\infty}^{\infty} \frac{q^{6n+3}}{1-q^{32n+16}} + \sum_{n=-\infty}^{\infty} \frac{q^{10n+5}}{1-q^{32n+16}}. \end{aligned} \tag{56}$$

Using a corollary of Ramanujan's ${}_1\Psi_1$ summation formula [1, Entry 17, p. 32]

$$\sum_{n=-\infty}^{\infty} \frac{z^n}{1-aq^n} = \frac{(az, q/az, q, q; q)_\infty}{(a, q/a, z, q/z; q)_\infty}, \qquad |q| < |z| < 1, \tag{57}$$

in Identity (56), we find that

$$\begin{aligned} &\sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \frac{q^{3n}+q^{5n}}{1-q^{16n}} - \sum_{\substack{n=-\infty \\ n\equiv 1 \pmod 2}}^{-1} \frac{q^{-11n}+q^{-13n}}{1-q^{-16n}} \\ &= \frac{(q^{32};q^{32})_\infty^2}{(q^{16};q^{32})_\infty^2} \left\{ q^3 \frac{(q^{22},q^{10};q^{32})_\infty}{(q^6,q^{26};q^{32})_\infty} + q^5 \frac{(q^{26},q^6;q^{32})_\infty}{(q^{10},q^{22};q^{32})_\infty} \right\}. \end{aligned} \tag{58}$$

Using Identity (1) in Identity (58), we obtain Identity (54). The proof of Identity (55) is similar to the proof of Identity (54), so we omit it here. □

5. Eisenstein Series Identities Associated with $I_1(q)$ and $I_2(q)$

In this section, we prove four Eisenstein series identities associated with $I_1(q)$ and $I_2(q)$ by using the Jacobi theta function θ_1. Differentiating both sides of Identity (13), and then setting $z = 0$, yields

$$\theta_1'(0|\tau) = 2q^{1/8}(q;q)_\infty^3,$$

where θ_1' denotes the partial derivative of θ_1 with respect to z.

Now we prove a lemma, which is fruitful in deriving Eisenstein series identities associated with $I_1(q)$ and $I_2(q)$.

Lemma 2. *We have*

$$\sum_{n=1}^{\infty}\frac{q^n-q^{7n}-q^{9n}+q^{15n}}{1-q^{16n}}\sin 2nz$$
$$=-\frac{\theta_1'(0|16\tau)\theta_1(8\pi\tau|16\tau)\theta_1(6\pi\tau|16\tau)\theta_1(2z|16\tau)}{4\theta_1(z+\pi\tau|16\tau)\theta_1(z-\pi\tau|16\tau)\theta_1(z+7\pi\tau|16\tau)\theta_1(z-7\pi\tau|16\tau)}, \tag{59}$$

and

$$\sum_{n=1}^{\infty}\frac{q^{3n}-q^{5n}-q^{11n}+q^{13n}}{1-q^{16n}}\sin 2nz$$
$$=-\frac{\theta_1'(0|16\tau)\theta_1(2\pi\tau|16\tau)\theta_1(8\pi\tau|16\tau)\theta_1(2z|16\tau)}{4\theta_1(z+3\pi\tau|16\tau)\theta_1(z-3\pi\tau|16\tau)\theta_1(z+5\pi\tau|16\tau)\theta_1(z-5\pi\tau|16\tau)}. \tag{60}$$

Proof. For simplicity, we use $J(z|\tau)$ to denote the logarithmic derivative of θ_1 with respect to z. Logarithmically differentiating (13) with respect to z, after some simplifications, we have

$$J\left(z+\frac{\pi\tau}{2}|\tau\right)=-i+4\sum_{n=1}^{\infty}\frac{q^{n/2}}{1-q^n}\sin 2nz.$$

Replacing τ by 16τ in the above equation, we deduce that

$$J\left(z+8\pi\tau|16\tau\right)=-i+4\sum_{n=1}^{\infty}\frac{q^{8n}}{1-q^{16n}}\sin 2nz.$$

Replacing z by $z-7\pi\tau$ in the above equation, we obtain

$$J\left(z+\pi\tau|16\tau\right)=-i+4\sum_{n=1}^{\infty}\frac{q^{8n}}{1-q^{16n}}\sin 2n(z-7\pi\tau).$$

Writing z as $-z$ in the above equation, we are led to the identity

$$J\left(z-\pi\tau|16\tau\right)=i+4\sum_{n=1}^{\infty}\frac{q^{8n}}{1-q^{16n}}\sin 2n(z+7\pi\tau).$$

Adding the previous two equations together and using the trigonometric identity

$$\sin 2n(z+7\pi\tau)+\sin 2n(z-7\pi\tau)=2\cos 14n\pi\tau\sin 2nz=(q^{7n}+q^{-7n})\sin 2nz,$$

in the resulting equation, we immediately deduce that

$$J\left(z-\pi\tau|16\tau\right)+J\left(z+\pi\tau|16\tau\right)=4\sum_{n=1}^{\infty}\frac{q^{n}+q^{15n}}{1-q^{16n}}\sin 2nz.$$

In a similar way, we find that

$$J\left(z-7\pi\tau|16\tau\right)+J\left(z+7\pi\tau|16\tau\right)=4\sum_{n=1}^{\infty}\frac{q^{7n}+q^{9n}}{1-q^{16n}}\sin 2nz.$$

Combining the previous two equations, we find that

$$\begin{aligned}&4\sum_{n=1}^{\infty}\frac{q^{n}-q^{7n}-q^{9n}+q^{15n}}{1-q^{16n}}\sin 2nz\\&=J\left(z-\pi\tau|16\tau\right)+J\left(z+\pi\tau|16\tau\right)-J\left(z-7\pi\tau|16\tau\right)-J\left(z+7\pi\tau|16\tau\right).\end{aligned}\tag{61}$$

Recall the following remarkable identity, which can be found in [16, 20]:

$$\begin{aligned}&J(x_1|\tau)+J(x_2|\tau)+J(x_3|\tau)-J(x_1+x_2+x_3|\tau)\\&\quad=\frac{\theta_1'(0|\tau)\theta_1(x_1+x_2|\tau)\theta_1(x_2+x_3|\tau)\theta_1(x_1+x_3|\tau)}{\theta_1(x_1|\tau)\theta_1(x_2|\tau)\theta_1(x_3|\tau)\theta_1(x_1+x_2+x_3|\tau)}.\end{aligned}$$

Replacing τ by 16τ in the above equation and then letting x_1 to $z-\pi\tau, x_2$ to $z+\pi\tau, x_3$ to $-z+7\pi\tau$, we obtain

$$\begin{aligned}&J\left(z-\pi\tau|16\tau\right)+J\left(z+\pi\tau|16\tau\right)-J\left(z-7\pi\tau|16\tau\right)-J\left(z+7\pi\tau|16\tau\right)\\&=-\frac{\theta_1'(0|16\tau)\theta_1(6\pi\tau|16\tau)\theta_1(8\pi\tau|16\tau)\theta_1(2z|16\tau)}{\theta_1(z+\pi\tau|16\tau)\theta_1(z-\pi\tau|16\tau)\theta_1(z+7\pi\tau|16\tau)\theta_1(z-7\pi\tau|16\tau)}.\end{aligned}$$

Combining the above equation and Identity (61), we get Identity (59). The proof of Identity (60) is similar to the proof of Identity (59), so we omit it here. □

Using Identities (59) and (60), we can obtain the following Eisenstein series identities.

Theorem 3. *Let $|q|<1$. Then, we have the following identities*

$$\sum_{n=1}^{\infty}\frac{n(q^{n}-q^{7n}-q^{9n}+q^{15n})}{1-q^{16n}}=\frac{q(q^{16};q^{16})_\infty^2(q^8;q^8)_\infty^2(q^6,q^{10};q^{16})_\infty}{(q,q^7,q^9,q^{15};q^{16})_\infty^2},\tag{62}$$

and

$$\sum_{n=1}^{\infty}\frac{n(q^{3n}-q^{5n}-q^{11n}+q^{13n})}{1-q^{16n}}=\frac{q^3(q^{16};q^{16})_\infty^2(q^8;q^8)_\infty^2(q^2,q^{14};q^{16})_\infty}{(q^3,q^5,q^{11},q^{13};q^{16})_\infty^2}.\tag{63}$$

Proof. Dividing both sides of Identity (59) by z and then letting $z \to 0$, we are led to

$$\sum_{n=1}^{\infty} \frac{n(q^n - q^{7n} - q^{9n} + q^{15n})}{1 - q^{16n}} = -\frac{\theta_1'(0|16\tau)^2 \theta_1(8\pi\tau|16\tau)\theta_1(6\pi\tau|16\tau)}{4\theta_1^2(\pi\tau|16\tau)\theta_1^2(7\pi\tau|16\tau)}.$$

Using Identity (13), we easily find that

$$\theta_1(\pi\tau|16\tau) = iq^{3/2}(q, q^{15}, q^{16}; q^{16})_\infty, \; \theta_1(6\pi\tau|16\tau) = iq^{-1}(q^6, q^{10}, q^{16}; q^{16})_\infty,$$
$$\theta_1(7\pi\tau|16\tau) = iq^{-3/2}(q^7, q^9, q^{16}; q^{16})_\infty, \; \theta_1(8\pi\tau|16\tau) = iq^{-2}(q^8, q^8, q^{16}; q^{16})_\infty.$$

Combining the above two equations, we obtain Identity (62). The proof of Identity (63) is similar to the proof of Identity (62), so we omit it here. □

Theorem 4. *Let $|q| < 1$. Then, we have the following identities*

$$\sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^n - q^{7n} - q^{9n} + q^{15n}}{1 - q^{16n}} = \frac{q(q^8; q^8)_\infty^2 (q^{48}; q^{48})_\infty (q^6, q^{10}; q^{16})_\infty (q, q^7, q^9, q^{15}; q^{16})_\infty}{(q^{16}; q^{16})_\infty (q^3, q^{21}, q^{27}, q^{45}; q^{48})}, \tag{64}$$

and

$$\sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^{3n} - q^{5n} - q^{11n} + q^{13n}}{1 - q^{16n}} = \frac{q^3(q^8; q^8)_\infty^2 (q^{48}; q^{48})_\infty (q^2, q^{14}; q^{16})_\infty (q^3, q^5, q^{11}, q^{13}; q^{16})_\infty}{(q^{16}; q^{16})_\infty (q^9, q^{15}, q^{33}, q^{39}; q^{48})}. \tag{65}$$

Proof. If p is a prime, we use $\left(\frac{\cdot}{p}\right)$ to denote the Legendre symbol modulo p. Setting $z = \frac{\pi}{3}$ in Identity (59) and noting that

$$\sin \frac{2n\pi}{3} = \frac{\sqrt{3}}{2} \left(\frac{n}{3}\right), \quad \theta_1(\frac{2\pi}{3}|\tau) = \sqrt{3} q^{1/8} (q^3; q^3)_\infty,$$

we find that

$$\sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^n - q^{7n} - q^{9n} + q^{15n}}{1 - q^{16n}} = -\frac{q^2 \theta_1'(0|16\tau)\theta_1(6\pi\tau|16\tau)\theta_1(8\pi\tau|16\tau)(q^{48}; q^{48})_\infty}{2\theta_1(\pi/3 + \pi\tau|16\tau)\theta_1(\pi/3 - \pi\tau|16\tau)\theta_1(\pi/3 + 7\pi\tau|16\tau)\theta_1(\pi/3 - 7\pi\tau|16\tau)}. \tag{66}$$

Recall the beautiful identity [17, Eq. (3. 1)]

$$\theta_1\left(\frac{\pi}{3} - z|\tau\right)\theta_1\left(\frac{\pi}{3} + z|\tau\right) = \frac{(q; q)_\infty^3}{(q^3; q^3)_\infty} \frac{\theta_1(3z|3\tau)}{\theta_1(z|\tau)}.$$

Using the above identity in Identity (66), we obtain Identity (64). The proof of Identity (65) is similar to the proof of Identity (64), so we omit it here. □

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