# CONGRUENCES FOR F-PARTITIONS WITH 4 REPETITIONS

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**Abstract**

Let $\phi_k(n)$ denote the number of F-partitions of $n$ that allow up to $k$ repetitions of an integer in any row. In this paper we represent the generating function for $\phi_4(n)$ in terms of $q$-products and dissect it to obtain several congruences for $\phi_4(n)$.

## 1. Introduction

A *partition* of a positive integer $n$ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ with $\sum_{i=1}^{k} \lambda_i = n$. It is often useful to represent a partition of $n$ in terms of a two-rowed Frobenius symbol

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix} \tag{1}$$

with

$$n = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i,$$

where the strictly decreasing sequences of non-negative integers $\{a_1, a_2, \cdots, a_r\}$ and $\{b_1, b_2, \cdots, b_r\}$ are obtained by enumerating the dots above and below the main diagonal by rows and columns, respectively, in the Ferrers–Young diagram of the partition. For example, the Ferrers–Young diagram of the partition $\lambda = (5, 4, 3, 1)$ of 13 is

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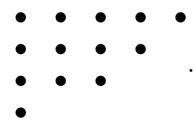

Enumerating the dots above and below the main diagonal by rows and columns, respectively, we can easily see that the partition $\lambda$ of 13 can be presented in Frobenius notation as

$$\begin{pmatrix} 4 & 2 & 0 \\ 3 & 1 & 0 \end{pmatrix}.$$

In [1], Andrews introduced the idea of generalized Frobenius partitions or, more simply, F-partitions which arise naturally as a combinatorial object associated to elliptic theta functions. A *generalized Frobenius partition* is an array as in (1) but the entries in the rows are allowed to be non-increasing. Andrews also discussed two general classes of F-partitions. In the first of these two classes, those F-partitions are considered that allow up to $k$ repetitions of an integer in any row. Let $\phi_k(n)$ denote the number of such F-partitions of $n$. For example, the 11 partitions enumerated by $\phi_3(4)$ are

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2\,0 \\ 0\,0 \end{pmatrix}, \begin{pmatrix} 0\,0 \\ 2\,0 \end{pmatrix}, \begin{pmatrix} 1\,1 \\ 0\,0 \end{pmatrix}, \begin{pmatrix} 0\,0 \\ 1\,1 \end{pmatrix}, \begin{pmatrix} 1\,0 \\ 1\,0 \end{pmatrix}, \begin{pmatrix} 1\,0\,0 \\ 0\,0\,0 \end{pmatrix}, \begin{pmatrix} 0\,0\,0 \\ 1\,0\,0 \end{pmatrix}.$$

For the second general class of F-partitions, $k$ copies $j_1, j_2, \cdots, j_k$ of each nonnegative integer $j$ are considered and an order relation between two copies $j_i$ and $l_h$ is defined by "$j_i < l_h$ if and only if $j < l$ or $j = l$ and $i < h$". Also $j_i$ is said to be distinct from $l_h$ unless $j = l$ and $i = h$. Further, $c\phi_k(n)$ represents the number of F-partitions of $n$ using these $k$ copies of integers with strict decrease in each row.

The generating function $\Phi_k(q)$ for $\phi_k(n)$ is given by [1],

$$\Phi_k(q) = \frac{\displaystyle\sum_{m_1,\, m_2,\, \ldots,\, m_{k-1}=-\infty}^{\infty} \zeta^{(k-1)m_1+(k-2)m_2+\ldots+m_{k-1}} q^{Q(m_1,\, m_2,\, \ldots,\, m_{k-1})}}{(q;q)_\infty^k}, \tag{2}$$

where

$$Q(m_1,\, m_2,\, \ldots,\, m_{k-1}) = m_1^2 + m_2^2 + \ldots + m_{k-1}^2 + \sum_{1 \le i < j \le k-1} m_i m_j,$$

$$(a;q)_\infty := \prod_{j=1}^{\infty} (1 - aq^{j-1}), \text{ and } \zeta = e^{2\pi i/(k+1)}.$$

In particular, Andrews also found the following elegant infinite product representations for $\Phi_1(q)$, $\Phi_2(q)$, and $\Phi_3(q)$.

$$\Phi_1(q) = \prod_{n=1}^{\infty} \frac{1}{1-q^n},$$

$$\Phi_2(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-q^{12n-2})(1-q^{12n-3})(1-q^{12n-9})(1-q^{12n-10})},$$

$$\Phi_3(q) = \prod_{n=1}^{\infty} \frac{(1-q^{12n-6})}{(1-q^{6n-1})(1-q^{6n-2})^2(1-q^{6n-3})^3(1-q^{6n-4})^2(1-q^{6n-5})(1-q^{12n})}.$$

The generating function for $c\phi_k(n)$ is given by [1],

$$\sum_{n=0}^{\infty} c\phi_k(n)q^n = \frac{\displaystyle\sum_{m_1,\, m_2,\, \ldots,\, m_{k-1}=-\infty}^{\infty} q^{Q(m_1,\, m_2,\, \ldots,\, m_{k-1})}}{(q;q)_\infty^k},$$

where

$$Q(m_1,\, m_2,\, \ldots,\, m_{k-1}) = m_1^2 + m_2^2 + \ldots + m_{k-1}^2 + \sum_{1 \le i < j \le k-1} m_i m_j.$$

In [1], it was also proved that

$$\phi_2(5n+3) \equiv c\phi_2(5n+3) \equiv 0 \pmod 5,$$
$$c\phi_k(n) \equiv 0 \pmod{k^2} \text{ if } k \text{ is a prime and does not divide } n.$$

Since its publication, a number of authors worked on these partition functions and uncovered a host of congruences, mostly for $c\phi_k(n)$. For example, Sellers [22] established that

$$\phi_3(3n+2) \equiv 0 \pmod 3.$$

Lovejoy [16] established modular forms whose Fourier coefficients are related to $c\phi_3(n)$ and proved the following congruences modulo 5, 7, 11 and 19 for $c\phi_3(n)$:

$$c\phi_3(45n+23) \equiv 0 \pmod 5,$$
$$c\phi_3(45n+41) \equiv 0 \pmod 5,$$
$$c\phi_3(63n+50) \equiv 0 \pmod 7,$$
$$c\phi_3(99n+95) \equiv 0 \pmod{11},$$
$$c\phi_3(171n+50) \equiv 0 \pmod{19}.$$

Baruah and Sarmah [4] represented the generating function for $c\phi_4(n)$ in terms of $q$-products and established the following congruences modulo powers of 4 for $c\phi_4(n)$:

$$c\phi_4(2n+1) \equiv 0 \pmod{4^2},$$
$$c\phi_4(4n+3) \equiv 0 \pmod{4^4},$$

$$c\phi_4(4n+2) \equiv 0 \pmod{4}.$$

Xia [25] proved the following congruences modulo 5 for $c\phi_4(n)$:

$$c\phi_4(20n+11) \equiv 0 \pmod{5}.$$

Hirschhorn and Sellers [11] proved the following characterization of $c\phi_4(10n+1)$ modulo 5:

$$c\phi_4(10n+1) \equiv \begin{cases} k+1 & \pmod{5} \text{ if } n = k(3k+1) \text{ for some integer } k, \\ 0 & \pmod{5} \text{ otherwise.} \end{cases}$$

From the above characterization they found the following infinite set of Ramanujan-like congruences modulo 5 satisfied by $c\phi_4(n)$. Let $p \geq 5$ be prime and let $r$ be an integer, $1 \leq r \leq p-1$, such that $12r+1$ is a quadratic non-residue modulo $p$. Then, for all $n \geq 0$,

$$c\phi_4(10pn+10r+1) \equiv 0 \pmod{5}.$$

Garvan and Sellers [8] proved several infinite families of congruences for $c\phi_k(n)$, where $k$ is allowed to grow arbitrarily large. In particular, they proved that, if $p$ is a prime, $r$ is an integer such that $0 < r < p$ and

$$c\phi_k(pn+r) \equiv 0 \pmod{p}$$

for all $n \geq 0$, then

$$c\phi_{pN+k}(pn+r) \equiv 0 \pmod{p}$$

for all $N \geq 0$ and $n \geq 0$. As a corollary, they proved a number of congruences for $c\phi_{pN+k}(pn+r)$ modulo $p$, where $p = 3$, 5, 7, and 11, for particular values of $k$. For some other congruences and families of congruences involving generalized Frobenius partitions we refer to [3, 7, 12, 15, 19, 20, 24, 26, 27].

Kolitsch [13, 14] introduced the function $\overline{c\phi}_k(n)$, which denotes the number of F-partitions of $n$ with $k$ colors whose order is $k$ under cyclic permutation of the $k$ colors. For example, the F-partitions enumerated by $\overline{c\phi}_2(2)$ are

$$\begin{pmatrix} 1_r \\ 0_r \end{pmatrix}, \begin{pmatrix} 1_g \\ 0_r \end{pmatrix}, \begin{pmatrix} 1_r \\ 0_g \end{pmatrix}, \begin{pmatrix} 1_g \\ 0_g \end{pmatrix}, \begin{pmatrix} 0_r \\ 1_r \end{pmatrix}, \begin{pmatrix} 0_r \\ 1_g \end{pmatrix}, \begin{pmatrix} 0_g \\ 1_r \end{pmatrix}, \text{ and } \begin{pmatrix} 0_g \\ 1_g \end{pmatrix},$$

where the subscripts represent the two colors of the non-negative integers. The generating function for $\overline{c\phi}_k(n)$ is given by [13],

$$\sum_{n=0}^{\infty} \overline{c\phi}_k(n) q^n = \frac{k \sum q^{Q(\mathbf{m})}}{(q;q)_\infty^k},$$

where the sum on the right extends over all vectors $\mathbf{m} = (m_1, m_2, \ldots, m_k)$ with $\mathbf{m}.\overline{1} = 1$ and $Q(\mathbf{m}) = \frac{1}{2}\sum_{i=1}^{k}(m_i - m_{i+1})^2$ wherein $\overline{1}$=$(1, 1, \ldots, 1)$ and $m_{k+1} = m_1$. Kolitsch [14] found that, for all integers $k \geq 2$,

$$\overline{c\phi}_k(n) \equiv 0 \pmod{k^2}.$$

Sellers [21, 23] established that

$$\overline{c\phi}_k(kn) \equiv 0 \pmod{k^3} \text{ for } k = 2,\ 3,\ 5,\ 7, \text{ and } 11.$$

Baruah and Sarmah [4] established the following congruences modulo powers of 4 for $\overline{c\phi}_4(n)$ :

$$\begin{aligned}\overline{c\phi}_4(2n) &\equiv 0 \pmod{4^3},\\ \overline{c\phi}_4(4n+3) &\equiv 0 \pmod{4^4},\\ \overline{c\phi}_4(4n) &\equiv 0 \pmod{4^4}.\end{aligned}$$

The existence of such a wide variety of results for $c\phi_k(n)$ and $\overline{c\phi}_k(n)$ for various values of $k$ inspires us to investigate the function $\phi_k(n)$ and to search for new results. The main objective of this paper is to represent the generating function for $\phi_4(n)$ in terms of $q$-products and dissect it to obtain a number of congruences for $\phi_4(n)$ modulo 2 and 5. In particular, we shall prove the following results.

**Theorem 1.** *We have*

$$\sum_{n=0}^{\infty}\phi_4(2n)q^n = \frac{f_2^{10}f_5}{f_1^9 f_4^2 f_{10}} + 2q\frac{f_2^{15}f_{20}^2}{f_1^{12}f_4^4f_{10}^2} - 8q\frac{f_4^6 f_{10}}{f_1^7 f_5}, \tag{3}$$

$$\sum_{n=0}^{\infty}\phi_4(2n+1)q^n = \frac{f_2^7 f_{10}^2}{f_1^6 f_4^2 f_5^2}, \tag{4}$$

*where $f_k$, $k \geq 1$, is defined as*

$$f_k := (q^k; q^k)_\infty.$$

**Theorem 2.** *For $n \geq 0$, we have*

$$\phi_4(4n+3) \equiv 0 \pmod{2}. \tag{5}$$

**Theorem 3.** *If $N$ is a positive integer which is not a multiple of 5, then we have*

$$\phi_4(2N+1) \equiv 0 \pmod{2}. \tag{6}$$

**Theorem 4.** *For $n \geq 0$, we have*

$$\phi_4(4n+1) \equiv \begin{cases} 1 & \pmod{2} \text{ if } n = \frac{5k(3k\pm1)}{2} \text{ for some integer } k,\\ 0 & \pmod{2} \text{ otherwise.}\end{cases} \tag{7}$$

**Theorem 5.** *For $n \geq 0$, we have*

$$\phi_4(10n+6) \equiv 0 \pmod 5. \tag{8}$$

**Theorem 6.** *For $n \geq 0$, we have*

$$\phi_4(10n+1) \equiv \begin{cases} k+1 & \pmod 5 \ \textit{if } n = k(3k+1) \textit{ for some integer } k, \\ 0 & \pmod 5 \ \textit{otherwise.} \end{cases}$$

We conclude this section with some well known identities that arise from Ramanujan's general theta function and a brief discussion on integer matrix exact covering systems as described in [6].

*Ramanujan's general theta function* $f(a,b)$ is given by [5, p.34, Eq.(18.1)],

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \text{ where } \quad |ab| < 1.$$

We also use the following two special cases of $f(a,b)$.

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^2 f_4^2}, \tag{9}$$

$$\psi(q) := f\left(q,q^3\right) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{f_2^2}{f_1}. \tag{10}$$

The product representations in the above two identities arise from Jacobi's triple product identity [5, p.34, Entry 19].
We also note that

$$\varphi(-q) = \frac{f_1^2}{f_2}. \tag{11}$$

An *exact covering system* is a partition of the set of integers into a finite set of arithmetic sequences. An *integer matrix exact covering system* is a partition of $\mathbb{Z}^n$, the set of all $n$-tuples with entries from $\mathbb{Z}$, into a lattice and a finite number of its translates without overlap.

Let

$$S = \sum_{x_1, x_2, \cdots, x_n = -\infty}^{\infty} f(x_1, x_2, \cdots, x_n).$$

We change the variables from $x_i$ to $y_i$ $(i = 1, 2, ..., n)$ by the transformation $y = Ax$, where $A$ is an integer matrix with $\det A \neq 0$,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \text{ and } y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

Then, as given in [4], $S$ can be written as a linear combination of $k$ parts using the integer matrix exact covering system $\left\{By + \frac{1}{d}Bc_r\right\}_{r=0}^{k-1}$ for $\mathbb{Z}^n$, where $B = \mathrm{sgn}(s_n(A))\dfrac{A^*}{d_{n-1}(A)}$ with $A^*$ = adjoint of $A$, $d_k(A) = k^{th}$ determinantal divisor of $A$, $s_n(A) = \frac{d_n(A)}{d_{n-1}(A)}$, $d = |s_n(A)|$, and $y \equiv c_r \pmod{d}$, $r = 0, 1, \cdots, k-1$ is the solution set of $By \equiv 0 \pmod{d}$.

## 2. Preliminaries

In this section, we list a number of lemmas that play important roles in the proofs of our main results.

**Lemma 1.** *If $ab = cd$, then*

$$f(a,b)\,f(c,d) = f(ad,bc)\,f(ac,bd) + af\left(c/a, a^2bd\right) f\left(d/a, a^2bc\right). \tag{12}$$

For a proof of Lemma 1, see [5, p.45, Entry 29].

**Lemma 2.** *We have*

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{13}$$

For a proof of Lemma 2, see [5, p.40, Entry 25].

**Lemma 3.** *We have*

$$\frac{f_1}{f_5} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q\frac{f_4^2 f_{40}}{f_8 f_{10}^2}. \tag{14}$$

For a proof of Lemma 3, see [10, 17].

**Lemma 4.** *We have*

$$\frac{1}{f_1^3 f_5} = \frac{f_4^4}{f_2^7 f_{10}} - 2q\frac{f_4^6 f_{20}^2}{f_2^9 f_{10}^3} + 5q\frac{f_4^3 f_{20}}{f_2^8} + 2q^2\frac{f_4^9 f_{40}^2}{f_2^{10} f_8^2 f_{10}^2 f_{20}}. \tag{15}$$

For a proof of Lemma 4, see [18].

**Lemma 5.** *We have*

$$\frac{f_5^5}{f_1^4 f_{10}^3} = \frac{f_5}{f_2^2 f_{10}} + 4q\frac{f_{10}^2}{f_1^3 f_2}, \tag{16}$$

$$\frac{f_2^3 f_5^2}{f_1^5 f_{10}^2} = \frac{f_5}{f_2^2 f_{10}} + 5q\frac{f_{10}^2}{f_1^3 f_2}. \tag{17}$$

For a proof of Lemma 5, see [2].

## 3. Proof of Theorem 1

We now present the proof of Theorem 1.

*Proof of Theorem 1.* From Equation (2), we have

$$\Phi_4(q) = \frac{S_4}{(q;\, q)_\infty^4}, \tag{18}$$

where

$$S_4 = \sum_{m_1,\, m_2,\, m_3=-\infty}^{\infty} \zeta^{3m_1+2m_2+m_3} q^{m_1^2+m_2^2+m_3^2+m_1m_2+m_2m_3+m_1m_3}$$

with $\zeta = e^{2\pi i/5}$. We change the variables from $m_1$, $m_2$, $m_3$ to $n_1$, $n_2$, $n_3$ using the integer matrix exact covering system

$$\left\{ \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

developed in [4]. Corresponding to this integer matrix exact covering system, we can write $S_4$ as a linear combination of four parts as

$$\begin{aligned}
S_4 = &\sum_{n_1,\, n_2,\, n_3=-\infty}^{\infty} \zeta^{2n_2+4n_3}\; q^{2n_1^2+2n_2^2+2n_3^2} \\
&+ \sum_{n_1,\, n_2,\, n_3=-\infty}^{\infty} \zeta^{3+2n_2+4n_3}\; q^{2n_1^2+2n_2^2+2n_3^2+2n_2+2n_3+1} \\
&+ \sum_{n_1,\, n_2,\, n_3=-\infty}^{\infty} \zeta^{2+2n_2+4n_3}\; q^{2n_1^2+2n_2^2+2n_3^2+2n_1+2n_3+1} \\
&+ \sum_{n_1,\, n_2,\, n_3=-\infty}^{\infty} \zeta^{1+2n_2+4n_3}\; q^{2n_1^2+2n_2^2+2n_3^2+2n_1+2n_2+1} \\
= &\left(\sum_{n_1=-\infty}^{\infty} q^{2n_1^2}\right)\left(\sum_{n_2=-\infty}^{\infty} \zeta^{2n_2} q^{2n_2^2}\right)\left(\sum_{n_3=-\infty}^{\infty} \zeta^{4n_3} q^{2n_3^2}\right) \\
&+ q\zeta^3 \left(\sum_{n_1=-\infty}^{\infty} q^{2n_1^2}\right)\left(\sum_{n_2=-\infty}^{\infty} \zeta^{2n_2} q^{2n_2^2+2n_2}\right)\left(\sum_{n_3=-\infty}^{\infty} \zeta^{4n_3} q^{2n_3^2+2n_3}\right) \\
&+ q\zeta^2 \left(\sum_{n_1=-\infty}^{\infty} q^{2n_1^2+2n_1}\right)\left(\sum_{n_2=-\infty}^{\infty} \zeta^{2n_2} q^{2n_2^2}\right)\left(\sum_{n_3=-\infty}^{\infty} \zeta^{4n_3} q^{2n_3^2+2n_3}\right)
\end{aligned}$$

$$+ q\zeta\left(\sum_{n_1=-\infty}^{\infty} q^{2n_1^2+2n_1}\right)\left(\sum_{n_2=-\infty}^{\infty} \zeta^{2n_2} q^{2n_2^2+2n_2}\right)\left(\sum_{n_3=-\infty}^{\infty} \zeta^{4n_3} q^{2n_3^2}\right). \tag{19}$$

Using Jacobi's triple product identity [5, p.34, Entry 19], we have

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \zeta^k q^{k^2} &= \prod_{k\geq 1}\left(1+\zeta^{-1}q^{2k-1}\right)\left(1+\zeta q^{2k-1}\right)\left(1-q^{2k}\right)\\ &= \left(-\zeta^{-1}q;\, q^2\right)_\infty \left(-\zeta q;\, q^2\right)_\infty \left(q^2;\, q^2\right)_\infty\\ &= f(\zeta q,\, \zeta^4 q). \end{aligned}\tag{20}$$

Using another version of Jacobi's triple product identity [9, p.11], we obtain

$$\begin{aligned}\sum_{k=-\infty}^{\infty} (-1)^k \zeta^k q^{\frac{k^2+k}{2}} &= \left(\zeta^{-1};\, q\right)_\infty (\zeta q;\, q)_\infty (q;\, q)_\infty\\ &= f(-\zeta^{-1},\, -\zeta q).\end{aligned}$$

Therefore, we have

$$\sum_{k=-\infty}^{\infty} \zeta^k q^{\frac{k^2+k}{2}} = f(\zeta^{-1},\, \zeta q) = f(\zeta^4,\, \zeta q). \tag{21}$$

Using Equations (20) and (21), we have

$$\sum_{n=-\infty}^{\infty} \zeta^{2n} q^{2n^2} = f(\zeta^2 q^2,\, \zeta^8 q^2) = f(\zeta^2 q^2,\, \zeta^3 q^2), \tag{22}$$

$$\sum_{n=-\infty}^{\infty} \zeta^{4n} q^{2n^2} = f(\zeta^4 q^2,\, \zeta^{16} q^2) = f(\zeta^4 q^2,\, \zeta q^2), \tag{23}$$

$$\sum_{n=-\infty}^{\infty} \zeta^{2n} q^{2n^2+2n} = f(\zeta^8,\, \zeta^2 q^4) = f(\zeta^3,\, \zeta^2 q^4), \tag{24}$$

$$\sum_{n=-\infty}^{\infty} \zeta^{4n} q^{2n^2+2n} = f(\zeta^{16},\, \zeta^4 q^4) = f(\zeta,\, \zeta^4 q^4). \tag{25}$$

We recall Equations (9) and (10) to note that

$$\varphi(q) := f(q,q) = \sum_{k=-\infty}^{\infty} q^{k^2}, \tag{26}$$

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{1}{2}\sum_{k=-\infty}^{\infty} q^{k(k+1)/2}. \tag{27}$$

Using Equations (22), (23), (24), (25), (26), and (27) in Equation (19), we have

$$S_4 = \varphi(q^2)f(\zeta^2q^2,\, \zeta^3q^2)f(\zeta^4q^2,\, \zeta q^2) + q\zeta^3\varphi(q^2)f(\zeta^3,\, \zeta^2q^4)f(\zeta,\, \zeta^4q^4)$$
$$+ 2q\zeta^2\psi(q^4)f(\zeta^2q^2,\, \zeta^3q^2)f(\zeta,\, \zeta^4q^4) + 2q\zeta\psi(q^4)f(\zeta^4q^2,\, \zeta q^2)f(\zeta^3,\, \zeta^2q^4). \quad (28)$$

Now

$$\begin{aligned}
&f(\zeta^2q^2,\, \zeta^3q^2)f(\zeta^4q^2,\, \zeta q^2)\\
&= \left(-\zeta^2q^2;\, q^4\right)_\infty \left(-\zeta^3q^2;\, q^4\right)_\infty \left(-\zeta^4q^2;\, q^4\right)_\infty \left(-\zeta q^2;\, q^4\right)_\infty \left(q^4;\, q^4\right)_\infty^2\\
&= \left(-\zeta q^2;\, q^4\right)_\infty \left(-\zeta^2q^2;\, q^4\right)_\infty \left(-\zeta^3q^2;\, q^4\right)_\infty \left(-\zeta q^4q^2;\, q^4\right)_\infty \left(-q^2;\, q^4\right)_\infty\\
&\qquad \times \frac{\left(q^4;\, q^4\right)_\infty^2}{\left(-q^2;\, q^4\right)_\infty}\\
&= \prod_{n=0}^{\infty} \left(1+\zeta q^{4n+2}\right)\left(1+\zeta^2 q^{4n+2}\right)\left(1+\zeta^3 q^{4n+2}\right)\left(1+\zeta^4 q^{4n+2}\right)\left(1+q^{4n+2}\right)\\
&\qquad \times \frac{\left(q^4;\, q^4\right)_\infty^2}{\left(-q^2;\, q^4\right)_\infty}\\
&= \prod_{n=0}^{\infty}\left(1+q^{20n+10}\right)\frac{\left(q^4;\, q^4\right)_\infty^2}{\left(-q^2;\, q^4\right)_\infty}\\
&= \left(-q^{10};\, q^{20}\right)_\infty \left(q^2;\, q^2\right)_\infty \left(q^8;\, q^8\right)_\infty\\
&= \frac{\left(q^{20};\, q^{20}\right)_\infty^2}{\left(q^{40};\, q^{40}\right)_\infty \left(q^{10};\, q^{10}\right)_\infty}\left(q^2;\, q^2\right)_\infty \left(q^8;\, q^8\right)_\infty\\
&= \frac{f_2f_8f_{20}^2}{f_{10}f_{40}}. \qquad (29)
\end{aligned}$$

Similarly, we find that

$$\begin{aligned}
&f(\zeta^3,\, \zeta^2q^4)f(\zeta,\, \zeta^4q^4)\\
&= \left(-\zeta^3;\, q^4\right)_\infty \left(-\zeta^2q^4;\, q^4\right)_\infty \left(-\zeta;\, q^4\right)_\infty \left(-\zeta^4q^4;\, q^4\right)_\infty \left(q^4;\, q^4\right)_\infty^2\\
&= \prod_{n=0}^{\infty}\left(1+\zeta^3q^{4n}\right)\left(1+\zeta^2q^{4n+4}\right)\left(1+\zeta q^{4n}\right)\left(1+\zeta^4q^{4n+4}\right)\left(q^4;\, q^4\right)_\infty^2\\
&= (1+\zeta)\left(1+\zeta^3\right)\prod_{n=0}^{\infty}\left(1+\zeta^3q^{4n+4}\right)\left(1+\zeta^2q^{4n+4}\right)\left(1+\zeta q^{4n+4}\right)\left(1+\zeta^4q^{4n+4}\right)\\
&\qquad \times \left(q^4;\, q^4\right)_\infty^2\\
&= -\frac{1}{\zeta^3}\prod_{n=0}^{\infty}\left(1+q^{20n+20}\right)\frac{\left(q^4;\, q^4\right)_\infty^2}{\left(-q^4;\, q^4\right)_\infty}\\
&= -\frac{1}{\zeta^3}\left(-q^{20};\, q^{20}\right)_\infty \frac{\left(q^4;\, q^4\right)_\infty^2}{\left(-q^4;\, q^4\right)_\infty}
\end{aligned}$$

$$= -\frac{1}{\zeta^3}\frac{f_{40}}{f_{20}}\frac{f_4^2 f_4}{f_8}$$
$$= -\frac{1}{\zeta^3}\frac{f_4^3 f_{40}}{f_8 f_{20}}. \tag{30}$$

Next, we take

$$A(q) = f(\zeta^3,\, \zeta^2 q^4) f(\zeta^4 q^2,\, \zeta q^2) + \zeta f(\zeta,\, \zeta^4 q^4) f(\zeta^2 q^2,\, \zeta^3 q^2).$$

Setting $a = \zeta$, $b = \zeta^4 q^2$, $c = \zeta^3 q^2$ and $d = \zeta^2$ in Equation (12), we have

$$A(q) = f(\zeta,\, \zeta^4 q^2) f(\zeta^2,\, \zeta^3 q^2). \tag{31}$$

Now

$$\begin{aligned}
&f(\zeta,\, \zeta^4 q^2) f(\zeta^2,\, \zeta^3 q^2)\\
&= \left(-\zeta;\, q^2\right)_\infty \left(-\zeta^4 q^2;\, q^2\right)_\infty \left(-\zeta^2;\, q^2\right)_\infty \left(-\zeta^3 q^2;\, q^2\right)_\infty \left(q^2;\, q^2\right)_\infty^2\\
&= \prod_{n=0}^{\infty} \left(1+\zeta q^{2n}\right)\left(1+\zeta^4 q^{2n+2}\right)\left(1+\zeta^2 q^{2n}\right)\left(1+\zeta^3 q^{2n+2}\right)\left(q^2;\, q^2\right)_\infty^2\\
&= (1+\zeta)\left(1+\zeta^2\right)\prod_{n=0}^{\infty}\left(1+\zeta q^{2n+2}\right)\left(1+\zeta^4 q^{2n+2}\right)\left(1+\zeta^2 q^{2n+2}\right)\left(1+\zeta^3 q^{2n+2}\right)\\
&\qquad \times \left(q^2;\, q^2\right)_\infty^2\\
&= -\frac{1}{\zeta}\prod_{n=0}^{\infty}\left(1+q^{10n+10}\right)\frac{\left(q^2;\, q^2\right)_\infty^2}{\left(-q^2;\, q^2\right)_\infty}\\
&= -\frac{1}{\zeta}\left(-q^{10};\, q^{10}\right)_\infty \frac{\left(q^2;\, q^2\right)_\infty^2}{\left(-q^2;\, q^2\right)_\infty}\\
&= -\frac{1}{\zeta}\frac{f_{20}}{f_{10}}\frac{f_2^3}{f_4}.
\end{aligned} \tag{32}$$

From Equations (31) and (32), we have

$$A(q) = f(\zeta^3,\, \zeta^2 q^4) f(\zeta^4 q^2,\, \zeta q^2) + \zeta f(\zeta,\, \zeta^4 q^4) f(\zeta^2 q^2,\, \zeta^3 q^2) = -\frac{1}{\zeta}\frac{f_{20}}{f_{10}}\frac{f_2^3}{f_4}. \tag{33}$$

Employing Equations (29), (30), and (33) in Equation (28), we have

$$\begin{aligned}
S_4 &= \varphi(q^2)\frac{f_2 f_8 f_{20}^2}{f_{10} f_{40}} + q\zeta^3 \varphi(q^2)\left(-\frac{1}{\zeta^3}\frac{f_4^3 f_{40}}{f_8 f_{20}}\right) + 2q\zeta\psi(q^4)\left(-\frac{1}{\zeta}\frac{f_2^3 f_{20}}{f_4 f_{10}}\right)\\
&= \frac{f_4^5}{f_2^2 f_8^2}\times\frac{f_2 f_8 f_{20}^2}{f_{10} f_{40}} - q\frac{f_4^5}{f_2^2 f_8^2}\times\frac{f_4^3 f_{40}}{f_8 f_{20}} - 2q\frac{f_8^2}{f_4}\times\frac{f_2^3 f_{20}}{f_4 f_{10}}\\
&= \frac{f_4^5 f_{20}^2}{f_2 f_8 f_{10} f_{40}} - q\frac{f_4^8 f_{40}}{f_2^2 f_8^3 f_{20}} - 2q\frac{f_2^3 f_8^2 f_{20}}{f_4^2 f_{10}}.
\end{aligned} \tag{34}$$

Using Equation (34) in Equation (18), we find that

$$\begin{aligned}\Phi_4(q) &= \frac{1}{f_1^4}\left[\frac{f_4^5 f_{20}^2}{f_2 f_8 f_{10} f_{40}} - q\frac{f_4^8 f_{40}}{f_2^2 f_8^3 f_{20}} - 2q\frac{f_2^3 f_8^2 f_{20}}{f_4^2 f_{10}}\right]\\ &= \frac{1}{f_1^4}\left[\frac{f_4^6 f_{10}^2}{f_2^2 f_8^2 f_{20}}\left(\frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q\frac{f_4^2 f_{40}}{f_8 f_{10}^2}\right) - 2q\frac{f_2^3 f_8^2 f_{20}}{f_4^2 f_{10}}\right]. \end{aligned} \tag{35}$$

Employing Equation (14) in Equation (35), we have

$$\begin{aligned}\Phi_4(q) &= \frac{1}{f_1^4}\left[\frac{f_4^6 f_{10}^2}{f_2^2 f_8^2 f_{20}} \times \frac{f_1}{f_5} - 2q\frac{f_2^3 f_8^2 f_{20}}{f_4^2 f_{10}}\right]\\ &= \frac{f_4^6 f_{10}^2}{f_1^3 f_2^2 f_5 f_8^2 f_{20}} - 2q\frac{f_2^3 f_8^2 f_{20}}{f_1^4 f_4^2 f_{10}}. \end{aligned} \tag{36}$$

Using Equations (13) and (15) in Equation (36), we have

$$\begin{aligned}\Phi_4(q) &= \frac{f_4^6 f_{10}^2}{f_2^2 f_8^2 f_{20}}\left[\frac{f_4^4}{f_2^7 f_{10}} - 2q\frac{f_4^6 f_{20}^2}{f_2^9 f_{10}^3} + 5q\frac{f_4^3 f_{20}}{f_2^8} + 2q^2\frac{f_4^9 f_{40}^2}{f_2^{10} f_8^2 f_{10}^2 f_{20}}\right]\\ &\quad - 2q\frac{f_2^3 f_8^2 f_{20}}{f_4^2 f_{10}}\left[\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right]\\ &= \left(\frac{f_4^{10} f_{10}}{f_2^9 f_8^2 f_{20}} + 2q^2\frac{f_4^{15} f_{40}^2}{f_2^{12} f_8^4 f_{20}^2} - 8q^2\frac{f_8^6 f_{20}}{f_2^7 f_{10}}\right)\\ &\quad + \left(5q\frac{f_4^9 f_{10}^2}{f_2^{10} f_8^2} - 2q\frac{f_4^{12} f_{20}}{f_2^{11} f_8^2 f_{10}} - 2q\frac{f_4^{12} f_{20}}{f_2^{11} f_8^2 f_{10}}\right). \end{aligned} \tag{37}$$

Equating the coefficients of $q^{2n}$ and then replacing $q^2$ by $q$ in Equation (37), we find that

$$\sum_{n=0}^{\infty}\phi_4(2n)q^n = \frac{f_2^{10} f_5}{f_1^9 f_4^2 f_{10}} + 2q\frac{f_2^{15} f_{20}^2}{f_1^{12} f_4^4 f_{10}^2} - 8q\frac{f_4^6 f_{10}}{f_1^7 f_5},$$

which is Equation (3).

Similarly, equating the coefficients of $q^{2n+1}$, dividing both sides of the resulting identity by $q$ and replacing $q^2$ by $q$ in Equation (37), we find that

$$\sum_{n=0}^{\infty}\phi_4(2n+1)q^n = 5\frac{f_2^9 f_5^2}{f_1^{10} f_4^2} - 4\frac{f_2^{12} f_{10}}{f_1^{11} f_4^2 f_5}. \tag{38}$$

Now, multiplying Equation (16) by 5 and Equation (17) by 4, and then subtracting the resulting equations, we have

$$5\frac{f_5^5}{f_1^4 f_{10}^3} - 4\frac{f_2^3 f_5^2}{f_1^5 f_{10}^2} = \frac{f_5}{f_2^2 f_{10}}. \tag{39}$$

Multiplying Equation (39) by $\dfrac{f_2^9 f_{10}^3}{f_1^6 f_4^2 f_5^3}$, we obtain

$$5\frac{f_2^9 f_5^2}{f_1^{10} f_4^2} - 4\frac{f_2^{12} f_{10}}{f_1^{11} f_4^2 f_5} = \frac{f_2^7 f_{10}^2}{f_1^6 f_4^2 f_5^2}. \tag{40}$$

Using Equation (40) in Equation (38), we find that

$$\sum_{n=0}^{\infty} \phi_4(2n+1)q^n = \frac{f_2^7 f_{10}^2}{f_1^6 f_4^2 f_5^2},$$

which is Equation (4). □

## 4. Proofs of Theorem 2-6

*Proof of Theorem 2.* Using Equations (9) and (11), we rewrite Equation (4) as

$$\begin{aligned}\sum_{n=0}^{\infty} \phi_4(2n+1)q^n &= \frac{f_2^7 f_{10}^2}{f_1^6 f_4^2 f_5^2} \\ &= \frac{\varphi(q)}{\varphi^2(-q)} \frac{f_{10}^2}{f_5^2}. \end{aligned} \tag{41}$$

From [9, p.311, Eq. (34.1.1)], we find that

$$\begin{aligned}\varphi(q) &= -\varphi(q^2) + 2\frac{(q^3;\, q^8)_\infty (q^5;\, q^8)_\infty (q^8;\, q^8)_\infty}{(q;\, q^8)_\infty (q^4;\, q^8)_\infty (q^7;\, q^8)_\infty} \\ &\equiv \varphi(q^2) \pmod{2}. \end{aligned} \tag{42}$$

Similarly, from [9, p.15, Eq. (1.10.1)], we have

$$\varphi^2(-q) \equiv \varphi^2(q^2) \pmod{2}. \tag{43}$$

Using Equations (42) and (43) in Equation (41), we find that

$$\sum_{n=0}^{\infty} \phi_4(2n+1)q^n \equiv \frac{f_{10}}{\varphi(q^2)} \pmod{2},$$

from which Equation (5) follows. □

*Proof of Theorem 3.* From Equation (41), we have

$$\sum_{n=0}^{\infty} \phi_4(2n+1)q^n = \frac{\varphi(q)}{\varphi^2(-q)} \frac{f_{10}^2}{f_5^2}. \tag{44}$$

From [9, p.311, Eq. (34.1.7)] and [9, p.313, Eq. (34.1.20)], respectively, we have

$$\varphi(q) \equiv \varphi(q^5) \pmod 2, \tag{45}$$

$$\varphi^2(-q) \equiv \varphi^2(-q^5) \pmod 2. \tag{46}$$

Using Equations (45) and (46) in Equation (44), we arrive at Equation (6). □

*Proof of Theorem 4.* Employing Equation (15) in Equation (4), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_4(2n+1)q^n &= \frac{f_4^6}{f_2^7} - 4q\frac{f_4^8 f_{20}^2}{f_2^9 f_{10}^2} + 10q\frac{f_4^5 f_{20} f_{10}}{f_2^8} + 4q^2\frac{f_4^{11} f_{40}^2}{f_2^{10} f_8^2 f_{10} f_{20}} \\ &\quad + 4q^2\frac{f_4^{10} f_{20}^4}{f_2^{11} f_{10}^4} - 20q^2\frac{f_4^7 f_{20}^3}{f_2^{10} f_{10}} - 8q^3\frac{f_4^{13} f_{20} f_{40}^2}{f_2^{12} f_8^2 f_{10}^3} \\ &\quad + 25q^2\frac{f_4^4 f_{10}^2 f_{20}^2}{f_2^9} + 20q^3\frac{f_4^{10} f_{40}^2}{f_2^{11} f_8^2} + 4q^4\frac{f_4^{16} f_{40}^4}{f_2^{13} f_8^4 f_{10}^2 f_{20}^2}.\end{aligned} \tag{47}$$

Equating the coefficients of $q^{2n}$ and replacing $q^2$ by $q$ in Equation (47), we find that

$$\begin{aligned}&\sum_{n=0}^{\infty}\phi_4(4n+1)q^n \\ &= \frac{f_2^6}{f_1^7} + 4q\frac{f_2^{11} f_{20}^2}{f_1^{10} f_4^2 f_5 f_{10}} + 4q\frac{f_2^{10} f_{10}^4}{f_1^{11} f_5^4} - 20q\frac{f_2^7 f_{10}^3}{f_1^{10} f_5} + 25q\frac{f_2^4 f_5^2 f_{10}^2}{f_1^9} + 4q^2\frac{f_2^{16} f_{20}^4}{f_1^{13} f_4^4 f_5^2 f_{10}^2} \\ &\equiv \frac{f_2^3}{f_1} + q\frac{f_{10}^3}{f_1} \pmod 2.\end{aligned} \tag{48}$$

Now, from Equation (15), we have

$$\frac{f_2^3}{f_1} + q\frac{f_{10}^3}{f_1} \equiv f_5 \pmod 2. \tag{49}$$

From Equations (48) and (49), we find that

$$\sum_{n=0}^{\infty}\phi_4(4n+1)q^n \equiv f_5 \pmod 2 \equiv 1 + \sum_{n=1}^{\infty} q^{\frac{5n(3n\pm1)}{2}} \pmod 2. \tag{50}$$

Now, Equation (7) follows from Equation (50). □

*Proof of Theorem 5.* We have

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_4(2n)q^n &= \frac{f_2^{10} f_5}{f_1^9 f_4^2 f_{10}} + 2q\frac{f_2^{15} f_{20}^2}{f_1^{12} f_4^4 f_{10}^2} - 8q\frac{f_4^6 f_{10}}{f_1^7 f_5} \\ &\equiv \frac{f_1^3}{f_5}\varphi(q) + 2q\frac{f_{20}}{f_5^2}\varphi(q) f_4^3 - 8q\frac{f_{10} f_{20}}{f_5^3} f_1^3 f_4 \pmod 5.\end{aligned} \tag{51}$$

From [9, p.341, Eq. (36.3.1)], we have the 5−dissection of $\varphi(q)$ as

$$\varphi(q) = L_0 + L_1 + L_4, \tag{52}$$

where $L_0 = \varphi(q^{25})$, $L_1 = 2qf(q^{15}, q^{35})$, and $L_4 = 2q^4 f(q^5, q^{45})$.

From [9, p.31, Eq. (3.2.5)], we have the 5−dissection of $f_1$ as

$$f_1 = E_0 + E_1 + E_2, \tag{53}$$

where

$$\begin{aligned}
E_0 &= f_{25}\frac{(q^{10}, q^{25})_\infty (q^{15}, q^{25})_\infty}{(q^5, q^{25})_\infty (q^{20}, q^{25})_\infty},\\
E_1 &= -qf_{25},\\
E_2 &= -q^2 f_{25}\frac{(q^5, q^{25})_\infty (q^{20}, q^{25})_\infty}{(q^{10}, q^{25})_\infty (q^{15}, q^{25})_\infty}.
\end{aligned}$$

Further, from [9, p.33, Eq. (3.2.6)], we have the 5−dissection of $f_1^3$ as

$$f_1^3 = J_0 + J_1, \tag{54}$$

where

$$\begin{aligned}
J_0 &= (q^{10}; q^{25})_\infty (q^{15}; q^{25})_\infty (q^{25}; q^{25})_\infty,\\
J_1 &= -3q(q^5; q^{25})_\infty (q^{20}; q^{25})_\infty (q^{25}; q^{25})_\infty.
\end{aligned}$$

Using Equations (52), (53), and (54) in Equation (51), we find that there are no terms of the form $q^{5n+3}$, $n \geq 0$, in the resulting congruence, from which Equation (8) follows. □

*Proof of Theorem 6.* We have

$$\begin{aligned}
\sum_{n=0}^{\infty} \phi_4(2n+1)q^n &= \frac{f_2^7 f_{10}^2}{f_1^6 f_4^2 f_5^2}\\
&\equiv \frac{f_2^{17}}{f_1^{16} f_4^2} \pmod 5.
\end{aligned}$$

For the remaining part of the proof, see [11, Theorem 1.3]. □

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